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Weighted utility theory with incomplete preferences

Edi Karni [a,*], Nan Zhou [b,1]

[a] Department of Economics, Johns Hopkins University, United States of America
[b] Johns Hopkins University, Advanced Academic Programs, United States of America

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ABSTRACT

This paper axiomatizes the representations of weighted utility theory with incomplete preferences. These include the general multiple weighted utility representation as well as special cases of multiple utilities or multiple weights only. Some behavioral implications are explored in the context of a portfolio selection problem, which illustrates the potential applicability of such models to a range of problems.

1. Introduction

1.1. Motivation and literature review

Two of the assumptions of expected utility theory seem less satisfactory than the others, those of completeness and independence. Completeness requires that decision makers are able to compare and express clear preferences between any two risky prospects, while independence requires that decision makers rank prospects only by their distinct characteristics, disregarding their common aspects.

That the completeness axiom may be too demanding was recognized from the outset by von Neumann and Morgenstern (1947) who say that "It is conceivable – and may even in a way be more realistic – to allow for cases where the individual is neither able to state which of two alternatives she prefers nor that they are equally desirable." Aumann (1962), who was the first to study expected utility theory without the completeness axiom, claims that "Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from a normative viewpoint." Later studies by Dubra et al. (2004), and, most recently, Galaabaatar and Karni (2012) all conclude that the departure from completeness axiom leads to expected multi-utility representations.

Experimental evidence, such as the Allais paradox, motivated developments in the 1980s of theories of decision making under risk that depart from the independence axiom. These theories include Quiggin's (1982) anticipated utility theory, Chew and MacCrimmon's (1979) weighted utility theory, Yaari's (1987) dual theory, Dekel's (1986) implicit weighted utility, and Gul's (1991) theory of disappointment aversion.[2]

Thus far, the only works that simultaneously depart from both the completeness and independence axioms are Maccheroni (2004) and Safra (2014). Maccheroni (2004) showed that without the completeness axiom, the representation theorem in Yaari's dual theory entails the existence of a set of probability transformation functions such that one risky prospect is preferred over another if and only if its rank-dependent expected value is larger according to every probability transformation function in that set. Safra (2014) studied a general model of decision making under risk that has the betweenness property[3] and showed that without completeness, the representation theorem entails the existence of a set of continuous functionals displaying betweenness such that one risky prospect is preferred over another if and only if it is assigned a higher value by every element in this set. Weighted utility theory, the subject of this work, also displays the betweenness property but is more structured and therefore calls for a different analysis.

The objective of this paper is to study weighted utility theory without the completeness axiom. Introduced by Chew and MacCrimmon (1979) and Chew (1983, 1989), weighted utility theory is based on a natural weakening of the independence axiom to a ratio substitution property, allowing the outcomes to hold different degrees of salience for the decision maker, captured in the representation by the namesake weight function. Incompleteness

* Corresponding author.
E-mail address: karni@jhu.edu (E. Karni).

1 We are grateful to Zvi Safra for his useful comments and suggestions.

2 See Karni and Schmeidler (1991) for a review of this literature.

3 Models of decision making under risk with the betweenness property include Chew (1983), Dekel (1986), and Gul (1991).

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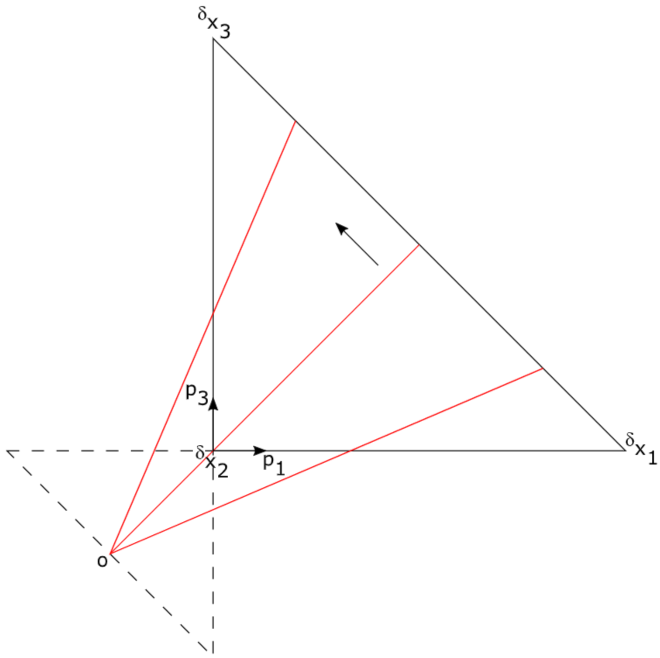

