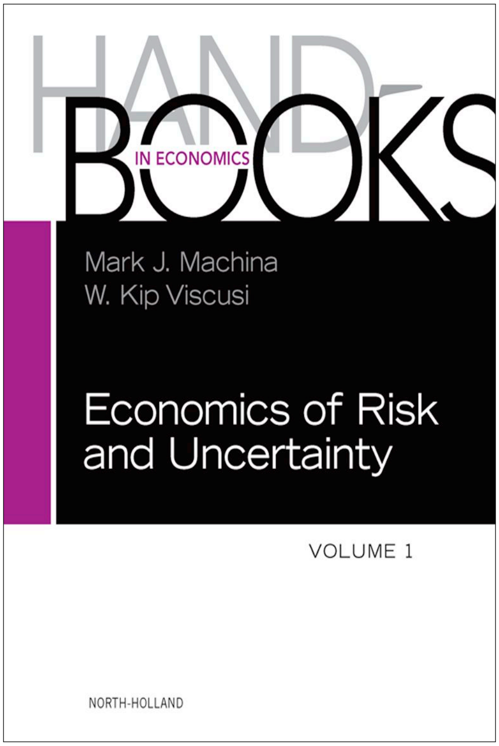

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CHAPTER 1

Axiomatic Foundations of Expected Utility and Subjective Probability

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Contents

Abstract

This chapter provides a critical review of the theories of decision making under risk and under uncertainty and the notion of choice-based subjective probabilities. It includes formal statements and discussions of the various models, including their analytical frameworks, the corresponding axiomatic foundations, and the representations.

Keywords

Expected Utility Theory, Subjective Expected Utility Theory, Subjective Probabilities, Incomplete Preferences, Bayesianism

JEL Classification Codes

D80, D81

1.1 INTRODUCTION

Expected utility theory consists of two main models. The first, expected utility under risk, is concerned with the evaluation of risky prospects, depicted as lotteries over an arbitrary set of outcomes, or prizes. Formally, let $X = \{x_1, \ldots, x_n\}$ be a set of outcomes

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and denote a risky prospect by $(x_1, p_1; \ldots; x_n, p_n)$, where, for each i, p_i denotes the probability of the outcomes x_i. According to the model of expected utility under risk, risky prospects are evaluated by the formula $\Sigma_{i=1}^n u(x_i) p_i$, where u is a real-valued function on X representing the decision maker's tastes. The second expected utility under uncertainty is concerned with the evaluation of random variables, or *acts*, representing alternative courses of action, whose distributions are not included in the data. Formally, let S be a set of states of nature and denote by F the set of acts (that is, the set of all mappings from S to X). According to the model of expected utility under uncertainty, an act f is evaluated using the formula $\Sigma_{i=1}^n u(x_i) \pi(f^{-1}(x_i))$, where π is a probability measure on S representing the decision maker's beliefs regarding the likelihoods of the *events* (that is, subsets of S).[1] The foundations of these models are preference relations on the corresponding choice sets whose structures justify the use of these formulas to evaluate and choose among risky prospects or acts.

1.1.1 Decision Making in the Face of Uncertainty and Subjective Probabilities: Interpretations and Methodology

The notion that an individual's degree of belief in the truth of propositions, or the likely realization of events, is quantifiable by probabilities is as old as the idea of probability itself, dating back to the second half of the 17th century. By contrast, the idea of inferring an individual's degree of belief in the truth of propositions, or the likely realization of events, from his/her choice behavior, and quantifying these beliefs by probabilities, took shape in the early part of the twentieth century. The idea that expected utility is the criterion for evaluating, and choosing among, risky alternatives dates back to the first half of the eighteenth century, whereas the axiomatization of this criterion is a modern concept developed in the mid-twentieth century.

Subjective expected utility theory is the result of the fusion of these two developments, which took place in the 1950s. It is based on three premises: (a) that decision making is (or ought to be) a process involving the evaluation of possible outcomes associated with alternative courses of action and the assessment of their likelihoods; (b) that the evaluation of outcomes and the assessment of their likelihoods are (or ought to be) quantifiable, by utilities and subjective probabilities, respectively, the former representing the decision maker's tastes, the latter his/her beliefs; and (c) that these ingredients of the decision-making process can be inferred from observed (or prescribed) patterns of choice and are (or should be) integrated to produce a criterion of choice.

The theories of subjective probabilities and expected utility are open to alternative, not mutually exclusive, interpretations. The positive interpretation, anchored in the revealed preference methodology, presumes that actual choice behavior abides by

[1] The random variable f induces a partition $\mathcal{P}$ of S, whose cells are the preimages of x_i under f (that is, $\mathcal{P} = \{f^{-1}(x_i) \mid i = 1, \ldots, n\}$).

the principles underlying the expected utility criterion and that subjective probabilities can be inferred from observing this behavior. The normative interpretation presumes that the principles underlying the expected utility criterion are basic tenets of rational choice and that rational choice implies, among others, the possibility of quantifying beliefs by probabilities.

The positive interpretation is an hypothesis about choice behavior, which, like other scientific theories, has testable implications. By contrast, the normative interpretation maintains that there are principles of rational behavior sufficiently compelling to make reasonable individuals use them to guide their decisions. Moreover, according to the normative interpretation the theory may be used to educate decision makers on how to organize their thoughts and information so as to make decisions that better attain their objectives. In the same vein, the theory provides "a set of criteria by which to detect, with sufficient trouble, any inconsistencies that may be among our beliefs, and to derive from beliefs we already hold such new ones as consistency demands." (Savage, 1954, p. 20).

1.1.2 A Brief History

From the very first, circa 1660, the idea of probability assumed dual meanings: the aleatory meaning, according to which probability is a theory about the relative frequency of outcomes in repeated trials; and the epistemological meaning, according to which probability is a theory of measurement of a person's "degree of belief" in the truth of propositions, or the likelihoods he assigns to events.[2] Both the "objective" and the "subjective" probabilities, as these meanings are commonly called, played important roles in the developments that led to the formulation of modern theories of decision making under risk and under uncertainty and of the theory of statistics. From their inception, the ideas of probabilities were tied to decisions. The aleatory interpretation is associated with choosing how to play games of chance; the epistemological interpretation is prominent in Pascal's analysis of a wager on the existence of God.[3]

The formal ideas of utility and expected utility (or "moral expectations") maximization as a criterion for evaluating gambles were originally introduced by Bernoulli (1738) to resolve the difficulty with using expected value posed by the St. Petersburg paradox. Bernoulli resolved the paradox by assuming a logarithmic utility function of wealth, whose essential property was "diminishing marginal utility." Bernoulli's preoccupation with games of chance justifies his taking for granted the existence of probabilities in the aleatory, or objective, sense. Bernoulli's and later treatments of the notion of utility in the eighteenth and nineteenth centuries have a distinct cardinal flavor. By contrast, the nature of modern utility theory is ordinal (that is, the utility is a numerical representation of ordinal preferences), and its empirical content is choice behavior. Put

[2] See Hacking (1984) for a detailed discussion.

[3] See Devlin's (2008) review of the correspondence between Pascal and Fermat concerning the division problem and Pascal (2010).

differently, the theory of choice is based on the premise that choice is, or ought to be, governed by ordinal preference relations on the set of alternatives, called the choice set. The theory's main concern is the study of the structures of choice sets and preference relations that allow the representation of the latter by utility functions.

The theories of individual choice under risk and under uncertainty are special branches of the general theory of choice, characterized by choice sets the elements of which are courses of action whose ultimate outcomes are not known at the time the choice must be made. Expected utility theory is a special instance of the theory of choice under objective and subjective uncertainty. In expected utility theory under objective uncertainty, or risk, the probabilities are a primitive concept representing the objective uncertainty. The theory's main concern is the representation of individual attitudes toward risk. Its basic premises are that (a) because the outcomes, x_i, are mutually exclusive, the evaluation of risky prospects entails separate evaluations of the outcomes, (b) these evaluations are quantifiable by a cardinal utility, u, and (c) the utilities of the alternative outcomes are aggregated by taking their expectations with respect to the objective probabilities, $p_1, \ldots, p_n$. Expected utility theory under subjective uncertainty is based on the presumption that the preference relation is itself a fusion of two underlying relations: (a) the relation "more likely than," on events, expressing the decision maker's beliefs regarding the likelihoods of the events and (b) the relation "more desirable risk than" depicting his/her evaluation of the outcomes and risk-attitudes. The beliefs, according to this theory, are quantifiable by (subjective) probabilities, $\pi(E)$, for every event $E \subseteq S$. The subjective probabilities play a role analogous to that of objective probability under objective uncertainty, thereby reducing the problem of decision making under uncertainty to decision making under risk, and permitting risk attitudes to be quantified by a utility function.

Throughout the historical processes of their respective evolutions, the idea of representing risk attitudes via cardinal utility was predicated on the existence of objective probabilities, and the notion of subjective probabilities presumed the existence of cardinal utility representing the decision maker's risk attitudes. In the early part of the twentieth century, Ramsey (1931) and de Finetti (1937) independently formalized the concept of choice-based subjective probability. Both assumed that, when betting on the truth of propositions, or on the realization of events, individuals seek to maximize their expected utility.[4] They explored conditions under which the degree of belief of a decision maker in the truth of a proposition, or event, may be inferred from his/her betting behavior and quantified by probability.

Invoking the axiomatic approach, which takes the existence of utilities as given, and assuming that individual choice is governed by expected utility maximization,

[4] In the case of de Finetti, the utility function is linear. Maximizing expected utility is, hence, equivalent to maximizing expected value.

Ramsey (1931) sketched a proof of the existence of subjective probabilities. According to Ramsey, "the degree of belief is a casual property of it, which can be expressed vaguely as the extent to which we are prepared to act on it" (Ramsey, 1931, p. 170).

Taking a similar attitude, de Finetti (1937) writes that "the degree of probability attributed by an individual to a given event is revealed by the conditions under which he would be disposed to bet on that event" (de Finetti, 1937). He proposed a definition of coherent subjective probabilities based on no arbitrage opportunities. D Finetti's model is based on the notion of expected value maximizing behavior, or linear utility.

The theory of expected utility under risk received its first axiomatic characterization with the publication of von Neumann and Morgenstern's (1947) *Theory of Games and Economic Behavior*. Von Neumann and Morgenstern analyzed the strategic behavior of players in noncooperative zero-sum games in which no pure strategy equilibrium exists. In such games, the equilibrium may require the employment of mixed strategies. The interest of von Neumann and Morgenstern in the decision making of players facing opponents who use a randomizing device to determine the choice of a pure strategy justifies their modeling the uncertainty surrounding the choice of pure strategies using probabilities in the aleatory sense of relative frequency. Invoking the axiomatic approach to depict the decision maker's preference relation on the set of objective risks, von Neumann and Morgenstern identified necessary and sufficient conditions for the existence of a utility function on a set of outcomes that captures the decision maker's risk attitudes, and represented his/her choice as expected utility maximizing behavior.

Building on, and synthesizing, the ideas of de Finetti and von Neumann and Morgenstern, Savage (1954) proposed the first complete axiomatic subjective expected utility theory. In his seminal work, titled *The Foundations of Statistics*, Savage introduced a new analytical framework and provided necessary and sufficient conditions for the existence and joint uniqueness of utility and probability, as well as the characterization of individual choice in the face of uncertainty as expected utility maximizing behavior. Savage's approach is pure in the sense that the notion of probability does not appear as a primitive concept in his model.

1.1.3 Belief Updating: Bayes' Rule

In 1763 Richard Price read to the Royal Society an essay titled "Essay Towards Solving a Problem in the Doctrine of Chances." In the essay, the Reverend Bayes (1764) outlined a method, since known as Bayes' rule, for updating probabilities in light of new information. Bayes' method does not specify how the original, or prior, probabilities to be updated are determined.

Savage's (1954) theory was intended to furnish the missing ingredient — the prior probability — necessary to complete Bayes' model. The idea is to infer from the decision maker's choice behavior the prior probabilities that represent his/her beliefs and, by so doing, to provide choice-based foundations for the existence of a Bayesian prior.

In Savage's theory, new information indicates that an event that a priori is considered possible is no longer so. The application of Bayes' rule requires that the probability of the complementary event be increased to 1, and the probabilities assigned to its subevents be increased equiproportionally.

The discussion that follows highlights two aspects of this theory. The first is the presumption that Savage's subjective probabilities represent the decision maker's beliefs and, consequently, constitute an appropriate concept of the Bayesian prior. The second is that the posterior preferences of Bayesian decision makers are obtained from their prior preferences solely by the application of Bayes' rule, independently of the decision maker's particular characteristics and risk attitudes.

1.1.4 The Analytical Framework

In the theories reviewed in this chapter, analytical frameworks consist of two primitive concepts: a *choice set*, whose elements are the alternatives the decision makers can potentially choose from, and a binary relation on the choice set, having the interpretation of the decision maker's *preference relation*. Henceforth, an abstract choice set is denoted by $\mathbb{C}$ and a preference relation on $\mathbb{C}$ is denoted by $\succcurlyeq$.[5] For all $c, c' \in \mathbb{C}$, $c \succcurlyeq c'$ means that the alternative c is at least as desirable as the alternative c'. The *strict preference relation* $\succ$ and the *indifference relation* $\sim$ are the asymmetric and symmetric parts of $\succcurlyeq$. Decision makers are characterized by their preference relations. A real-valued function V on $\mathbb{C}$ is said to *represent* $\succcurlyeq$ *on* $\mathbb{C}$ if $V(a) \geq V(b)$ if and only if $a \succcurlyeq b$ for all $a, b \in \mathbb{C}$.

The main concern of the discussion that ensues has to do with the representations of preference relations. The nature of these representations depends on the interaction between the structure of the choice set and that of the preference relations. Throughout, the structure of the choice set reflects physical or material properties of the alternatives under consideration and is presumed to be objective, in the sense of having the same meaning for all decision makers regardless of their personal characteristics. The preference structure is depicted by a set of axioms. In the normative interpretation, these axioms are regarded as tenets of rational choice and should be judged by their normative appeal. In the positive interpretation, these axioms are principles that are supposed to govern actual choice behavior and should be evaluated by their predictive power.