Fig. 1. Weighted utility.

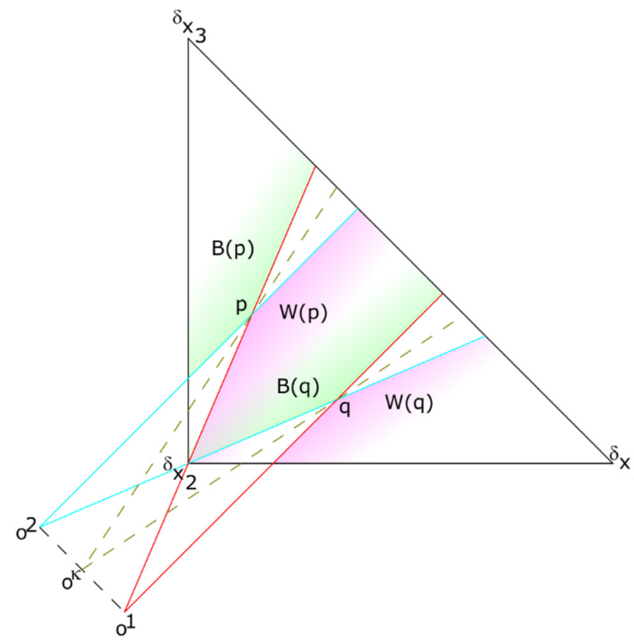

Fig. 2. Multiple utilities.

in weighted utility theory may thus be the result not only of indecisive tastes, captured by a set of utility functions that rank the outcomes differently, but also of conflicting perceptions of the alternatives presented, captured by a set of weight functions that represent different transformations of the probabilities, or some combination of both. We begin by analyzing the general multiple weighted expected utility model, and follow with the two special cases of multiple utilities paired with a single weight function, or a single utility paired with multiple weights. We also discuss some behavioral implications of the model in the context of portfolio choice.

1.2. An informal review

To set the stage and develop some intuition, we begin with an informal review. Let $X = \{x_1, \ldots, x_n\}$ be the set of outcomes, and denote the set of lotteries over X by $\Delta(X) = \{p \in \mathbb{R}^n_+ : \sum_{x \in X} p(x) = 1\}$.[4] Denote by δ_x the degenerate lottery that assigns $x \in X$ unit probability mass. Let $\succ$ be a strict preference relation over $\Delta(X)$, that is, an irreflexive and transitive binary relation which may or may not be negatively transitive. If $\succ$ is negatively transitive, implying completeness, but violates the independence axiom and instead satisfies only the weaker substitution axiom of Chew (1989), then there exist a utility function u and a positive valued weight function w mapping X to $\mathbb{R}$, such that, for all $p, q \in \Delta(X)$,

$$p \succ q \Longleftrightarrow \frac{\sum_{x \in X} p(x) w(x) u(x)}{\sum_{x \in X} p(x) w(x)} > \frac{\sum_{x \in X} q(x) w(x) u(x)}{\sum_{x \in X} q(x) w(x)}. \quad (1)$$

For example, if $n = 3$ and $\delta_{x_3} \succ \delta_{x_2} \succ \delta_{x_1}$, the indifference map induced by (1) is depicted in Fig. 1. The indifference curves all emanate from a *source point* o lying outside the simplex.

Fig. 1 depicts a decision maker that attaches greater weight to the extreme outcomes x_1 and x_3 than to the median outcome x_2, indicating that the former have more influence on her evaluation of any particular lottery than their probability would justify. The degree of risk aversion would vary across the simplex and thus the decision maker would exhibit Allais-type behavior, being willing to take risks when she feels she has nothing to lose that she would otherwise avoid if her alternatives were more attractive. The extent of this distortion depends on the proximity of the source point o to the simplex and as it is moved farther away from the diagram, approaches the parallel indifference map of expected utility.

Now suppose that $\succ$ is also incomplete. As in multiple expected utility models with independence such as those of Dubra et al. (2004), and Galaabaatar and Karni (2012), the preference relation cannot be meaningfully characterized with indifference curves, as two lotteries that are not strictly comparable are not necessarily equivalent. Consider the lottery p in Fig. 2, let $B(p) = \{r \in \Delta(X) : r \succ p\}$ and $W(p) = \{r \in \Delta(X) : r \prec p\}$ respectively denote the upper and lower contour sets of p, and observe that they are demarcated by rays emanating from a pair of distinct source points o^1 and o^2. Unlike in classic weighted utility theory, these rays are not indifference curves, but indicate only that no two lotteries lying on a single ray are strictly comparable, a relation which is not transitive and hence not an equivalence relation.

Nevertheless these *incomparability curves* do inherit many of the properties of indifference curves from the weighted utility setup. As each set of such curves converges at a source point, each in turn has a weighted linear utility representation as in (1), with the two sources o^1 and o^2 respectively corresponding to utility and weight pairs (u^1, w) and (u^2, w). As the diagram indicates, for any lottery to be strictly preferred to p it must lie above both of the incomparability curves intersecting p, and thus the preference relation has a multiple weighted expected utility representation,

[4] For each $p, q \in \Delta(X)$ and $\alpha \in [0, 1]$, define $\alpha p + (1 - \alpha)q \in \Delta(X)$ by $(\alpha p + (1 - \alpha)q)(x) = \alpha p(x) + (1 - \alpha)q(x)$ for all $x \in X$. Then $\Delta(X)$ is a convex subset of the linear space $\mathbb{R}^n$.

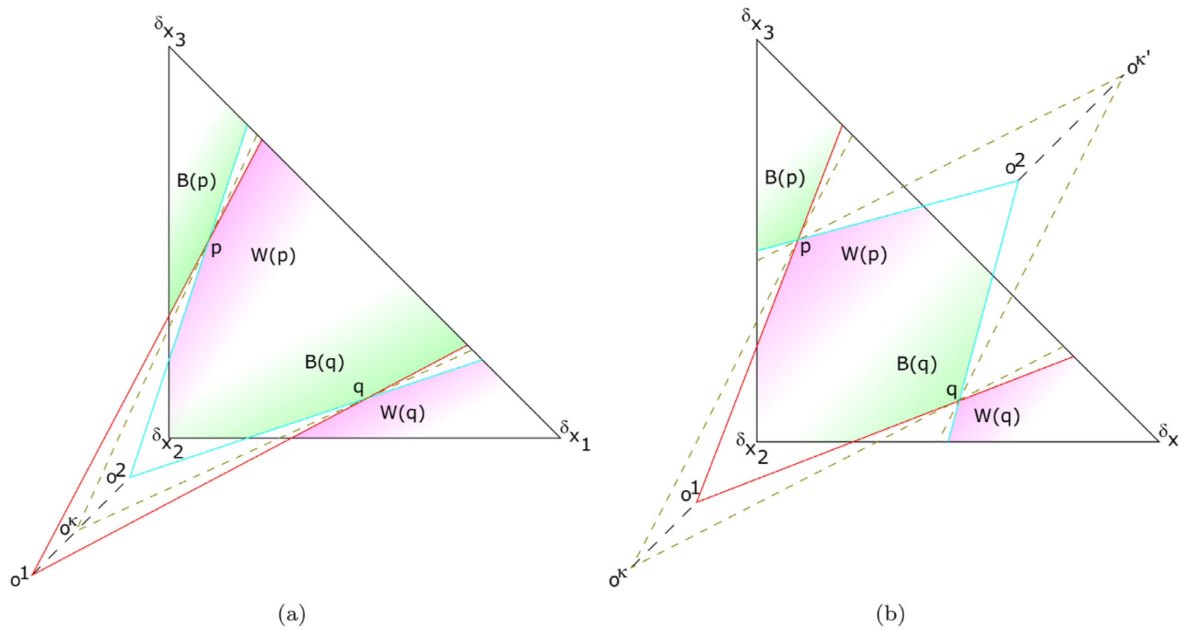

Fig. 3. Multiple weights.