1.2 EXPECTED UTILITY UNDER RISK

This section reviews the theory of decision making under risk due to the original contribution of von Neumann and Morgenstern (1947). In particular, it describes the necessary and sufficient conditions for the representation of a preference relation on risky alternatives by an expected utility functional.

[5] Formally, a binary relation is a subset of $\mathbb{C} \times \mathbb{C}$.

1.2.1 The Analytical Framework

Expected utility theory under risk is a special case of a more abstract choice theory in which the choice set, $\mathbb{C}$, is a convex subset of a linear space. The following examples show that distinct specifications of the objects of choice in expected utility theory under risk are but specific instances of $\mathbb{C}$.

Example 1. Let X be an arbitrary set of outcomes, and consider the set $\Delta(X)$ of all the simple distributions on X.[6] Elements of $\Delta(X)$ are referred to as *lotteries*. For any two lotteries, p and q and $\alpha \in [0,1]$, define the convex combination $\alpha p + (1-\alpha)\, q \in \Delta(X)$ by $(\alpha p + (1-\alpha)\, q)\,(x) = \alpha p(x) + (1-\alpha)\, q(x)$, for all $x \in X$. Under this definition, $\Delta(X)$ is a convex subset of the finite dimensional linear space $\mathbb{R}^n$.

Example 2. Let $M(X)$ denote the set of all the probability measures on the measure space $(X, \mathcal{X})$. For any two elements, P and Q of $M(X)$ and $\alpha \in [0,1]$, define the convex combination $\alpha P + (1-\alpha)\, Q \in M(X)$ by $(\alpha P + (1-\alpha)\, Q)\,(T) = \alpha P(T) + (1-\alpha)\, Q(T)$ for all T in the σ-algebra, $\mathcal{X}$, on X. Then $M(X)$ is a convex subset of the linear space of measures on the measurable space $(X, \mathcal{X})$.

Example 3. Let $\mathcal{F}$ denote the set of cumulative distribution functions on the real line. For any two elements F and G of $\mathcal{F}$ and $\alpha \in [0,1]$, define the convex combination $\alpha F + (1-\alpha)\, G \in \mathcal{F}$ by $(\alpha F + (1-\alpha)\, G)\,(x) = \alpha F(x) + (1-\alpha)\, G(x)$ for all $x \in \mathbb{R}$. Then $\mathcal{F}$ is a convex subset of the linear space of real-valued functions on $\mathbb{R}$.

1.2.2 The Characterization of the Preference Relations

In expected utility theory under risk, preference relations, $\succcurlyeq$ on $\mathbb{C}$, are characterized by three axioms. The first, the weak-order axiom, requires that the preference relation be complete and transitive. Completeness means that all elements of $\mathbb{C}$ are comparable in the sense that, presented with a choice between two alternatives in $\mathbb{C}$, the decision maker is (or should be) able to indicate that one alternative is at least as desirable to him as the other. In other words, the axiom rules out that the decision maker find some alternatives noncomparable and, as a result, is unable to express preferences between them. Transitivity requires that choices be consistent in the elementary sense that if an alternative a is at least as desirable as another alternative b which, in turn, is deemed at least as desirable as a third alternative c then alternative c is not (or should not be) strictly preferred over a. Formally:

(A.1) (**Weak order**) $\succcurlyeq$ on $\mathbb{C}$ is *complete* (that is, for all $a, b \in \mathbb{C}$, either $a \succcurlyeq b$ or $b \succcurlyeq a$) and *transitive* (that is, for all $a, b, c \in \mathbb{C}$, $a \succcurlyeq b$ and $b \succcurlyeq c$ imply $a \succcurlyeq c$).

[6] A probability distribution is simple if it has finite support.

Note that the weak-order axiom does not require the use of properties of $\mathbb{C}$ that derive from its being a convex set in a linear space. This is not the case with the next two axioms.

The second axiom, the Archimedean axiom, imposes a sort of continuity on the preference relation. It requires implicitly that no alternative in $\mathbb{C}$ be infinitely more, or less, desirable than any other alternative. Specifically, let $\alpha a + (1-\alpha)\, c \in \mathbb{C}$ be interpreted as a lottery that assigns the probabilities α and $(1-\alpha)$ to the alternatives a and c, respectively. The axiom requires that, for no $a, b, c \in \mathbb{C}$ such that a is strictly preferred to b and b is strictly preferred to c, even a small chance of c makes the lottery $\alpha a + (1-\alpha)\, c$ less desirable than getting b with certainty, or that even a small chance of winning a makes the lottery $\alpha a + (1-\alpha)\, c$ more desirable than getting b with certainty. Formally:

(A.2) (**Archimedean**) For all $a, b, c \in \mathbb{C}$, if $a \succ b$ and $b \succ c$ then there exist $\alpha, \beta \in (0,1)$ such that $\alpha a + (1-\alpha)\, c \succ b$ and $b \succ \beta a + (1-\beta)\, c$.

The third axiom, the independence axiom, imposes a form of separability on the preference relation. It requires that the comparison of any two alternatives in $\mathbb{C}$ be based on their distinct aspects (that is, the decision maker disregards all aspects that are common to the two alternatives). Let the alternatives $\alpha a + (1-\alpha)\, c$ and $\alpha b + (1-\alpha)\, c$, be interpreted as lotteries assigning the probabilities α to a and b, respectively, and the probability $(1-\alpha)$ to the alternative c. Considering these two alternatives, the decision maker figures that both yield c with probability $(1-\alpha)$, so in this respect they are the same, and that with probability α they yield, respectively, the alternatives a and b, so in this respect they are different. The axiom asserts that the preference between the alternatives $\alpha a + (1-\alpha)\, c$ and $\alpha b + (1-\alpha)\, c$ is the same as that between a and b. Put differently, the independence axiom requires that the preference between the lotteries a and b be the same whether they are compared directly or embedded in larger, compound, lotteries that are otherwise identical. Formally:

(A.3) (**Independence**) For all $a, b, c \in \mathbb{C}$ and $\alpha \in (0,1]$, $a \succcurlyeq b$ if and only if $\alpha a + (1-\alpha)\, c \succcurlyeq \alpha b + (1-\alpha)\, c$.

1.2.3 Representation

A real-valued function V on a linear space $\mathcal{L}$ is *affine* if $V\left(\alpha \ell + (1-\alpha)\, \ell'\right) = \alpha V(\ell) + (1-\alpha)\, V\left(\ell'\right)$, for all $\ell, \ell' \in \mathcal{L}$ and $\alpha \in [0,1]$. It is said to be *unique up to positive affine transformation* if and only if any other real-valued, affine function $\hat{V}$ on $\mathcal{L}$ representing $\succcurlyeq$ on $\mathbb{C}$ satisfies $\hat{V}(\cdot) = \beta V(\cdot) + \gamma$, where $\beta > 0$.

The first representation theorem gives necessary and sufficient conditions for the existence of an affine representation of $\succcurlyeq$ on $\mathbb{C}$ and describes its uniqueness properties.[7]

[7] For a variation of this result in which $\mathbb{C}$ is a general mixture set, the Archimedean axiom is replaced by mixture continuity and the independence axiom is weakened, see Herstein and Milnor (1953).

Theorem 1.1 (von Neumann and Morgenstern). *Let $\succcurlyeq$ be a binary relation on $\mathbb{C}$. Then $\succcurlyeq$ satisfies the weak-order, Archimedean, and independence axioms if and only if there exists a real-valued, affine function, U, on $\mathbb{C}$ that represents $\succcurlyeq$. Moreover, U is unique up to a positive affine transformation.*

Clearly, any monotonic increasing (order preserving) transformation of U also represents the preference relation. However, such transformations are not necessarily affine. The uniqueness up to positive affine transformation means that the transformations under consideration preserve the affinity property of the representation.

The next theorem applies Theorem 1.1 to the set of simple probability distributions, $\Delta(X)$, in Example 1. It invokes the fact that $\Delta(X)$ is a convex set, gives necessary and sufficient conditions for existence, and the uniqueness (up to a positive affine transformation) of a utility function on the set of outcomes, X, whose expected value with respect to simple probability distributions in $\Delta(X)$ represents the preference relation on $\Delta(X)$.

Theorem 1.2 (Expected utility for simple probability measures) *Let $\succcurlyeq$ be a binary relation on $\Delta(X)$. Then $\succcurlyeq$ satisfies the weak-order, Archimedean, and independence axioms if and only if there exists a real-valued function, u, on X such that $\succcurlyeq$ is represented by:*

$$p \mapsto \sum_{x \in \{x \in X \mid p(x) > 0\}} u(x)p(x).$$

Moreover, u is unique up to a positive affine transformation.

1.2.4 Strong Continuity and Expected Utility Representation for Borel Probability Measures

The expected utility representation may be extended to more general probability measures. Doing so requires that the preference relation display stronger continuity than that captured by the Archimedean axiom.[8]

Let X be a finite dimensional Euclidean space, and let M be the set of all probability measures on $(X, \mathcal{B})$, where $\mathcal{B}$ denotes the Borel σ-algebra on X.[9] Assume that M is endowed with the topology of weak convergence.[10] Suppose that $\succcurlyeq$ is continuous in the topology of weak convergence. Formally,

(A.2') (**Continuity**) For all $P \in M$, the sets $\{Q \in M \mid Q \succcurlyeq P\}$ and $\{Q \in M \mid P \succcurlyeq Q\}$ are closed in the topology of weak convergence.

[8] See also Fishburn (1970).

[9] A Borel σ-algebra is the smallest σ-algebra that contains the open sets of X. Any measure μ defined on the Borel σ-algebra, $\mathcal{B}$, is called a Borel measure.

[10] The topology of weak convergence is the coarsest topology on M such that for every continuous and bounded real-valued function f on X the map $P \to \int_X f(x)dP(x)$ is continuous. In this topology, a sequence $\{P_n\}$ converges to P if $\int_X f(x)dP_n(x)$ converges to $\int_X f(x)dP(x)$ for every continuous and bounded real-valued function f on X.

Replacing the Archimedean axiom (A.2) with (A.2′) results in a stronger expected utility representation given in the following theorem. In particular, the utility function in the representation is continuous and bounded.[11]

Theorem 1.3 (Expected utility for Borel probability measures) *Let $\succcurlyeq$ be a preference relation on M. Then $\succcurlyeq$ satisfies the weak-order, continuity and independence axioms if and only if there exists a real-valued, continuous, and bounded function u on X such that $\succcurlyeq$ is represented by:*

$$P \mapsto \int_X u(x)dP(x).$$

Moreover, u is unique up to a positive affine transformation.

Let $X = \mathbb{R}$, the set of real numbers. For every $P \in M$, define the distribution function F by $F(x) = P(-\infty, x]$ for all $x \in \mathbb{R}$. Denote by $\mathcal{F}$ the set of all distribution functions so defined and let $\succcurlyeq$ be a preference relation on $\mathcal{F}$. Then, by Theorem 1.3, $\succcurlyeq$ satisfies the weak-order, continuity, and independence axioms if and only if there exists a real-valued, affine function u on X such that $\succcurlyeq$ is represented by $F \mapsto \int_{-\infty}^{\infty} u(x)dF(x)$. Moreover, u is unique up to a positive affine transformation.

1.3 EXPECTED UTILITY UNDER UNCERTAINTY AND SUBJECTIVE PROBABILITIES

This section reviews three models of decision making under uncertainty, the models of Savage (1954), Anscombe and Aumann (1963), and Wakker (1989). The models differ in the specification of the choice sets and the corresponding preference structures, but their objective is the same: the simultaneous determination of a utility function that quantifies the decision maker's tastes and a probability measure that quantifies his/her beliefs.

1.3.1 Savage's Analytical Framework

In the wake of Savage's (1954) seminal work, it is commonplace to model decision making under uncertainty by constructing a choice set using two primitive sets: a set, S, of *states of the nature* (or states, for brevity), and a set, C, whose elements are referred to as *consequences*. The choice set, F, is the set of mappings from the set of states to the set of consequences. Elements of F are referred to as *acts* and have the interpretation of courses of action.

States are resolutions of uncertainty, "a description of the world so complete that, if true and known, the consequences of every action would be known" (Arrow, 1971,

[11] A continuous weak order satisfying the independence axiom is representable by a continuous linear functional. Theorem 1.3 follows from Huber (1981) Lemma 2.1.

p. 45). Implicit in the definition of the state space is the notion that there is a unique *true* state. Subsets of the set of states are *events*. An event is said to *obtain* if it includes the true state.

A consequence is a description of anything that might happen to the decision maker. The set of consequences is arbitrary. A combination of an act, f, chosen by the decision maker, and a state, s, "selected" by nature determines a unique consequence, $c\left(f, s\right) \in C$.

Decision makers are characterized by preference relations, $\succcurlyeq$ on F, having the usual interpretation, namely, $f \succcurlyeq g$ means that the act f is at least as desirable as the act g.

1.3.2 The Preference Structure

Savage's (1954) subjective expected utility model postulates a preference structure that permits: (a) the numerical expression of the decision maker's valuation of the consequences by a utility function; (b) the numerical expression of the decision maker's degree of beliefs in the likelihoods of events by a finitely additive, probability measure; and (c) the evaluation of acts by the mathematical expectations of the utility of their consequences with respect to the subjective probabilities of the events in which these consequences materialize. In this model, the utility of the consequences is independent of the underlying events, and the probabilities of events are independent of the consequences assigned to these events by the acts.

The statement of Savage's postulates uses the following notation and definitions. Given an event E and acts f and h, $f_E h$ denotes the act defined by $(f_E h)(s) = f(s)$ if $s \in E$, and $(f_E h)(s) = h(s)$ otherwise. An event E is *null* if $f_E h \sim f'_E h$ for all acts f and f', otherwise it is *nonnull*. A constant act is an act that assigns the same consequence to all events. Constant acts are denoted by their values (that is, if $f(s) = x$ for all s, the constant act f is denoted by x).

The first postulate asserts that the preference relation is a weak order. Formally:

P.1 (**Weak order**) The preference relation is a transitive and complete binary relation on F.

The second postulate requires that the preference between acts depend solely on the consequences in the events in which the values of the two acts being compared are distinct.[12] Formally:

P.2 (**Sure-Thing Principle**) For all acts, f, f', h, and h' and every event E, $f_E h \succcurlyeq f'_E h$ if and only if $f_E h' \succcurlyeq f'_E h'$.

The Sure-Thing Principle makes it possible to define preferences conditional on events as follows: For every event E, and all $f, f' \in F$, $f \succcurlyeq_E f'$ if $f \succcurlyeq f'$ and $f(s) = f'(s)$ for every s not in E. The third postulate asserts that, conditional on any

[12] The second postulate introduces separability reminiscent of that encountered in the independence axiom.

nonnull events, the ordinal ranking of consequences is independent of the conditioning events. Formally:

P.3 (**Ordinal Event Independence**) For every nonnull event E and all constant acts, x and y, $x \succcurlyeq y$ if and only if $x_E f \succcurlyeq y_E f$ for every act f.

In view of P.3 it is natural to refer to an act that assigns to an event E a consequence that ranks higher than the consequence it assigns to the complement of E as a *bet* on E. The fourth postulate requires that the betting preferences be independent of the specific consequences that define the bets. Formally:

P.4 (**Comparative Probability**) For all events E and E' and constant acts x, y, x', and y' such that $x \succ y$ and $x' \succ y'$, $x_E y \succcurlyeq x_{E'} y$ if and only if $x'_E y' \succcurlyeq x'_{E'} y'$.

Postulates P.1–P.4 imply the existence of a transitive and complete relation on the set of all events that has the interpretation of "at least as likely to obtain as," representing the decision maker's beliefs as *qualitative probability*.[13] Moreover, these postulates also imply that the decision maker's risk attitudes are event independent.

The fifth postulate renders the decision-making problem and the qualitative probability nontrivial. It requires the existence of constant acts between which the decision maker is not indifferent.

P.5 (**Nondegeneracy**) For some constant acts x and x', $x \succ x'$.

The sixth postulate introduces a form of continuity of the preference relation. It asserts that no consequence is either infinitely better or infinitely worse than any other consequence. Put differently, it requires that there be no consequence that, were it to replace the payoff of an act on a nonnull event, no matter how unlikely, would reverse a strict preference ordering of two acts. Formally:

P.6 (**Small-Event Continuity**) For all acts f, g, and h, satisfying $f \succ g$, there is a finite partition $(E_i)_{i=1}^n$ of the state space such that, for all i, $f \succ h_{E_i} g$ and $h_{E_i} f \succ g$.

A probability measure is *nonatomic* if every nonnull event may be partitioned into two non null subevents. Formally, π is a nonatomic probability measure on the set of states if for every event E and number $0 < \alpha < 1$, there is an event $E' \subset E$ such that $\pi\left(E'\right) = \alpha\pi(E)$. Postulate P.6 implies that there are infinitely many states of the world and that if there exists a probability measure representing the decision maker's beliefs, it must be nonatomic. Moreover, the probability measure is defined on the set of all events, hence it is *finitely additive* (that is, for every event E, $0 \le \pi(E) \le 1$, $\pi(S) = 1$ and for any two disjoint events, E and E', $\pi\left(E \cup E'\right) = \pi(E) + \pi\left(E'\right)$).

[13] A binary relation $\succeq$ on an algebra of events, $\mathcal{A}$, in S is a *qualitative probability* if (a) $\succeq$ is complete and transitive; (b) $E \succeq \varnothing$, for all $E \in \mathcal{A}$; (c) $S \succ \varnothing$; and (d) for all $E, E', E'' \in \mathcal{A}$, if $E \cap E'' = E' \cap E'' = \varnothing$ then $E \succeq E'$ if and only if $E \cup E'' \succeq E' \cup E''$.

The seventh postulate is a monotonicity requirement asserting that if the decision maker considers an act strictly better (worse) than each of the payoffs of another act, taken as constant acts, on a given nonnull event, then the former act is conditionally strictly preferred (less preferred) than the latter. Formally:

P.7 (**Dominance**) For every event E and all acts f and f', if $f \succ_E f'(s)$ for all s in E then $f \succcurlyeq_E f'$ and if $f'(s) \succ_E f$ for all s in E then $f' \succcurlyeq_E f$.

Postulate P.7 is not necessary to obtain a subjective expected utility representation of *simple acts* (that is, acts with finite range). However, it is necessary if the model is to include nonsimple acts and it is sometimes regarded as a purely technical condition. However, as shown in Section 1.4, if the preference relation is incomplete, this condition has important implications for choice behavior.

1.3.3 Subjective Expected Utility Representation

Savage's (1954) theorem establishes that the properties described by the postulates P.1–P.7 are necessary and sufficient conditions for the representation of the preference relation by the expectations of a utility function on the set of consequences with respect to a probability measure on the set of all events. The utility function is unique up to a positive affine transformation, and the probability measure is unique, nonatomic, and finitely additive.

Theorem 1.4 (Savage) *Let $\succcurlyeq$ be a preference relation on F. Then $\succcurlyeq$ satisfies postulates P.1–P.7 if and only if there exists a unique, nonatomic, finitely additive probability measure π on S and a real-valued, bounded, function u on C such that $\succcurlyeq$ is represented by*

$$f \mapsto \int_S u\left(f(s)\right) d\pi(s).$$

Moreover, u is unique up to a positive affine transformation, and $\pi(E) = 0$ if and only if E is null.

In Savage's theory the set of consequences is arbitrary. Therefore, to define quantitative probabilities on the algebra of all events the set of states must be rich in the sense that it is, at least, infinitely countable. In many applications, however, it is natural to model the decision problem using finite state spaces. For example, to model betting on the outcome of a horse race, it is natural to define a state as the order in which the horses cross the finish line, rendering the state space finite. To compensate for the loss of richness of the state space, two approaches were suggested in which the set of consequences is enriched. Anscombe and Aumann (1963) assumed that the consequences are simple lotteries on an arbitrary set of prizes. Wakker (1989) assumed that the set of consequences is a connected separable topological space. We consider these two approaches next, beginning with the model of Anscombe and Aumann (1963).

1.3.4 The Model of Anscombe and Aumann

Let S be a finite set of states, and let the set of consequences, $\Delta(X)$, be the set of lotteries on an arbitrary set of prizes, X. The choice set, H, consists of all the mappings from S to $\Delta(X)$. Elements of H are acts whose consequences are lottery tickets. The choice of an act, h, by the decision maker and a state, s, by nature entitles the decision maker to participate in the lottery $h(s)$ to determine his/her ultimate payoff, which is some element of the set X. Following Anscombe and Aumann (1963), it is useful to think about states as the possible outcomes of a horse race. An act is a bet on the outcome of a horse race whose payoffs are lottery tickets. Using this metaphor, they refer to elements of H as *horse lotteries* and elements of $\Delta(X)$ as *roulette lotteries.*

Invoking the formulation of Fishburn (1970), for any two elements, f and g, of H and $\alpha \in [0, 1]$, define the convex combination $\alpha f + (1 - \alpha) g \in H$ by $\left(\alpha f + (1 - \alpha) g\right)(s) = \alpha f(s) + (1 - \alpha) g(s)$, for all $s \in S$. Under this definition, H is a convex subset of a finite dimensional linear space.[14] To interpret this definition, consider the act $\alpha f + (1 - \alpha) g$. It is useful to think of this act as a compound lottery. The first stage is a lottery that assigns the decision maker the acts f or g, with probabilities α and $(1 - \alpha)$, respectively. Once the true state, s, is revealed, the decision maker's ultimate payoff is determined by either the lottery $f(s)$ or the lottery $g(s)$, depending on the act assigned to him in the first stage. According to this interpretation, the true state is revealed after the act is selected by the outcome of the first-stage lottery. Together, the true state, s, and the act $\alpha f + (1 - \alpha) g$ determine a lottery $\left(\alpha f + (1 - \alpha) g\right)(s)$ in $\Delta(X)$.

Consider next the lottery $\alpha f(s) + (1 - \alpha) g(s) \in \Delta(X)$. This lottery may be interpreted as a compound lottery in which the true state is revealed first and the decision maker then gets to participate in the lottery $f(s)$ or $g(s)$ according to the outcome of a second-stage lottery that assigns him $f(s)$ the probability α and $g(s)$ the probability $(1 - \alpha)$.

By definition of the choice set, these two compound lotteries are identical, implying that the decision maker is indifferent between finding out which act is assigned to him before or after the state of nature is revealed.[15] Drèze (1985) interprets this indifference as reflecting the decision maker's belief that he/she cannot affect the likelihood of the states, for if he/she could, knowing in advance the act he/she is assigned, he/she would tilt the odds in his/her favor. This opportunity is forgone if the decision maker learns which act is assigned to him only after the state has already been revealed. Hence, if a decision maker believes that he can affect the likelihood of the states, he should strictly prefer the act $\alpha f + (1 - \alpha) g$ over the act $\left(\alpha f(s) + (1 - \alpha) g(s)\right)_{s \in S}$. According to

[14] The linear space is $\mathbb{R}^{|S| \times n}$, $n < \infty$, where n denotes the number of elements in the supports of the roulette lotteries.
[15] Anscombe and Aumann (1963) imposed this indifference in their reversal of order axiom.

Drèze, implicit in the specification of the choice set is the assumption that the likely realization of the states is outside the control of the decision maker.[16]

Let $\succcurlyeq$ be a weak order on H satisfying the Archimedean and independence axioms. Since H is a convex subset of a linear space, application of Theorem 1.1 implies that $\succcurlyeq$ has an additively separable representation.[17]

Corollary 1.5 (State-dependent expected utility) *Let $\succcurlyeq$ be a binary relation on H, then $\succcurlyeq$ is a weak order satisfying the Archimedean and independence axioms if and only if there exists a real-valued function, w, on $\Delta(X) \times S$, affine in its first argument, such that $\succcurlyeq$ is represented by*

$$h \mapsto \sum_{s \in S} w\left(h(s)\, ,\, s\right).$$

Moreover, w is unique up to cardinal unit-comparable transformation (that is, if $\hat{w}$ represents $\succcurlyeq$ in the additive form, then $\hat{w}(\cdot, s) = bw(\cdot, s) + a(s)$, for all $s \in S$).

Presumably, the value $w\left(p, s\right)$ is a fusion of the decision maker's belief regarding the likelihood of the state s and his/her evaluation of the roulette lottery p. If the beliefs are independent of the payoffs, the valuations of the roulette lotteries are independent of the likelihood of the states in which they materialize, and the ex ante valuation of p equal its ex post valuation discounted by its likelihood, then $w\left(p, s\right)$ can be decomposed into a product $U(p)\pi(s)$, where U is a real-valued, affine function on $\Delta(X)$ and π is a probability measure on S. For such a decomposition to be well defined, the preference structure must be tightened.

To obtain the aforementioned decomposition of the function w, Anscombe and Aumann (1963) amended the von Neumann and Morgenstern model with two axioms. The first requires that the decision maker's risk attitudes (that is, his/her ranking of roulette lotteries) be independent of the state.[18] This axiom captures the essence of postulates P.3 and P.4 of Savage (1954), which assert that the preference relation exhibits state independence in both the ordinal and cardinal sense. The statement of this axiom requires the following additional notations and definitions: For all $h \in H$ and $p \in \Delta(X)$, let $h_{-s}p$ be the act obtained by replacing the s-coordinate of h, $h(s)$, with p. A state s is *null* if $h_{-s}p \sim h_{-s}q$, for all $p, q \in \Delta(X)$. A state is *nonnull* if it is not null. Formally:

(A.4) (**State independence**) For all nonnull $s, s' \in S$, and $h \in H$, $h_{-s}p \succcurlyeq h_{-s}q$ if and only if $h_{-s'}p \succcurlyeq h_{-s'}q$.

[16] Invoking the analytical framework of Anscombe and Aumann (1963), Drèze (1961, 1985) departed from their "reversal of order in compound lotteries" axiom. This axiom asserts that the decision maker is indifferent between the acts $\alpha f + (1-\alpha)\, g$ and $\left(\alpha f(s) + (1-\alpha)\, g(s)\right)_{s \in S}$. Drèze assume instead that decision makers may strictly prefer knowing the outcome of a lottery before the state of nature becomes known. In other words, according to Drèze, $\alpha f + (1-\alpha)\, g \succ \left(\alpha f(s) + (1-\alpha)\, g(s)\right)_{s \in S}$. The representation entails the maximization of subjective expected utility over a convex set of subjective probability measures.

[17] See Kreps (1988).

[18] In conjunction with the other axioms, this requirement is equivalent to the following monotonicity axiom: For all $h, h' \in H$, if $h(s) \succcurlyeq h'(s)$ for all $s \in S$, where $h(s)$ and $h'(s)$ are constant acts, then $h \succcurlyeq h'$.

The second axiom, which is analogous to Savage's P.5, requires that the decision maker not be indifferent among all acts. Formally:

(A.5) (**Nontriviality**) There are acts h and h' in H such that $h \succ h'$.

Preference relations that have the structure depicted by the axioms (A.1)–(A.5) have subjective expected utility representations.

Theorem 1.6 (Anscombe and Aumann) *Let $\succcurlyeq$ be a binary relation on H. Then $\succcurlyeq$ is a weak-order satisfying Archimedean, independence, state independence and nontriviality if and only if there exists a real-valued function, u, on X and a probability measure π on S such that $\succcurlyeq$ is represented by*

$$h \mapsto \sum_{s \in S} \pi(s) \sum_{x \in X} u(x) h(s)(x).$$

Moreover, u is unique up to a positive affine transformation, π is unique, and $\pi(s) = 0$ if and only if s is null.