with a set of utilities $\mathcal{U} = \{u^1, u^2\}$.

$$p \succ q \iff \frac{\sum_{x\in X} p(x)w(x)u(x)}{\sum_{x\in X} p(x)w(x)} > \frac{\sum_{x\in X} q(x)w(x)u(x)}{\sum_{x\in X} q(x)w(x)}, \ \forall u \in \mathcal{U}. \tag{2}$$

Two other aspects of this setup are noteworthy. Firstly, every point on the line segment connecting o^1 and o^2 also projects a set of incomparability curves, always lying between the rays projected by the two endpoints. Any such point o^κ would thus also be a source point and correspond to some utility u^κ, which could be included within $\mathcal{U}$ without altering the preference relation it represents. The location of o^κ between o^1 and o^2 implies that u^κ would be some convex combination of u^1 and u^2, and thus could not contradict any ordering jointly established by these utilities. This leads us to conclude that, just as in multiple expected utility models with independence, the representation will only be unique up to some closed convex hull, though as the utilities here are not linear we will need to adopt a slightly different approach to establish this result.

Secondly, the line segment connecting the source points o^1 and o^2 is parallel to that connecting the best and worst outcomes δ_{x_3} and δ_{x_1}. Hence these sources are equidistant from the simplex and represent different utilities paired with the same weight function. As the incomparability curves projected from o^1 are everywhere steeper than those projected from o^2, u^1 is uniformly more risk averse than u^2. This naturally leads us to consider the dual case, where a single utility function is paired with multiple weight functions.

Fig. 3(a) depicts such a case, where there are a pair of source points o^1 and o^2 corresponding to utility-weight pairs (u, w^1) and (u, w^2). Here the utility functions are identical, as the incomparability curves drawn from both sources through δ_{x_2} coincide and thus rank the median outcome identically, but as o^2 is closer to the simplex, w^2 represents a greater deviation from the uniform weights of expected utility theory. Fig. 3(b) depicts a similar case, where there are again two sources o^1 and o^2, and two corresponding utility-weight pairs (u, w^1) and (u, w^2), but here o^2 is located on the other side of the simplex. This produces incomparability curves that fan in rather than out, and indicating that x_2 is weighted more heavily than the extreme outcomes, rather than less. The preferences depicted in either incomparability map would have a representation consisting of the single utility u and multiple weights $\mathcal{W} = \{w^1, w^2\}$.

$$p \succ q \iff \frac{\sum_{x\in X} p(x)w(x)u(x)}{\sum_{x\in X} p(x)w(x)} > \frac{\sum_{x\in X} q(x)w(x)u(x)}{\sum_{x\in X} q(x)w(x)}, \ \forall w \in \mathcal{W}. \tag{3}$$

Note that in the multiple weight case depicted in Fig. 3(a), analogously to the multiple utility case depicted in Fig. 2, we may include in $\mathcal{W}$ the weight function w^κ corresponding to any point o^κ on the line segment connecting o^1 and o^2 without altering the preferences. However, attempting the same in Fig. 3(b) would be invalid, as it would produce source points lying within the simplex. In this case, we can instead include any source points lying on the line defined by o^1 and o^2 but not on the segment connecting them, effectively connecting o^1 to o^2 through the point at infinity, as any of these would produce incomparability curves that lie between those projected from the endpoints and hence their inclusion would not alter the representation.