1.3.5 Wakker's Model

Let S be a finite set of states, and let C be a connected separable topological space.[19] Elements of C are consequences and are denoted by c. The choice set, A, consists of all acts (that is, mappings from S to C). As usual, an act, a, and a state, s, determine a consequence, $c(a, s) = a(s)$. Decision makers are characterized by a preference relation, $\succcurlyeq$, on A.

Wakker (1989) assumes that a preference relation, $\succcurlyeq$, is a continuous weak order (that is, $\succcurlyeq$ on A is continuous if and only if, for all $a \in A$, the sets $\{a' \in A \mid a' \succcurlyeq a\}$ and $\{a' \in A \mid a \succcurlyeq a'\}$ are closed in the product topology on $C^{|S|}$).

To grasp the main innovation of Wakker's approach, it is useful to contrast it with the approach of Anscombe and Aumann (1963). They exploit the ordinal ranking of roulette lotteries to obtain cardinal utility representation of the "values" of the outcomes. The cardinality means that a difference between the utilities of two outcomes is a meaningful measure of the "intensity" of preference between them. Without roulette lotteries, the intensity of preferences must be gauged by other means.

Wakker measures the intensity of preferences among consequences in a given state by the compensating variations in the consequences in other states. To see how this works, for each $a \in A$, denote by $a_{-s}c$ the act obtained from a by replacing its payoff in state s, namely, the consequence $a(s)$, by the consequence c. Let s be a nonnull state and consider the following preferences among acts. The indifferences $a'_{-s}c' \sim a_{-s}c$ and $a_{-s}c'' \sim a'_{-s}c'''$ indicate that, in state s, the intensity of preference of c'' over c''' is the same as the intensity of preference of c over c', in the sense that both just compensate for the

[19] A topological space is connected if the only sets that are both open and closed are the empty set and the space itself, or equivalently, if the space is not the union of two nonempty disjoint open subsets. A topological space is separable if it contains a countable dense subset.

difference in the payoffs in the other states, represented by the subacts a'_{-s} and a_{-s}. The difference in the payoffs represented by the subacts a'_{-s} and a_{-s} is used as a "measuring rod" to assess the difference in values between c and c' and between c'' and c''' in state s.

Consider next another nonnull state, t, and let a''_{-t} and a'''_{-t} be subacts such that $a''_{-t}c' \sim a'''_{-t}c$. The difference in the payoffs represented by the subacts a''_{-t} and a'''_{-t} measures the difference in values between c and c' in state t. The intensity of preference of c over c' in state t is the compensating variation for the difference in the payoffs of the subacts a''_{-t} and a'''_{-t}. Using this as a measuring rod, it is possible to check whether the difference in values between c'' and c''' in state t also constitutes a compensating variation for difference in the payoffs of the subacts a''_{-t} and a'''_{-t}. If it does then we conclude that the intensity of preferences between c'' and c''' in state t is the same as that between c and c'. Since the only restriction imposed on s and t is that they be nonnull, we conclude that the intensity of preferences between consequences is state independent.

The next axiom asserts that the "intensity of preferences" as measured by compensating variations are state independent.[20] Although the axiom is stated using the weak preference relation instead of the indifference relation, the interpretation is similar—namely, that the preference relation displays no contradictory preference intensity between consequences in distinct nonnull states. Formally:

(W) (State-Independent Preference Intensity) For all $a, a', a'', a''' \in A$, $c, c', c'', c''' \in C$ and nonnull $s, t \in S$, if $a'_{-s}c' \succcurlyeq a_{-s}c$, $a_{-s}c'' \succcurlyeq a'_{-s}c'''$ and $a''_{-t}c' \succcurlyeq a'''_{-t}c$ then $a''_{-t}c'' \succcurlyeq a'''_{-t}c'''$.

State-independent preference intensity incorporates two properties of the preference relation. First, the relative ranking of acts that agree on the payoff in one state is independent of that payoff. This property, dubbed coordinate independence, is analogous to, albeit weaker than, Savage's (1954) Sure-Thing Principle. Second, the intensity of preference between consequences is independent of the state. This property is analogous to the Anscombe and Aumann (1963) state-independence axiom.

Wakker (1989) shows that a continuous weak order satisfying state-independent preference intensity admits a subjective expected utility representation.

Theorem 1.7 (Wakker) *Let $\succcurlyeq$ be a binary relation on A. Then $\succcurlyeq$ is a continuous weak-order displaying state-independent preference intensity if and only if there exists a real-valued, continuous function, u, on C and a probability measure, μ, on S such that $\succcurlyeq$ is represented by*

$$a \mapsto \sum_{s \in S} \mu(s) u\left(a(s)\right).$$

Moreover, if there are at least two nonnull states, then u is unique up to a positive affine transformation, π is unique, and $\pi(s) = 0$ if and only if s is null.

[20] State-independent preference intensity is equivalent to the requirement that the preference relation display no contradictory tradeoffs. For details, see Wakker (1989), in particular, Lemma IV.2.5.

1.3.6 Beliefs and Probabilities

Beginning with Ramsey (1931) and de Finetti (1937) and culminating with Savage (1954), with rare exceptions, choice-based notions of subjective probabilities are treated as an aspect of the representation of a decision maker's preference relation, that *defines* his/her degree of belief regarding the likelihood of events. In other words, while presuming to represent the decision maker's beliefs about the likelihood of events, according to this approach it is immaterial whether in fact the decision maker actually entertains such beliefs.

A different approach considers the decision maker's beliefs to be a cognitive phenomenon that feeds into the decision-making process. According to this approach, the subjective probabilities are meaningful only to the extent that they *measure* the decision maker's actual beliefs.

To highlight the difference between these two notions of subjective probability, it is convenient to think of the decision maker as a black box, which when presented with pairs of alternatives, selects the preferred alternative or express indifference between the two. Invoking this metaphor, imagine a black box into which we upload probabilities of events, utilities of consequences, and a set of commands instructing the box to select the alternative that yields the highest expected utility. Is it possible to recover the probabilities and utilities that were uploaded into the black box from the observation of its choice pattern?

To answer this question, consider the following example. Let there be two states, say 1 and 2, and upload into the box equal probabilities to each state. Suppose that acts are state-contingent monetary payoffs. Assume that for the same payoff, the utility in state 1 is twice that of state 2. Instructed to calculate expected utility, the box assigns the act that pays x_1 in state 1 and x_2 dollars in state 2 the expected utility $0.5 \times 2u(x_1) + 0.5 \times u(x_2)$, where u is the utility function uploaded into the black box. Clearly, the beliefs of the black box are represented by the uniform probability distribution on the set of states $\{1, 2\}$.

To infer the probabilities from the black box's choice behavior, apply any of the standard elicitation methods. For instance, applying the quadratic scoring rule method, we ask the black box to select the value of α, which determines a bet whose payoff is $-r\alpha^2$ in state 1 and $-r(1-\alpha)^2$ in state 2, $r > 0$. As r tends to zero the optimal value of α is an estimate of the probability of state 1. In this example, as r tends to zero, the optimal α tends to 1/3. Clearly, this probability is not what was uploaded into the black box, and consequently does not measure the black box's beliefs. However, selecting acts so as to maximize the expected utility according to the formula $\frac{2}{3} \times u(x_1) + \frac{1}{3} \times u(x_2)$ induces choices identical to those implied by the original set of instructions. Thus, the output of the scoring rule may be taken as a definition of the black box's beliefs.

The source of the difficulty, illustrated by this example, is that the utility function and probability measure that figure in the representations in Savage's theorem and in the theorem of Anscombe and Aumann are *jointly unique* (that is, the probability is unique given the utility and the utility is unique, up to a positive affine transformations, given the probability). Thus, the uniqueness of the probabilities in the aforementioned theorems depends crucially on assigning the consequences utility values that are independent of the underlying events. In other words, the uniqueness of the probability depends on the convention that maintains constant acts are constant utility acts. This convention, however, is not implied by the axiomatic structure and, consequently, lacks testable implications for the decision maker's choice behavior. In other words, this example illustrates the fact that the subjective probability in the theories of Savage (1954), Anscombe and Aumann (1963), and all other models that invoke Savage's analytical framework, are arbitrary theoretical concepts devoid of choice-theoretic meaning. The structures of the preference relations, in particular postulates P.3 and P.4 of Savage (1954) and the state-independence axiom of Anscombe and Aumann (1963), require that the preference relation be state independent. However, neither by itself nor in conjunction with the other axioms do state-independent preferences imply that the utility function is state-independent. Put differently, state-independent preferences do not rule out that the events may affect the decision maker's well-being other than simply through their assigned consequences.[21]

In view of this discussion, it is natural to ask whether and why it is important to search for probabilities that measure the decision maker's beliefs. The answer depends on the applications of the theory one has in mind. If the model is considered to be a positive or a normative theory of decision making in the face of uncertainty, then the issue of whether the probabilities represent the decision maker's beliefs is indeed secondary. The only significance of the subjective expected utility representation is its additively separable functional form. However, as shown in Corollary 1.5, additive separability can be obtained with fewer restrictions on the structure of the underlying preferences. In the Anscombe and Aumann (1963) framework, state independence is unnecessary for the representation. Similarly, Savage's postulate P.3 is unnecessary.[22] Insisting that the preference relation exhibits state-independence renders the model inappropriate for the analysis of important decision problems, such as the demand for health and life insurance, in which the state itself may affect the decision maker's risk attitudes. Unless the subjective probabilities are a meaningful measurement of beliefs, the imposition of state-independent preferences seems unjustifiable.

Insofar as providing choice-based foundations of Bayesian statistics is concerned, which is the original motivation of Savage's (1954) work, the failure to deliver subjective probabilities that represent the decision maker's beliefs is fatal. In fact, if one does not

[21] On this point, see Schervish et al. (1990), Nau (1995), Seidenfeld et al. (1995), Karni (1996), Drèze (1961,1987).
[22] See Hill (2010).

insist that the subjective probability measure the decision maker's beliefs, it seems more convenient to interpret the additively separable representation of the preference relation as an expected utility representation with respect to a uniform prior and state-dependent utility functions.[23] Using the uniform prior for Bayesian statistical analysis seems at least as compelling as—and more practical than—using subjective probabilities obtained by an arbitrary normalization of the utility function.

A somewhat related aspect of subjective expected utility theory that is similarly unsatisfactory concerns the interpretation of null events. Ideally, an event should be designated as null, and be ascribed zero probability, if and only if the decision maker believes it to be impossible. In the models of Savage (1954) and Anscombe and Aumann (1963), however, an event is defined as null if the decision maker displays indifference among all acts whose payoffs agree on the complement of the said event. This definition does not distinguish events that the decision maker perceives as impossible from events on which all possible outcomes are equally desirable. It is possible, therefore, that events that the decision maker believes possible, or even likely, are defined as null and assigned zero probability. Imagine, for example, a passenger who is about to board a flight. Suppose that, having no relatives that he cares about, the passenger is indifferent to the size of his/her estate in the event that he dies. For such a passenger, a fatal plane crash is, by definition, a null event, and is assigned zero probability, even though he recognizes that the plane could crash. This problem renders the representation of beliefs by subjective probabilities dependent on the implicit, and unverifiable, assumption that in every event some outcomes are strictly more desirable than others. If this assumption is not warranted, the procedure may result in a misrepresentation of beliefs.

1.3.7 State-Dependent Preferences

The requirement that the (conditional) preferences be state (or event) independent imposes significant limitations on the range of applications of subjective expected utility theory. Disability insurance policies, or long-term care insurance plans, are acts whose consequences — the premia and indemnities — depend on the realization of the decision maker's state of health. In addition to affecting the decision maker's well-being — which, as the preceding discussion indicates, is not inconsistent with the subjective expected utility models — alternative states of health conceivably influence his/her risk attitudes as well as his/her ordinal ranking of the consequences. For instance, loss of ability to work may affect a decision maker's willingness to take financial risks; a leg injury may reverse his/her preferences between going hiking and attending a concert. These scenarios, which are perfectly reasonable, violate Anscombe and Aumann's (1963) state independence and Savage's (1954) postulates P.3 and P.4. Similar observations apply to the choice of life and health insurance policies.

[23] This is the essence of the state-dependent expected utility in Corollary 1.5.

Imposing state independence as a general characteristic of preference relations is problematic, to say the least. Moreover, as the preceding discussion shows, state independence cannot be justified as a means of defining choice-based representation of decision makers' beliefs.

To disentangle utility from subjective probabilities, or tastes from beliefs, in a meaningful way, it is necessary to observe the decision maker's response to shifts in the state probabilities. Such observations are precluded in Savage's (1954) framework. To overcome this difficulty, the literature pursued two distinct approaches to modeling state-dependent preferences and state-dependent utility functions. The first entails abandoning the revealed-preference methodology, in its strict form, and considering verbal expressions of preferences over hypothetical alternatives. The second presumes the availability of actions by which decision makers may affect the likelihoods of events. The first approach is described next. The second approach, which requires a different analytical framework, is discussed in the following section.

Unlike in the case of state-independent preferences, when the preference relations are state-dependent, it is impossible to invoke the convention that the utility function is state independent. To overcome the problem of the indeterminacy of the subjective probabilities and utilities when the preference relation is state dependent, several models based on hypothetical preferences have been proposed. Fishburn (1973), Drèze and Rustichini (1999), and Karni (2007) depart from the revealed-preference methodology, invoking instead preference relations on conditional acts (that is, preference relations over the set of acts conditional on events).

Karni and Schmeidler (1981) introduce a preference relation on hypothetical lotteries whose prizes comprise outcome–state pairs.[24] Let $\Delta(X \times S)$ denote the set of (hypothetical) lotteries on the set of outcome-state pairs, $X \times S$. Because the lotteries in $\Delta(X \times S)$ imply distinct, hence incompatible, marginal distributions on the state space, preferences among such lotteries are introspective and may be expressed only verbally. For example, a decision maker who has to choose between watching a football game in an open stadium or staying at home and watching the game on TV is supposed to be able to say how he would have chosen if the weather forecast predicted an 80% chance of showers during the game and how he would have chosen if the forecast were for 35% chance of showers during the game.

Let $\hat{\succcurlyeq}$ denote an introspective preference relation on $\Delta(X \times S)$. Assume that decision makers are capable of conceiving such hypothetical situations and evaluating them by the same cognitive processes that govern their actual decisions. Under these assumptions, the verbal expression of preferences provides information relevant for the determination of the probabilities and utilities. Specifically, suppose that the preference

[24] Karni (1985) provides a unified exposition of this approach and the variation due to Karni et al. (1983) described in following paragraphs.

relation, $\hat{\succcurlyeq}$, on $\Delta(X \times S)$ satisfies the axioms of expected utility and is consistent with the actual preference relation $\succcurlyeq$ on the set of acts. To grasp the meaning of consistency of the hypothetical and actual preference relations, define $\ell \in \Delta(X \times S)$ to be *nondegenerate* if $\sum_{x \in X} \ell(x, s) > 0$ for all $s \in S$. Let H be the set of acts as in the model of Anscombe and Aumann (1963). Define a mapping, Ψ, from the subset of nondegenerate lotteries in $\Delta(X \times S)$ to H by: $\Psi(\ell(x, s)) = \ell(x, s) / \sum_{x \in X} \ell(x, s)$, for all $(x, s) \in X \times S$. In other words, every nondegenerate lottery in $\Delta(X \times S)$ is transformed by Ψ into an act in H by assigning to each $x \in X$ the probability of x under ℓ conditional on s.

Next recall that in Savage's analytical framework, the interpretation of a null state is ambiguous, because the definition does not distinguish between attitudes toward states that are considered impossible and events in which all outcomes are equally preferred. The availability of outcome-state lotteries makes it possible to define a state as *obviously null*, if it is null according to the usual definition and, in addition, there exist ℓ and ℓ' in $\Delta(X \times S)$ that *agree outside* s (that is, $\ell(x, s') = \ell'(x, s')$ for all $x \in X$ and $s' \neq s$) and $\ell \hat{\succ} \ell'$. A state is *obviously nonnull* if it is nonnull according to the usual definition.

Let $\succcurlyeq$ denote the actual preference relation on H. Intuitively, the preference relations $\succcurlyeq$ and $\hat{\succcurlyeq}$ are consistent if they are induced by the same utilities and the difference between them is due solely to the differences in the subjective probabilities. This idea is captured by the (strong) consistency axiom of Karni and Schmeidler (1981). Formally:

(KS) (**Strong consistency**) For all $s \in S$ and nondegenerate lotteries ℓ and ℓ' in $\Delta(X \times S)$, if $\ell(x, s') = \ell'(x, s')$ for all $x \in X$ and $s' \neq s$, and $\Psi(\ell) \succ \Psi(\ell')$, then $\ell \hat{\succ} \ell'$. Moreover, if s is obviously nonnull, then, for all nondegenerate ℓ and ℓ' in $\Delta(X \times S)$ such that $\ell(x, s') = \ell'(x, s')$, for all $x \in X$ and $s' \neq s$, $\ell \hat{\succ} \ell'$ implies $\Psi(\ell) \succ \Psi(\ell')$.

By the von Neumann-Morgenstern theorem, the expected utility representation of the introspective preferences, $\hat{\succcurlyeq}$, yields state-dependent utility functions, $u(\cdot, s)$, $s \in S$. Consistency implies that the same utility functions are implicit in the additive representation of the actual preferences in Corollary 1.5. This fact makes it possible to identify all of the subjective probabilities implicit in the valuation functions, $w(\cdot, s)$, in the state dependent expected utility representation. Formally, strong consistency implies that $\pi(s) = w(x, s) / u(x, s)$, $s \in S$, is independent of x. Moreover, the actual preference relation, $\succcurlyeq$ on H, has the expected utility representation, $f \mapsto \sum_{s \in S} \pi(s) \sum_{x \in X} u(x, s) f(s)(x)$, and the preference relation $\hat{\succcurlyeq}$ is represented by $\ell \mapsto \sum_{s \in S} \sum_{x \in X} u(x, s) \ell(x, s)$, where ℓ denotes a hypothetical outcome-state lottery. The function u, which is the same in both representations, is unique up to cardinal, unit-comparable transformation. Moreover, if all states are either obviously null or obviously nonnull then the probability π is unique, satisfying $\pi(s) = 0$ if s is obviously null, and $\pi(s) > 0$ if s is obviously nonnull.

Karni and Mongin (2000) observe that if the decision maker's beliefs are a cognitive phenomenon, quantifiable by a probability measure, then the subjective probability that figures in the representation above is the numerical expression of these beliefs. The explanation for this result is that the hypothetical lotteries incorporate distinct marginal distributions on the state space. Thus, the introspective preference relation captures the effects of shifting probabilities across states, thereby making it possible to measure the relative valuations of distinct outcomes. This aspect of the model does not exist in Savage's analytical framework.

A weaker version of this approach, based on restricting consistency to a subset of hypothetical lotteries that have the same marginal distribution on S, due to Karni et al. (1983), yields a subjective expected utility representation with state-dependent utility preferences. However, the subjective probabilities in this representation are contingent on the arbitrary choice of the marginal probabilities on the state space and are, therefore, themselves arbitrary. Consequently, the utility functions capture the decision maker's state-dependent risk attitudes but do not necessarily represent his/her evaluation of the consequences in the different states. Wakker (1987) extends the theory of Karni, Schmeidler, and Vind to include the case in which the set of consequences is a connected topological space.

Another way of defining subjective probabilities when the utilities are state dependent is to redefine the probabilities and utilities as follows: Let the state space be finite and the set of consequences be the real numbers, representing sums of money. Invoking the model of Wakker (1987), suppose that the preference relation is represented by a subjective expected utility function, $\sum_{s\in s} \pi(s) u\left(f(s), s\right)$. Suppose further that, for all $s \in S, u(\cdot, s)$, is differentiable and strictly monotonic, increasing in its first argument. Denote the derivative by $u'(\cdot, s)$ and define $\pi(s) = \hat{\pi}(s) u'(0, s) / \sum_{s\in S} \hat{\pi}(s) u'(0, s)$ and $\hat{u}\left(f(s), s\right) = u\left(f(s), s\right) / u'(0, s)$. Then the preference relation is represented by $\sum_{s\in s} \hat{\pi}(s) \hat{u}\left(f(s), s\right)$. This approach was developed by Nau (1995), who refers to $\hat{\pi}$ as risk-neutral probabilities, and by Karni and Schmeidler (1993).

Skiadas (1997) proposes a model, based on hypothetical preferences, that yields a representation with state-dependent preferences. In Skiadas' model, acts and states are primitive concepts, and preferences are defined on act–event pairs. For any such pair, the consequences (utilities) represent the decision maker's expression of his/her holistic valuation of the act. The decision maker is not supposed to know whether the given event occurred. Hence the decision maker's evaluation of the acts reflects, in part, his/her anticipated feelings, such as disappointment aversion.

1.4 BAYESIAN DECISION THEORY AND THE REPRESENTATION OF BELIEFS

The essential tenets of Bayesian decision theory are that (a) new information affects the decision maker's preferences, or choice behavior, through its effect on his/her beliefs rather than his/her tastes and (b) posterior probabilities, representing the decision maker's posterior beliefs, are obtained by updating the probabilities representing his/her prior beliefs using Bayes' rule. The critical aspect of Bayesian decision theory is, therefore, the existence and uniqueness of subjective probabilities, prior and posterior, representing the decision maker's prior and posterior beliefs that abide by Bayes' rule.

As was argued in Section 1.3.5, in decision theories invoking Savage's analytical framework, the unique representation of beliefs is predicated on a convention, that constant acts are constant utility acts, that is not implied by the preference structures. Consequently, these models do not provide choice-based foundations of Bayesian theory. A complete Bayesian decision theory anchored in the revealed preference methodology requires an alternative analytical framework, such as the one advanced in Karni (2011, 2011a) reviewed next.

1.4.1 The Analytical Framework

Let Θ be a finite set whose elements, *effects*, are physical phenomena, on which decision makers may place bets, that may or may not affect their well-being. Let A be a set whose elements, called *actions*, describe initiatives by which decision makers believe they can affect the likelihoods of ensuing effects.[25] Let B denote the set of all the real-valued mappings on Θ. Elements of B are referred to as *bets* and have the interpretation of effect-contingent monetary payoffs. Let $\bar{X}$ be a finite set of *signals* that the decision maker may receive before taking actions and choosing bets. The signals may be informative or noninformative.[26] Let X denote the set of informative signals and denote by o the noninformative signal. Hence, $\bar{X} = X \cup \{o\}$. The choice set, $\mathcal{I}$, consists of information-contingent plans, or *strategies*, for choosing actions and bets. Formally, a strategy $I \in \mathcal{I}$ is a function $I : \bar{X} \to A \times B$. Decision makers are characterized by a preference relation on $\mathcal{I}$.

The following example lends concrete meaning to the abstract terms mentioned above. Consider a resident of New Orleans facing the prospect of an approaching hurricane. The decision maker must make a plan that, contingent on the weather report, may include boarding up his/her house, moving his/her family to a shelter, and betting on the storm's damage (that is, taking out insurance on his/her property). The uncertainty is resolved once the weather forecast is obtained, the plan is put into effect, the storm passes, its path and force determined, and the ensuing damage is verified.

In this example, effects correspond to the potential material and bodily damage, and actions are the initiatives (e.g., boarding up the house, moving to a shelter) the decision maker can take to mitigate the damage. Bets are alternative insurance policies and observations are weather forecasts. The uncertainty in this example is resolved in

[25] A is assumed to be a connected topological space. For example, elements of A may describe levels of effort in which case A may be an interval in the real numbers.

[26] Receiving no signal is equivalent to receiving noninformative signal.

two stages. In the first stage, a weather forecast is obtained, and the action and the bet prescribed by the strategy are put into effect. In the second stage, the path and force of the hurricane are determined, the associated damage is realized, and the insurance indemnity is paid.

Consider the issue of subjective probabilities. At the point at which he/she contemplates his/her strategies, the decision maker entertains beliefs about two aspects of uncertainty. The first concerns the likelihood of alternative weather reports and, conditional on these reports, the likelihood of subsequent path-force combinations of the approaching hurricane. The second is the likelihood of the ensuing levels of damage (the effects). Clearly, the likelihoods of the latter are determined by those of the former, coupled with the actions that were taken, in the interim, by the decision maker.

As usual, a consequence depicts those aspects of the decision problem that affect the decision maker's ex post well-being. In this model, a *consequence* is a triplet (a, r, θ), representing the action, the monetary payoff of the bet, and the effect. The set of all consequences is given by the Cartesian product $C = A \times \mathbb{R} \times \Theta$.

Uncertainty in this model is resolved in two stages. In the first stage, an observation, $x \in \bar{X}$, obtains and the action and bet prescribed by each strategy for that observation are determined. In the second stage, the effect is realized and the payoff of the bet determined. Let Ω be the set of all functions from the set of actions to the set of effects (that is, $\Omega := \{\omega : A \to \Theta\}$). Elements of Ω depict the resolution of uncertainty surrounding the effects. A complete resolution of uncertainty is a function, s, from $\mathcal{I}$ to C. The *state space* is the set of all such functions, given by $S = \bar{X} \times \Omega$. Each state $s = (x, \omega)$ is an intersection of an *informational event* $\{x\} \times \Omega$ and a *material event* $\bar{X} \times \{\omega\}$. In other words, a state has two dimensions, corresponding to the two stages of the resolution of uncertainty — the purely informational dimension, x, and the material dimension, ω. Informational events do not affect the decision maker's well-being directly, whereas material events may. In general, states are abstract resolutions of uncertainty. In some situations, however, it is natural to attribute to the states concrete interpretations. In the example of the hurricane, the informational events are weather forecasts and the material events correspond to the path and force of the hurricane.

The Bayesian model requires a definition of a σ-algebra, $\mathcal{E}$, on S and a unique probability measure, P, on the measurable space $(S, \mathcal{E})$, such that (a) the conditioning of P on the noninformative signal o represents the decision maker's prior beliefs and (b) the conditioning of P on informative signals $x \in X$ represents the decision maker's posterior beliefs.

Denote by $I_{-x}(a, b)$ the strategy in which the x -coordinate of I, $I(x)$, is replaced by (a, b). The truncated strategy I_{-x} is referred to as a *substrategy*. For every given $x \in \bar{X}$, denote by $\succcurlyeq^x$ the induced preference relation on $A \times B$ defined by $(a, b) \succcurlyeq^x (a', b')$ if and only if $I_{-x}(a, b) \succcurlyeq I_{-x}(a', b')$. The induced preference relation $\succcurlyeq^o$ is referred to as the *prior* preference relation; the preference relations $\succcurlyeq^x, x \in X$, are the *posterior* preference relations. An observation, x, is *essential* if $(a,b) \succ^x (a',b')$, for some $(a,b), (a',b') \in A \times B$. Assume that all elements of $\bar{X}$ are essential.

For every $a \in A$ and $x \in \bar{X}$, define a binary relation $\succcurlyeq_a^x$ on B as follows: for all $b, b' \in B$, $b \succcurlyeq_a^x b'$ if and only if $(a,b) \succcurlyeq^x (a,b')$. An effect, θ, is said to be *nonnull given the observation–action pair* (x,a) if $(b_{-\theta} r) \succ_a^x (b_{-\theta} r')$, for some $b \in B$ and $r, r' \in \mathbb{R}$; otherwise, it is *null given the observation–action pair* (x,a). Let Θ(a,x) denote the set of nonnull effects given (x,a).