Finally, we consider the general case that incorporates both multiple utilities and multiple weights, as depicted in Fig. 4. Here the four source points $\Omega = \{o^{11}, o^{12}, o^{21}, o^{22}\}$ correspond to pairs of utility and weight functions $\mathcal{V} = \{(u^1, w^1), (u^1, w^2), (u^2, w^1), (u^2, w^2)\}$ and the preferences depicted have the representation

$$p \succ q \iff \frac{\sum_{x\in X} p(x)w(x)u(x)}{\sum_{x\in X} p(x)w(x)} > \frac{\sum_{x\in X} q(x)w(x)u(x)}{\sum_{x\in X} q(x)w(x)}, \quad \forall (u, w) \in \mathcal{V}. \tag{4}$$

Here the set of utility-weight pairs is separable, as $\mathcal{V} = \mathcal{U} \times \mathcal{W} = \{u^1, u^2\} \times \{w^1, w^2\}$, though this need not be the case generally. Any point lying in the convex hull of Ω would map to a utility-weight pair that could be included in $\mathcal{V}$ without altering the preferences represented. Therefore, this representation admits any of the models considered so far as special cases, with the single utility or single weight cases in (2) and (3) if respectively $\mathcal{U}$ or $\mathcal{W}$ are singletons, weighted utility if $\mathcal{V}$ is a singleton, multiple expected utility if every element of $\mathcal{W}$ is a constant function, and finally expected utility if all of these hold.

The next section introduces the basic model. Section 3 details the general multiple weighted expected utility model, with

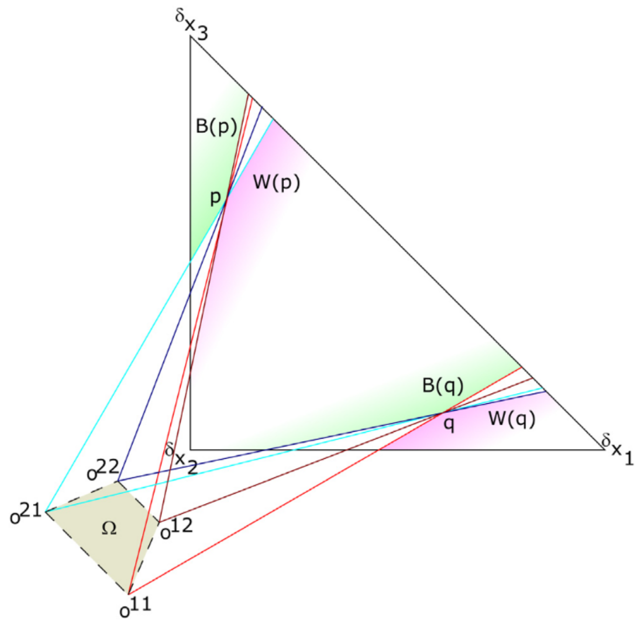

Fig. 4. Multiple weighted expected utility.

the special cases of a single utility or single weight covered in Section 4. Section 5 discusses behavioral implications of the theory and Section 6 investigates potential experimental designs. Concluding remarks appear in Section 7 and the proofs are collected in Section 8.

2. Analytical framework

2.1. Preference structure

For $n \geq 3$, let $X = \{x_1, \ldots, x_n\}$ be a set of outcomes,[5] and $\Delta(X) = \{p \in \mathbb{R}^n_+ : \sum_{x \in X} p(x) = 1\}$ the set of lotteries over X. Let $\succ$ be a binary relation on $\Delta(X)$, which we refer to as a *strict preference relation*. The preference relation $\succ$ is said to be bounded if there are best and worst outcomes $\overline{x}, \underline{x} \in X$ such that $\delta_{\overline{x}} \succ p \succ \delta_{\underline{x}}$, for all $p \in \Delta(X) \setminus \{\delta_{\overline{x}}, \delta_{\underline{x}}\}$, which we assume throughout.[6] Number the elements in X in nondecreasing order of preference, so that $\underline{x} = x_1$ and $\overline{x} = x_n$.