1.4.2 Preferences on Strategies and their Representation

Suppose that $\succcurlyeq$ on $\mathcal{I}$ is a continuous weak order.[27] The next axiom, coordinate independence, requires that preferences between strategies be independent of the coordinates on which they agree. It is analogous to, but weaker, than Savage's (1954) Sure-Thing Principle.[28] Formally,

(K1) (**Coordinate independence**) *For all* $x \in \bar{X}, I, I' \in \mathcal{I}$, *and* $(a,b), (a',b') \in A \times B, I_{-x}(a,b) \succcurlyeq I'_{-x}(a,b)$ *if and only if* $I_{-x}(a',b') \succcurlyeq I'_{-x}(a',b')$.

Two effects, θ and θ', are said to be *elementarily linked* if there are actions $a, a' \in A$ and observations $x, x' \in \bar{X}$ such that $\theta, \theta' \in \Theta(a,x) \cap \Theta(a',x')$. Two effects θ and θ' are said to be *linked* if there exists a sequence of effects $\theta = \theta_0, \ldots, \theta_n = \theta'$ such that θ_j and θ_{j+1} are elementarily linked, $j = 0, \ldots, n-1$. The set of effects, Θ, is linked if every pair of its elements is linked.

The next axiom requires that the "intensity of preferences" for monetary payoffs contingent on any given effect be independent of the action and the observation. This is analogous to the assumption that the decision maker's risk attitudes are independent of the action and the observation (i.e., if the consequences were lotteries then their ordinal ranking would be action and observation independent). This axiom applies Wakker's (1989) idea that the preference relation exhibits no contradictory payoffs across states, to actions and observation. Formally,

(K2) (**Independent betting preferences**) *For all* $(a,x), (a',x') \in A \times \bar{X}, b, b', b'', b''' \in B, \theta \in \Theta(a,x) \cap \Theta(a',x')$, *and* $r, r', r'', r''' \in \mathbb{R}$, *if* $(b_{-\theta} r) \succcurlyeq_a^x (b'_{-\theta} r'), (b'_{-\theta} r'') \succcurlyeq_a^x (b_{-\theta} r''')$, *and* $(b''_{-\theta} r') \succcurlyeq_{a'}^{x'} (b'''_{-\theta} r)$ *then* $(b''_{-\theta} r'') \succcurlyeq_{a'}^{x'} (b'''_{-\theta} r''')$.

The idea behind this axiom is easier to grasp by considering a specific instance in which $(b_{-\theta}, r) \sim_a^x (b'_{-\theta}, r')$, $(b_{-\theta} r'') \sim_a^x (b'_{-\theta} r''')$ and $(b''_{-\theta} r') \sim_{a'}^{x'} (b'''_{-\theta} r)$. The first pair of indifferences indicates that, given a and x, the difference in the payoffs b and b' contingent on the effects other than θ, measures the intensity of preferences between the payoffs r and r' and r'' and r''', contingent on θ. The indifference $(b''_{-\theta} r') \sim_{a'}^{x'} (b'''_{-\theta} r)$ indicates

[27] Let $\mathcal{I}$ be endowed with the product topology, then $\succcurlyeq$ is continuous in this topology.
[28] See Wakker (1989) for details.

that given another action–observation pair, (a', x') the intensity of preferences between the payoffs r and r' contingent on θ is measured by the difference in the payoffs the bets b'' and b''' contingent on the effects other than θ. The axiom requires that, in this case, the difference in the payoffs b'' and b''' contingent on the effects other than θ is also a measure of the intensity of the payoffs r'' and r''' contingent on θ. Thus, the intensity of preferences between two payoffs given θ is independent of the actions and the observations.

To link the decision maker's prior and posterior probabilities the next axiom asserts that, for every $a \in A$ and $\theta \in \Theta$, the prior probability of θ given a is the sum over X of the joint probability distribution on $X \times \Theta$ conditional on θ and a. Let $I^{-o}(a, b)$ denote the strategy that assigns the action–bet pair (a, b) to every observation other than o (that is, $I^{-o}(a, b)$ is a strategy such that $I(x) = (a, b)$ for all $x \in X$). Formally;

(K.3) (**Belief consistency**) *For every $a \in A$, $I \in \mathcal{I}$ and $b, b' \in B$, $I_{-o}(a, b) \sim I_{-o}(a, b')$ if and only if $I^{-o}(a, b) \sim I^{-o}(a, b')$.*

The interpretation of belief consistency is as follows. The decision maker is indifferent between two strategies that agree on X and, in the event that no new information becomes available, call for the implementation of the alternative action-bet pairs (a, b) or (a, b') if and only if he is indifferent between two strategies that agree on o and call for the implementation of the same action-bet pairs (a, b) or (a, b') regardless of the observation. Put differently, given any action, the preferences on bets conditional on there being no new information are the same as those when new information may not be used to select the bet. Hence, in and of itself, information is worthless.

Bets whose payoffs completely offset the direct impact of the effects are dubbed *constant utility bets*. The present analytical framework renders this notion a choice-base phenomenon. To grasp this claim, recall that actions affect decision makers in two ways: directly through their utility cost and indirectly by altering the probabilities of the effects. Moreover, only the indirect impact depends on the observations. In the case of constant utility bets, and only in this case, the intensity of the preferences over the actions is observation independent. This means that the indirect influence of the actions is neutralized, which can happen only if the utility associated with constant utility bets is invariable across the effects. Formally,

Definition 1.8 *A bet $\bar{b} \in B$ is a constant utility bet according to $\succcurlyeq$ if, for all $I, I', I'', I''' \in \mathcal{I}$, $a, a', a'', a''' \in A$ and $x, x' \in \bar{X}$, $I_{-x}(a, \bar{b}) \sim I'_{-x}(a', \bar{b})$, $I_{-x}(a'', \bar{b}) \sim I'_{-x}(a''', \bar{b})$ and $I''_{-x'}(a, \bar{b}) \sim I'''_{-x'}(a', \bar{b})$ imply $I''_{-x'}(a'', \bar{b}) \sim I'''_{-x'}(a''', \bar{b})$ and $\cap_{(x,a) \in X \times A}\{b \in B \mid b \sim_a^x \bar{b}\} = \{\bar{b}\}$.*

As in the interpretation of independent betting preferences, think of the preferences $I_{-x}(a, \bar{b}) \sim I'_{-x}(a', \bar{b})$ and $I_{-x}(a'', \bar{b}) \sim I'_{-x}(a''', \bar{b})$ as indicating that, given $\bar{b}$ and x, the preferential difference between the substrategies I_{-x} and I'_{-x} measures the intensity of preference of a over a' and of a'' over a'''. The indifference $I''_{-x'}(a, \bar{b}) \sim I'''_{-x'}(a', \bar{b})$ implies that, given $\bar{b}$, and another observation x', the preferential difference between the substrategies $I''_{-x'}$ and $I'''_{-x'}$ is another measure of the intensity of preference of a over a'.

Then it must be true that the same difference also measures the intensity of preference of a'' over a'''. Thus, the intensity of preferences between actions is observation independent, reflecting solely the direct disutility of action. In other words, the indirect effect has been neutralized. The requirement that $\cap_{(x,a)\in X\times A}\{b\in B \mid b\sim_a^x \bar{b}\}=\{\bar{b}\}$ implicitly asserts that actions and observations affect the probabilities of the effects and that these actions and observations are sufficiently rich that $\bar{b}$ is well defined.[29]

Let $B^{cu}(\succcurlyeq)$ be a subset of all constant utility bets according to $\succcurlyeq$. In general, this set may be empty. This is the case if the range of the utilities of the monetary payoffs across effects does not overlap. Here it is assumed that $B^{cu}(\succcurlyeq)$ is nonempty. The set $B^{cu}(\succcurlyeq)$ is said to be *inclusive* if for every $(x,a)\in X\times A$ and $b\in B$ there is $\bar{b}\in B^{cu}(\succcurlyeq)$ such that $b\sim_a^x \bar{b}$.[30]

Invoking the notion of constant utility bets, the next axiom requires that the trade-offs between the actions and the substrategies be independent of the constant utility bets. Formally;

(K.4) (**Trade-off independence**) For all $I,I'\in\mathcal{I}$, $x\in\bar{X}$, $a,a'\in A$ and $\bar{b},\bar{b}'\in B^{cu}(\succcurlyeq)$, $I_{-x}\left(a,\bar{b}\right)\succcurlyeq I'_{-x}\left(a',\bar{b}\right)$ if and only if $I_{-x}\left(a,\bar{b}'\right)\succcurlyeq I'_{-x}\left(a',\bar{b}'\right)$.

Finally, the direct effect (that is, the cost) of actions, measured by the preferential difference between $\bar{b}$ and $\bar{b}'$ in $B^{cu}(\succcurlyeq)$, must be observation independent. Formally:

(K.5) (**Conditional monotonicity**) For all $\bar{b},\bar{b}'\in B^{cu}(\succcurlyeq)$, $x,x'\in\bar{X}$, and $a,a'\in A$, $\left(a,\bar{b}\right)\succcurlyeq^x\left(a',\bar{b}'\right)$ if and only if $\left(a,\bar{b}\right)\succcurlyeq^{x'}\left(a',\bar{b}'\right)$.

The next theorem, due to Karni (2011), asserts the existence of a subjective expected utility representation of the preference relation $\succcurlyeq$ on $\mathcal{I}$ and characterizes the uniqueness properties of its constituent utilities and the probabilities. For each $I\in\mathcal{I}$, let $\left(a_{I(x)},b_{I(x)}\right)$ denote the action–bet pair corresponding to the x coordinate of I — that is, $I(x)=\left(a_{I(x)},b_{I(x)}\right)$.

Theorem 1.9 (Karni) *Let $\succcurlyeq$ be a preference relation on $\mathcal{I}$, and suppose that $B^{cu}(\succcurlyeq)$ is inclusive. Then $\succcurlyeq$ is a continuous weak order satisfying coordinate independence, independent betting preferences, belief consistency, trade-off independence, and conditional monotonicity if and only if there exist continuous, real-valued functions $\{u(\cdot,\theta)\mid\theta\in\Theta\}$ on $\mathbb{R}$, $v\in\mathbb{R}^A$, and a family, $\{\pi(\cdot,\cdot\mid a)\mid a\in A\}$, of joint probability measures on $\bar{X}\times\Theta$ such that $\succcurlyeq$ on $\mathcal{I}$ is represented by*

$$I\mapsto\sum_{x\in\bar{X}}\mu(x)\left[\sum_{\theta\in\Theta}\pi\left(\theta\mid x,a_{I(x)}\right)u\left(b_{I(x)}(\theta),\theta\right)+v\left(a_{I(x)}\right)\right], \qquad (1.1)$$

where $\mu(x)=\sum_{\theta\in\Theta}\pi(x,\theta\mid a)$ for all $x\in\bar{X}$ is independent of a, $\pi(\theta\mid x,a)=\pi(x,\theta\mid a)/\mu(x)$ for

[29] To render the definition meaningful, it is assumed that, given $\bar{b}$, for all $a,a',a'',a'''\in A$ and $x,x'\in\bar{X}$ there are $I,I',I'',I'''\in\mathcal{I}$ such that the indifferences $I_{-x}\left(a,\bar{b}\right)\sim I'_{-x}\left(a',\bar{b}\right)$, $I_{-x}\left(a'',\bar{b}\right)\sim I'_{-x}\left(a''',\bar{b}\right)$ and $I''_{-x'}\left(a,\bar{b}\right)\sim I'''_{-x'}\left(a',\bar{b}\right)$ hold.

[30] Inclusiveness of $B^{cu}(\succcurlyeq)$ simplifies the exposition.

all $(x,a) \in \bar{X} \times A$, $\pi(\theta \mid o, a) = \frac{1}{1-\mu(o)} \sum_{x \in X} \pi(x, \theta \mid a)$ *for all* $a \in A$*, and, for every* $\bar{b} \in B^{cu}(\succcurlyeq)$, $u\left(\bar{b}(\theta), \theta\right) = u\left(\bar{b}\right)$*, for all* $\theta \in \Theta$.

Moreover, if $\{\hat{u}(\cdot,\theta) \mid \theta \in \Theta\}$, $\hat{v} \in \mathbb{R}^A$ and $\{\hat{\pi}(\cdot,\cdot \mid a) \mid a \in A\}$ *is another set of utilities and a family of joint probability measures representing* $\succcurlyeq$ *in the sense of (1.1), then* $\hat{\pi}(\cdot,\cdot \mid a) = \pi(\cdot,\cdot \mid a)$ *for every* $a \in A$ *and there are numbers* $m > 0$ *and* k, k' *such that* $\hat{u}(\cdot,\theta) = m\hat{u}(\cdot,\theta) + k$ *,* $\theta \in \Theta$ *and* $\hat{v} = mv + k'$.

Although the joint probability distributions $\pi(\cdot,\cdot \mid a)$, $a \in A$ depend on the actions, the distribution μ is independent of a, consistent with the formulation of the decision problem, according to which the choice of actions is contingent on the observations. In other words, if new information becomes available, it precedes the choice of action. Consequently, the dependence of the joint probability distributions $\pi(\cdot,\cdot \mid a)$ on a captures solely the decision maker's beliefs about his/her ability to influence the likelihood of the effects by his/her choice of action.

1.4.3 Action-Dependent Subjective Probabilities on *S*

The family of joint probability distributions on observations and effects that figure in the representation (1.1) of the preference relation can be projected on the underlying state space to obtain a corresponding family of action-dependent, subjective probability measures. Moreover, this family of measures is the only such family that is consistent with the (unique) joint probability distributions on observations and effects. To construct the aforementioned family of probability measures partition the state space twice. First, partition the state space to *informational events*, $\mathcal{Y}$, corresponding to the observations (that is, let $\mathcal{Y} = \{\{x\} \times \Omega \mid x \in \bar{X}\}$). Second, for each action, partition the state space into *material events* corresponding to the effects. To construct the material partitions, fix $a \in A$ and, for every $\theta \in \Theta$, let $\mathcal{T}_a(\theta) := \{\omega \in \Omega \mid \omega(a) = \theta\}$. Then $\mathcal{T}_a = \{\bar{X} \times \mathcal{T}_a(\theta) \mid \theta \in \Theta\}$ is a (finite) material partition of S.