If the strict preference relation $\succ$ is negatively transitive, then its negation $\neg(p \succ q)$ defines the complete and transitive weak preference relation $p \preccurlyeq q$. The multiplicity of the utility representation thus depends on this assumption being relaxed. Defining the *incomparability relation* $p \asymp q$ as the conjunction of $\neg(p \succ q)$ and $\neg(p \prec q)$, we obtain a relation that is not necessarily transitive and thus not necessarily an equivalence relation, unlike the indifference relation this would normally define under completeness.[7] Intuitively, the inability to rank a pair of alternatives does not necessarily mean that the decision maker considers them to be equivalent, but rather may imply that she evaluates them by multiple criteria that disagree on their ranking.

We assume throughout that $\succ$ is a continuous strict partial order.

(A.1) **(Strict Partial Order)** The preference relation $\succ$ is irreflexive and transitive.

The following axiom is a slight strengthening of the usual Archimedean axiom, disposing with the requirement that a strict ranking $p \succ q \succ r$ exists, and admitting the possibility that two of these lotteries may be incomparable instead.

(A.2) **(Strong Archimedean)** For all $p, q, r \in \Delta(X)$ if $p \succ q$ then there is $\alpha \in (0, 1)$ such that $\alpha p + (1-\alpha)r \succ q$ and if $p \prec q$, there is $\alpha' \in (0, 1)$ such that $\alpha' p + (1-\alpha')r \prec q$.

The next axiom asserts that a probability mixture of two lotteries must be ranked between them. It characterizes a class of models including expected, weighted, and implicit weighted utility theory.[8]

(A.3) **(Betweenness)** For all $p, q \in \Delta(X)$ and $\alpha \in (0, 1)$, $p \succ q$ if and only if $p \succ \alpha p + (1-\alpha)q \succ q$.

For every $\alpha \in [0, 1]$, let $\zeta_\alpha := \alpha\delta_{x_n} + (1-\alpha)\delta_{x_1}$, then if $p \asymp \zeta_\alpha$, then we can interpret α as a utility value that may be assigned to p. For every $p \in \Delta(X)$, let $A(p) = \{\alpha \in [0, 1] : p \asymp \zeta_\alpha\}$ denote the range of utility values assigned to p, measured along the line connecting the best and worst outcomes. The following proposition establishes that each of these utility ranges is a closed interval.

Proposition 1. *For all $p \in \Delta(X)$, there are $\underline{\alpha}, \overline{\alpha} \in [0, 1]$ such that $A(p) = [\underline{\alpha}, \overline{\alpha}]$.*

In the standard expected utility and multi-utility models, applying the independence axiom at this step produces the desired representations.

2.2. Partial substitution

At the core of weighted utility theory is the weak substitution axiom that replaces the independence axiom.[9]

(Weak Substitution) For all $p, q \in \Delta(X)$, $p \sim q$ if and only if for every $\beta \in (0, 1)$ there is $\gamma \in (0, 1)$ such that $\beta p + (1-\beta)r \sim \gamma q + (1-\gamma)r$ for all $r \in \Delta(X)$.

The weak substitution axiom can be equivalently expressed as a ratio substitution property.

(Ratio Substitution) For all $p, q \in \Delta(X)$, $p \sim q$ if and only if there is $\tau > 0$ such that for every $\beta \in (0, 1)$, $\beta p + (1-\beta)r \sim \frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)}$ for all $r \in \Delta(X)$.

That these are equivalent can be shown by setting $\tau = \frac{\gamma/(1-\gamma)}{\beta/(1-\beta)}$. This odds ratio is interpreted as the weight of p relative to that of q. If $\succ$ is complete, a weighted linear utility function can thus be obtained by finding, for each $x_i \in X$, the unique α_i such that $\delta_{x_i} \sim \zeta_{\alpha_i}$, and τ_i satisfying ratio substitution between these two lotteries, and for any $p \in \Delta(X)$ repeatedly applying weak substitution to obtain

$$p := \sum_{i=1}^{n} p_i \delta_{x_i} \sim \frac{p_1\tau_1\zeta_{\alpha_1} + \sum_{i=2}^{n} p_i\delta_{x_i}}{p_1\tau_1 + \sum_{i=2}^{n} p_i} \sim \cdots \sim \frac{\sum_{i=1}^{n} p_i\tau_i\zeta_{\alpha_i}}{\sum_{i=1}^{n} p_i\tau_i} = \zeta_{\frac{\sum_{i=1}^{n} p_i\tau_i\alpha_i}{\sum_{i=1}^{n} p_i\tau_i}} := \zeta_{\alpha_p}. \quad (5)$$

5 If $n = 1$ then X is a singleton and there is no decision to be made, and if $n = 2$ we will only have best and worst outcomes, so that the entire space of lotteries will be strictly ranked.

6 Generally speaking, $\succ$ is bounded if there are $\overline{p}, \underline{p} \in \Delta(X)$ such that $\overline{p} \succ p \succ \underline{p}$ for all $p \in \Delta(X) \setminus \{\overline{p}, \underline{p}\}$. However, anticipating the monotonicity of the strict preference relation described below, there is no essential loss in our definition.

7 We may still define weak preference and indifference relations that have the usual properties by following Galaabaatar and Karni (2013) and letting $p \succcurlyeq q$ if $r \succ p$ implies $r \succ q$, and $p \sim q$ if $p \succcurlyeq q$ and $p \preccurlyeq q$.

8 See Dekel (1986) for an example and Chew (1989) for a review of this class of models.

9 See Chew (1989).

By betweenness, the above implies that for any $p, q \in \Delta(X)$, $p \succ q \Leftrightarrow \alpha_p > \alpha_q$, so that we obtain a weighted utility representation by setting $u(x_i) = \alpha_i$ and $w(x_i) = \tau_i$ for $i = 1, \ldots, n$. The critical step in this construction lies in exploiting the transitivity of the indifference relation $\sim$.

For preferences $\succ$ that are not necessarily complete, we consider a modification that replaces the indifference relation $\sim$ with the incomparability relation $\asymp$.

(A.4) (Partial Substitution) For all $p, q \in \Delta(X)$, $p \asymp q$ if and only if for every $\beta \in (0,1)$ there is $\gamma \in (0,1)$ such that $\beta p + (1-\beta)r \asymp \gamma q + (1-\gamma)r$, for all $r \in \Delta(X)$.

It is noteworthy that if $\succ$ satisfies betweenness (A.3) and partial substitution (A.4) then a betweenness property holds for the incomparability relation $\asymp$.[10] The next lemma establishes the analogous ratio substitution property in our setup.

Lemma 1. *If $\succ$ satisfies (A.1)–(A.4) then, for all $p, q \in \Delta(X)$, $p \asymp q$ if and only if there is $\tau > 0$ such that for every $\beta \in (0,1)$, $\beta p + (1-\beta)r \asymp \frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)}$, for all $r \in \Delta(X)$.*

For any pair of incomparable lotteries, define the set of substitution odds ratios as

$$T(p,q) = \left\{ \tau > 0 : \beta p + (1-\beta)r \asymp \frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)}, \right.$$
$$\left. \forall \beta \in (0,1), r \in \Delta(X) \right\}. \tag{6}$$

By Lemma 1, $T(p,q) \neq \varnothing$ if and only if $p \asymp q$. Under completeness, weak substitution implies that for every $\beta \in (0,1)$ we have a unique $\gamma \in (0,1)$, which can be seen by picking any $r \succ q$ and applying betweenness. Therefore the odds ratio τ must be unique as well. This is not the case here, however, as $\asymp$ is intransitive and hence $T(p,q)$ is not necessarily a singleton. Consequently, lotteries may have a range of weights in addition to a range of utility values.