For every given action, define next a σ-algebra of events. Formally, let $\mathcal{E}_a$ be the σ-algebra on S generated by $\mathcal{Y} \wedge \mathcal{T}_a$, the join of $\mathcal{Y}$ and $\mathcal{T}_a$, whose elements are *events*.[31] Hence, events are unions of elements of $\mathcal{Y} \wedge \mathcal{T}_a$.

Consider the measurable spaces $(S, \mathcal{E}_a)$, $a \in A$. Define a probability measure η_a on $\mathcal{E}_a$ as follows: $\eta_a(E) = \sum_{x \in Z} \sum_{\theta \in \Upsilon} \pi(x, \theta \mid a)$ for every $E = Z \times \mathcal{T}_a(\Upsilon)$, where $Z \subseteq \bar{X}$, and $\mathcal{T}_a(\Upsilon) = \cup_{\theta \in \Upsilon} \mathcal{T}_a(\theta)$, $\Upsilon \subseteq \Theta$. Then, by representation (1.1), η_a is unique and the subjective probabilities, $\eta_a(E_I)$, of the informational events $E_I := \{Z \times \Omega \mid Z \subseteq \bar{X}\}$ are independent of a. Denote these probabilities by $\eta(E_I)$.

For every given a, consider the collection of material events $\mathcal{M}_a := \{\mathcal{T}_a(\Upsilon) \mid \Upsilon \subseteq \Theta\}$. By representation (1.1), the prior probability measure on $\mathcal{M}_a$ is given by $\eta_a(\mathcal{T}_a(\Upsilon) \mid o) = \sum_{\theta \in \Upsilon} \pi(\theta \mid o, a)$ and, for every $x \in X$, the posterior probability measure on $\mathcal{M}_a$ is given by $\eta_a(\mathcal{T}_a(\Upsilon) \mid x) = \sum_{\theta \in \Upsilon} \pi(\theta \mid x, a)$. Theorem 1.9 may be

[31] The join of two partitions is the coarsest common refinement of these partitions.

restated in terms of these probability measures as follows: Let $\succcurlyeq$ be a preference relation on $\mathcal{I}$, and suppose that $B^{cu}(\succcurlyeq)$ is inclusive. Then $\succcurlyeq$ is a continuous weak order satisfying coordinate independence, independent betting preferences, belief consistency, trade-off independence, and conditional monotonicity if and only if $\succcurlyeq$ on $\mathcal{I}$ is represented by

$$I \mapsto \sum_{x \in \bar{X}} \eta(x) \left[\sum_{\theta \in \Theta} u\left(b_{I(x)}(\theta)\ ,\ \theta\right) \eta_{a_{I(x)}}\left(\mathcal{T}_{a_{I(x)}}(\theta) \mid x\right) + v\left(a_{I(x)}\right) \right],$$

where the functions u on $\mathbb{R} \times \Theta$ and v on A, are as in Theorem 1.9 and, for each $a \in A$, η_a is a unique probability measure on the measurable space $(S, \mathcal{E}_a)$, such that $\eta_a(\{x\} \times \Omega) = \eta_{a'}(\{x\} \times \Omega) = \eta(x)$, for all $a, a' \in A$ and $x \in \bar{X}$.

The existence of a unique collection of measure spaces $\{(S, \mathcal{E}_a, \eta_a) \mid a \in A\}$ is sufficiently rich to allow action-dependent probabilities to be defined for every event that matters to the decision maker, for all conceivable choices among strategies he might be called upon to make. Hence, from the viewpoint of Bayesian decision theory, the family of action-dependent subjective probability measures is complete in the sense of being well defined for every conceivable decision problem that can be formulated in this framework. However, there is no guarantee that these subjective probability measures are mutually consistent. Karni (2011a) provides necessary and sufficient conditions for the existence of a unique probability space that supports all these action-dependent measures in the sense that $\eta_a(E)$ coincides with this measure for every $a \in A$ and $E \in \mathcal{E}_a$.

1.5 EXPECTED UTILITY THEORY WITH INCOMPLETE PREFERENCES

Perhaps the least satisfactory aspect of decision theory in general and expected utility theories under risk and under uncertainty in particular, is the presumption that decision makers can always express preferences, or choose between alternatives in a coherent manner. Von Neumann and Morgenstern expressed doubts concerning this aspect of the theory. "It is conceivable — and may even in a way be more realistic — to allow for cases where the individual is neither able to state which of two alternatives he prefers nor that they are equally desirable" (von Neumann and Morgenstern, 1947, p. 19). Aumann goes even further, writing "Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from a normative viewpoint" (Aumann, 1962, p. 446). The obvious way to address this issue while maintaining the other aspects of the theory of rational choice is to relax the assumption that the preference relations are complete.

1.5.1 Expected Multi-Utility Representation Under Risk

Aumann (1962) was the first to model expected utility under risk without the completeness axiom.[32] Invoking the algebraic approach, Shapley and Baucells (1998) characterized in complete expected utility preferences over risky prospects whose domain is lotteries. They showed that relaxing the completeness axiom while maintaining the other aspects of the theory, one risky prospect, or lottery, is weakly preferred over another only if its expected utility is greater for a set of von Neumann–Morgenstern utility functions. Dubra et al. (2004) used the topological approach to obtain an analogous result for incomplete expected utility preferences over risky prospects whose domain is a compact metric space. Formally, let X be an arbitrary compact metric space whose elements are outcomes, and denote by $\mathcal{P}(X)$ the set of all Borel probability measures on X.

Theorem 1.10 (Dubra, Maccheroni and Ok) *A reflexive and transitive binary relation, $\succcurlyeq$ on $\mathcal{P}(X)$, satisfying the independence axiom and weak continuity*[33] *if and only if there exists a convex set $\mathcal{U}$ of real-valued, continuous functions on X such that for all $P, Q \in \mathcal{P}(X)$,*

$$P \succcurlyeq Q \Leftrightarrow \int_X u(x) dP(x) \geq \int_X u(x)\, dQ(x) \text{ for all } u \in \mathcal{U}. \tag{1.2}$$

Let $\langle \mathcal{U} \rangle$ be the closure of the convex cone generated by all the functions in $\mathcal{U}$ and all the constant functions on X. Dubra et al. (2004) show that if $\mathcal{U}$ in (1.2) consists of bounded functions, then it is unique in the sense that if $\mathcal{V}$ is another nonempty set of continuous and bounded real-valued functions on X representing the preference relation $\succcurlyeq$ as in (1.2), then $\langle \mathcal{U} \rangle = \langle \mathcal{V} \rangle$.

The interpretation of the expected multi-utility representation under risk is that the preference relation is incomplete because the decision maker does not have a clear sense of his/her tastes or, more precisely, his/her risk attitudes. The range of risk attitudes that the decision maker might entertain is represented by the utility functions in $\mathcal{U}$, which makes it impossible for him/her to compare risky prospects that are ranked differently by different elements of $\mathcal{U}$.

1.5.2 Expected Multi-Utility Representation Under Uncertainty

Under uncertainty, the inability of a decision maker to compare all acts may reflect his/her lack of a clear sense of his/her risk attitudes, his/her lack of ability to arrive at precise assessment of the likelihoods of events, or both. The incompleteness of preferences in this case entails multi-prior expected multi-utility representations. In the context

[32] See discussion of Aumann's contribution in Dubra et al. (2004).

[33] Weak continuity requires that, for any convergent sequences (P_n) and (Q_n) in $\mathcal{P}(X)$, $P_n \succcurlyeq Q_n$ for all n implies that $lim_{n\to\infty} P_n \succcurlyeq lim_{n\to\infty} Q_n$.

of Anscombe and Aumann's (1963) analytical framework, one act, f, is preferred over another act, g, if and only if there is a nonempty set, Φ, of pairs, (π, U), consisting of a probability measure, π, on the set of states, S, and real-valued affine function, U on $\Delta(X)$, such that

$$\sum_{s\in S}\pi(s)U\left(f(s)\right) \geq \sum_{s\in S}\pi(s)U\left(g(s)\right), \text{ for all } (\pi, U) \in \Phi. \tag{1.3}$$

The issue of incomplete preferences in the context of decision making under uncertainty was first addressed by Bewley (2002).[34] The incompleteness of the preference relation in Bewley's model is due solely to the incompleteness of beliefs. In terms of representation (1.3), Bewley's work corresponds to $\Phi = \Pi \times \{U\}$, where Π is a closed convex set of probability measures on the set of states and U is a real-valued affine function on $\Delta(X)$.

Seidenfeld et al. (1995), Nau (2006) and Ok et al. (2012) study the representation of preference relations that accommodate incompleteness of both beliefs and tastes. Seidenfeld et al. (1995) axiomatize the case in which the representation entails $\Phi = \{(\pi, U)\}$. Ok et al. (2012) axiomatize a preference structure in which the source of incompleteness is either beliefs or tastes, but not both.[35] Galaabaatar and Karni (2013) axiomatize incompleteness in both beliefs and tastes, providing necessary and sufficient conditions for multi-prior expected multi-utility representations of preferences. They obtain Bewley's Knightian uncertainty and expected multi-utility representation with complete beliefs as special cases. Because this contribution is more satisfactory from the axiomatic point of view (that is, its simplicity and transparency), it is reviewed below.

Consider a strict preference relation, $\succ$ on H. The set H is said to be $\succ$ *-bounded* if it includes best and worst elements (that is, there exist h^M and h^m in H such that $h^M \succ h \succ h^m$, for all $h \in H - \{h^M, h^m\}$).[36] For every $h \in H$, let $B(h) := \{f \in H \mid f \succ h\}$ and $W(h) := \{f \in H \mid h \succ f\}$ denote the upper and lower contour sets of h, respectively. The relation $\succ$ is convex if the upper contour set is convex.

Since the main interest here is the representation of incomplete preferences, instead of the weak-order axiom, assume the following weaker requirement:

(A.1') (**Strict partial order**) The preference relation $\succ$ is transitive and irreflexive.

Let $\succ$ be a binary relation on H. Then, analogous to Corollary 1.5, H is $\succ$ -bounded strict partial order satisfying the Archimedean and independence axioms if and only if

[34] Bewley's original work first appeared in 1986 as a Cowles Foundation Discussion Paper no. 807.

[35] In terms of representation (1.3), Ok et al. (2012) consider the cases in which $\Phi = \Pi \times \{U\}$ or $\Phi = \{\pi\} \times \mathcal{U}$.

[36] The difference between the preference structure above and that of expected utility theory is that the induced relation $\neg\left(f \succ g\right)$ is reflexive but not necessarily transitive (it is not necessarily a preorder). Moreover, it is not necessarily complete. Thus, $\neg\left(f \succ g\right)$ and $\neg\left(g \succ f\right)$, do not imply that f and g are indifferent (i.e., equivalent), rather they may be *noncomparable*. If f and g are noncomparable, we write $f \bowtie g$.

there exists a nonempty, convex, and closed set $\mathcal{W}$ of real-valued functions, w, on $X \times S$, such that

$$\sum_{s\in S}\sum_{x\in X} h^{M}(x,s)w(x,s) > \sum_{s\in S}\sum_{x\in X} h(x,s)w(x,s) > \sum_{s\in S}\sum_{x\in X} h^{m}(x,s)w(x,s)$$

for all $h \in H - \{h^{M}, h^{m}\}$ and $w \in \mathcal{W}$, and for all $h, h' \in H$, $h \succ h'$ if and only if

$$\sum_{s\in S}\sum_{x\in X} h(x,s)w(x,s) > \sum_{s\in S}\sum_{x\in X} h'(x,s)w(x,s) \text{ for all } w \in \mathcal{W}. \tag{1.4}$$

To state the uniqueness properties of the representation, the following notations and definitions are needed. Let δ_s be the vector in $\mathbb{R}^{|X|\cdot|S|}$ such that $\delta_s(t,x) = 0$ for all $x \in X$ if $t \neq s$ and $\delta_s(t, x) = 1$ for all $x \in X$ if $t = s$. Let $D = \{\theta\delta_s \mid s \in S, \theta \in \mathbb{R}\}$. Let $\mathcal{U}$ be a set of real-valued functions on $\mathbb{R}^{|X|\cdot|S|}$. Fix $x^0 \in X$, and for each $u \in \mathcal{U}$ define a real-valued function, $\hat{u}$, on $\mathbb{R}^{|X|\cdot|S|}$ by $\hat{u}(x,s) = u(x,s) - u\left(x^0, s\right)$ for all $x \in X$ and $s \in S$. Let $\widehat{\mathcal{U}}$ be the normalized set of functions corresponding to $\mathcal{U}$ (that is, $\widehat{\mathcal{U}} = \{\hat{u} \mid u \in \mathcal{U}\}$). We denote by $\langle\widehat{\mathcal{U}}\rangle$ the closure of the convex cone in $\mathbb{R}^{|X|\cdot|S|}$ generated by all the functions in $\widehat{\mathcal{U}}$ and D. With this in mind, the uniqueness of the representation requires that if $\mathcal{W}'$ be another set of real-valued, affine functions on H that represents $\succ$ in the sense of (4), then $\langle\widehat{\mathcal{W}'}\rangle = \langle\widehat{\mathcal{W}}\rangle$.

This representation is not the most parsimonious, as the set $\mathcal{W}$ includes functions that are redundant (that is, their removal does not affect the representation). Henceforth, consider a subset of essential functions, $\mathcal{W}^o \subset \mathcal{W}$, that is sufficient for the representation. Define the sets of essential component functions $\mathcal{W}^o_s := \{w(\cdot, s) \mid w \in \mathcal{W}^o\},\ s \in S$.

As in the state-dependent expected utility representation, decomposing the functions $w\left(p,s\right)$ into subjective probabilities and utilities requires tightening the structure of the preferences. To state the next axiom, which is a special case of Savage's (1954) postulate P.7, the following notation is useful. For each $f \in H$ and every $s \in S$, let f^s denote the constant act whose payoff is $f(s)$ in every state (that is, $f^s\left(s'\right) = f(s)$ for all $s' \in S$). The axiom requires that if an act, g, is strictly preferred over every constant act, f^s, obtained from the act f, then g is strictly preferred over f. Formally,

(A.6) (**Dominance**) For all $f, g \in H$, if $g \succ f^s$ for every $s \in S$, then $g \succ f$.

Galaabataar and Karni (2013) show that a preference relation is a strict partial order satisfying Archimedean, independence, and dominance if and only if there is a nonempty convex set of affine utility functions on $\Delta(X)$ and, corresponding to each utility function, a convex set of probability measures on S such that, when presented with a choice between two acts, the decision maker prefers the act that yields a higher expected utility according to every utility function and every probability measure in the corresponding set.

Let the set of probability–utility pairs that figure in the representation be $\Phi := \{(\pi, U) \mid U \in \mathcal{U},\ \pi \in \Pi^U\}$. Each $(\pi, U) \in \Phi$ defines a hyperplane $w := \pi \cdot U$. Denote by $\mathcal{W}$ the set of all these hyperplanes, and define $\langle \Phi \rangle = \langle \mathcal{W} \rangle$.

Theorem 1.11 (Galaabaatar and Karni) *Let $\succ$ be a binary relation on H. Then H is $\succ$-bounded and $\succ$ is nonempty bounded strict partial order satisfying the Archimedean, independence, and dominance axioms if and only if there exists a nonempty, and convex set, $\mathcal{U}$, of real-valued, affine functions on $\Delta(X)$, and closed and convex sets $\Pi^U, U \in \mathcal{U}$, of probability measures on S such that, for all $h \in H$ and $(\pi, U) \in \Phi$,*

$$\sum_{s \in S} \pi(s) U(h^M(s)) > \sum_{s \in S} \pi(s) U(h(s)) > \sum_{s \in S} \pi(s) U(h^m(s))$$

and for all $h, h' \in H$,

$$h \succ h' \Leftrightarrow \sum_{s \in S} \pi(s) U(h(s)) > \sum_{s \in S} \pi(s) U(h'(s)) \text{ for all } (\pi, U) \in \Phi, \tag{1.5}$$

where $\Phi = \{(\pi, U) \mid U \in \mathcal{U}, \pi \in \Pi^U\}$. Moreover, if $\Phi' = \{(\pi', V) \mid V \in \mathcal{V},\ \pi' \in \Pi^V\}$ is another set of real-valued, affine functions on $\Delta(X)$ and sets of probability measures on S that represent $\succ$ in the sense of (1.5), then $\langle \Phi' \rangle = \langle \Phi \rangle$ *and $\pi(s) > 0$ for all s.*

1.5.2.1 Special Cases

Galaabaatar and Karni (2013) analyze three special cases. The first involves complete separation of beliefs from tastes (that is, $\Phi = \mathcal{M} \times \mathcal{U}$, where $\mathcal{M}$ is a nonempty convex set of probability measures on S, and $\mathcal{U}$ is a nonempty, closed, and convex set of real-valued, affine functions on $\Delta(X)$).

To grasp the next result, recall that one of the features of Anscombe and Aumann's (1963) model is the possibility it affords for transforming uncertainty into risk by comparing acts to their reduction to lotteries under alternative measures on $\Delta(S)$. In particular, there is a subjective probability measure, α^* on S, that governs the decision-maker's choice. In fact, every act, f, is indifferent to the constant act f^{α^*} obtained by the reduction of the compound lottery represented by (f, α^*).[37] It is, therefore, natural to think of an act as a tacit compound lottery in which the probabilities that figure in the first stage are the subjective probabilities that govern choice behavior. When the set of subjective probabilities that govern choice behavior is not a singleton, an act f corresponds to a set of implicit compound lotteries, each of which is induced by a (subjective) probability measure. The set of measures represents the decision maker's indeterminate

[37] For each act–probability pair $(f, \alpha) \in H \times \Delta(S)$, we denote by f^a the constant act defined by $f^{\alpha}(s) = \Sigma_{s' \in S} \alpha_s f(s')$ for all $s \in S$.

beliefs. If, in addition, the reduction of compound lotteries assumption is imposed, then (f,α) is equivalent to its reduction, f^{a}.

The next axiom asserts that $g \succ f$ is sufficient for the reduction of (g,α) to be preferred over the reduction of (f,α) for all α in the aforementioned set of measures. Formally:

(A.7) (**Belief consistency**) For all $f,g \in H, g \succ f$ implies $g^{\alpha} \succ f^{\alpha}$ for all $\alpha \in \Delta(S)$ such that $f' \succ h^{p}$ implies $\neg(h^{p} \succ (f')^{\alpha})$ (for any $p \in \Delta(X), f' \in H$).

The next theorem characterizes the "product representation." For a set of functions, $\mathcal{U}$ on X, we denote by $\langle\mathcal{U}\rangle$ the closure of the convex cone in $\mathbb{R}^{|X|}$ generated by all the functions in $\mathcal{U}$ and all the constant functions on X.

Theorem 1.12 (Galaabaatar and Karni) *Let $\succ$ be a binary relation on H, then H is $\succ$-bounded and $\succ$ nonempty bounded strict partial order satisfying the Archimedean, independence dominance and belief consistency axioms if and only if there exist nonempty sets, $\mathcal{U}$ and $\mathcal{M}$, of real-valued, affine functions on $\Delta(X)$ and probability measures on S, respectively, such that, for all $h \in H$ and $(\pi,U) \in \mathcal{M}\times\mathcal{U}$,*

$$\sum_{s\in S}\pi(s)U(h^{M}(s)) > \sum_{s\in S}\pi(s)U(h(s)) > \sum_{s\in S}\pi(s)U(h^{m}(s))$$

and for all $h,h' \in H, h \succ h'$ if and only if

$$\sum_{s\in S}\pi(s)U(h(s)) > \sum_{s\in S}\pi(s)U(h'(s)) \text{ for all } (\pi,U) \in \mathcal{M}\times\mathcal{U}.$$

Moreover, if $\mathcal{V}$ and $\mathcal{M}'$ are another pair of sets of real-valued functions on X and probability measures on S that represent $\succ$ in the above sense, then $\langle\mathcal{U}\rangle = \langle\mathcal{V}\rangle$ and $cl(conv(\mathcal{M})) = cl(conv(\mathcal{M}'))$, where $cl(conv(\mathcal{M}))$ is the closure of the convex hull of $\mathcal{M}$. In addition, $\pi(s) > 0$ for all $s \in S$ and $\pi \in \mathcal{M}$.

The model of Knightian uncertainty requires a formal definition of complete tastes. To provide such a definition, it is assumed that the conditional on the state the strict partial orders induced by $\succ$ on H exhibits negative transitivity.[38] Formally:

(A.8) (**Conditional negative transitivity**) For all $s \in S$, $\succ_{s}$ is negatively transitive.

Define the weak conditional preference relation, $\succsim_{s}$ on $\Delta(X)$ as follows: For all $p,q \in \Delta(X)$, $p \succsim_{s} q$ if $\neg(q \succ_{s} p)$. Then $\succsim_{s}$ is complete and transitive.[39] Let $\succ^{c}$ be the restriction of $\succ$ to the subset of constant acts, H^{c}, in H. Then $\succ^{c} = \succ_{s}$ for all $s \in S$. Define $\succsim^{c}$ on H^{c} as follows: For all $p,q \in H^{c}$, $p \succsim^{c} q$ if $\neg(q \succ p)$. Then $\succsim^{c} = \succsim_{s}$ for all $s \in S$.

[38] A strict partial order, $\succ$ on a set D, is said to exhibit negative transitivity if for all $x,y,z \in D, \neg(x \succ y)$ and $\neg(y \succ z)$ imply $\neg(x \succ z)$.

[39] See Kreps' (1988) proposition (2.4).

Conditional negative transitivity implies that the weak preference relation $\succsim^c$ on H^c is complete.[40]

The Galaabaatar and Karni (2013) version of Knightian uncertainty can be stated as follows:

Theorem 1.13 (Knightian uncertainty) *Let $\succ$ be a binary relation on H. Then H is $\succ$-bounded, and $\succ$ is nonempty, strict partial order satisfying the Archimedean, independence, dominance, and conditional negative transitivity if and only if there exists a nonempty set, $\mathcal{M}$, of probability measures on S and a real-valued, affine function U on $\Delta(X)$ such that*

$$\sum_{s\in S} U\left(h^M(s)\right)\pi(s) > \sum_{s\in S} U(h(s))\,\pi(s) > \sum_{s\in S} U\left(h^m(s)\right)\pi(s)$$

for all $h \in H$ and $\pi \in \mathcal{M}$, and for all $h, h' \in H$, $h > h'$ if and only if $\sum_{s\in S} U(h(s))\,\pi(s) \succ \sum_{s\in S} U(h'(s))\pi(s)$ for all $\pi \in \mathcal{M}$. *Moreover, U is unique up to a positive affine transformation, the closed convex hull of $\mathcal{M}$ is unique, and for all $\pi \in \mathcal{M}$, $\pi(s) > 0$ for any s.*

The dual of Knightian uncertainty is the case in which the incompleteness of the decision-maker's preferences is due solely to the incompleteness of his/her tastes. To define the notion of coherent beliefs, denote by h^p the constant act in H whose payoff is p for every $s \in S$. For each event E, $pEq \in H$ is the act whose payoff is p for all $s \in E$ and q for all $s \in S - E$. Denote by $p\alpha q$ the constant act whose payoff, in every state, is $\alpha p + (1 - \alpha)\,q$. A *bet* on an event E is the act pEq, whose payoffs satisfy $p \succ q$.

A decision maker who prefers the constant act $p\alpha q$ to the bet pEq is presumed to believe that α exceeds the likelihood of E. A preference relation $\succ$ on H is said to exhibit *coherent beliefs,* if for all events E and $p, q, p', q' \in \Delta(X)$ such that $h^p \succ h^q$ and $h^{p'} \succ h^{q'}$, $p\alpha q \succ pEq$ if and only if $p'\alpha q' \succ p'Eq'$, and $pEq \succ p\alpha q$ if and only if $p'Eq' \succ p'\alpha q'$. It is noteworthy that a decision maker whose preference relation satisfies strict partial order, Archimedean, independence, and dominance exhibits coherent beliefs.

Belief completeness is captured by the following axiom:

(A.9) (**Complete beliefs**) For all events E and $\alpha \in [0, 1]$, either $p^M\alpha p^m \succ p^M Ep^m$ or $p^M Ep^m \succ p^M\alpha' p^m$ for all $\alpha > \alpha'$.

If the beliefs are complete, then the incompleteness of the preference relation on H is due entirely to the incompleteness of tastes.

The next theorem is the subjective expected multi-utility version of the Anscombe–Aumann (1963) model corresponding to the situation in which the decision-maker's beliefs are complete.

Theorem 1.14 (Subjective expected multi-utility) *Let $\succ$ be a binary relation on H. Then H is $\succ$-bounded and $\succ$ is a nonempty strict partial order satisfying the Archimedean,*

[40] This is the assumption of Bewley (2002).

independence, dominance, and complete beliefs if and only if there exists a nonempty set, $\mathcal{U}$, of real-valued, affine functions on $\Delta(X)$ and a probability measure π on S such that

$$\sum_{s\in S} U\left(h^M(s)\right)\pi(s) > \sum_{s\in S} U(h(s))\,\pi(s) > \sum_{s\in S} U\left(h^m(s)\right)\pi(s), \tag{1.6}$$

for all $h \in H$ and $U \in \mathcal{U}$ and for all $h, h' \in H$, $h \succ h'$ if and only if $\sum_{s\in S} U(h(s))\,\pi(s) > \sum_{s\in S} U\left(h'(s)\right)\pi(s)$ for all $U \in \mathcal{U}$. Moreover, if $\mathcal{V}$ is another set of real-valued, affine functions on $\Delta(X)$ that represent $\succ$ in the above sense then $\langle\mathcal{V}\rangle = \langle\mathcal{U}\rangle$. The probability measure, π, is unique and $\pi(s) > 0$ if and only if s is nonnull.

1.6 CONCLUSION

For more than half a century, expected utility theory has been the paradigmatic model of decision making under risk and under uncertainty. The expected utility model acquired its dominant position because it is founded on normatively compelling principles, and its representation has an appealing functional form. The functional form captures two subroutines that are presumed to be activated when decision makers must choose among alternative courses of action — tastes for the consequences and beliefs regarding their likely realizations — and integrates them to obtain a decision criterion. In addition to providing analytical tools for the study of decision making under uncertainty, the theory was also intended to furnish choice-based foundations of the prior probabilities in Bayesian statistics.

This chapter reviewed the main models that formalized these ideas, emphasizing the interaction among the structure of the choice sets and the axiomatic structure of the preference relations. In addition, the chapter includes a critical evaluation of the expected utility models and a discussion of alternative developments intended to address some of their perceived shortcomings.

Almost from the start, doubts were raised about the descriptive validity of the central tenets of expected utility theory — the independence axiom in the case of decision making under risk and the Sure-Thing Principle in the case of decision making under uncertainty. The weight of the experimental evidence, suggesting that decision makers violate the independence axiom and the Sure-Thing Principle in a systematic manner, inspired the developments of alternative models that depart from these axioms. The development and study of these nonexpected utility theories have been the main concern of decision theory during the last quarter of century and led to the development of an array of interesting and challenging ideas and models. These developments are treated in other chapters in this handbook.

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