Proposition 2. *For all $p, q \in \Delta(X)$ such that $p \asymp q$, there are $\underline{\tau}, \overline{\tau} > 0$ such that $T(p,q) = [\underline{\tau}, \overline{\tau}]$.*

We can now attempt to replicate the construction of the utility representation as in (5). For every $i = 1, \ldots, n$, consider picking some $\alpha_i \in A(\delta_{x_i})$ and $\tau_i \in T(\delta_{x_i}, \zeta_{\alpha_i})$, and then repeatedly applying partial substitution to yield

$$p := \sum_{i=1}^{n} p_i \delta_{x_i} \asymp \frac{p_1\tau_1\zeta_{\alpha_1} + \sum_{i=2}^{n} p_i\delta_{x_i}}{p_1\tau_1 + \sum_{i=2}^{n} p_i} \asymp \cdots \asymp \frac{\sum_{i=1}^{n} p_i\tau_i\zeta_{\alpha_i}}{\sum_{i=1}^{n} p_i\tau_i}$$
$$= \zeta_{\frac{\sum_{i=1}^{n} p_i\tau_i\alpha_i}{\sum_{i=1}^{n} p_i\tau_i}} := \zeta_{\alpha_p}. \tag{7}$$

However, as $\asymp$ is intransitive, (7) does not necessarily imply that $p \asymp \zeta_{\alpha_p}$. Intuitively, if $\succ$ is negatively transitive, then every α_i and τ_i is unique, so that we can obtain for any p the unique α_p by simply taking the weighted convex combination as in (5). Under incompleteness, while we know that each $x_i \in X$ has utility range $A(\delta_{x_i})$, if we arbitrarily select $\{\alpha_1, \ldots, \alpha_n\} \in \prod_{i=1}^{n} A(\delta_{x_i})$, these values need not be assigned by the same utility function, and hence the α_p produced by (7) need not belong to $A(p)$. It is the converse, that every $\alpha_p \in A(p)$ can be constructed from some $\{\alpha_1, \ldots, \alpha_n\} \in \prod_{i=1}^{n} A(\delta_{x_i})$, that we need to show to ensure that the preference relation can be represented by a set of weighted linear utility functions.

Intuitively, (A.4) states that, for any pair of incomparable lotteries $p \asymp q$, there is some odds ratio $\tau \geq 0$ that represents the relative weight of p to that of q, although this is now not necessarily unique which allows for the representation to be multi-weight as well as multi-utility. The axiom requires that the decision maker applies this weighting consistently throughout the space of lotteries, ensuring that any non-linearity is indeed due to weighting outcomes differently rather than some other explanation such as disappointment aversion or rank dependence. Specifically, given two incomparable lotteries $p \asymp q$, the set of odds ratios $T(p,q)$ cannot depend on the third lottery r that they are mixed with. Therefore it cannot be the case for $r, s \in \Delta(X)$ that every $\tau \geq 0$ such that for every $\beta \in (0,1)$, $\beta p + (1-\beta)r \asymp \frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)}$, that there is some $\beta \in (0,1)$ such that $\beta p + (1-\beta)r \succ \frac{\beta\tau q + (1-\beta)s}{\beta\tau + (1-\beta)}$ or $\beta p + (1-\beta)r \prec \frac{\beta\tau q + (1-\beta)s}{\beta\tau + (1-\beta)}$. That is, the relative importance that the decision maker attaches to p over q cannot depend on whether they are being mixed with r or s. In Section 6 we describe an experiment design to test partial substitution axiom.

2.3. Source space

As discussed in the introduction, a preference relation with a multiple weighted expected utility representation can be visualized as a set of lotteries with incomparability curves projected from a set of source points lying outside the simplex. Suppose we have $p, q \in \Delta(X)$ such that $p \asymp q$, and some $\tau \in T(p,q)$. By definition, for every $\beta \in (0,1)$ and $r \in \Delta(X)$, the line defined by $\beta p + (1-\beta)r$ and $\frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)}$ is an incomparability curve. All of these curves converge at some source point o. As its location can depend on neither β nor r, we have that

$$\begin{aligned} o &= \frac{1}{\beta(1-\tau)}[\beta p + (1-\beta)r] \\ &\quad - \frac{\beta\tau + (1-\beta)}{\beta(1-\tau)}\left[\frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)}\right] \\ &= \frac{p - \tau q}{1-\tau}. \end{aligned} \tag{8}$$

Define the source space Ω as the collection of all such source points,

$$\Omega = \left\{ o = \frac{p - \tau q}{1-\tau} : p \asymp q, \tau \in T(p,q) \right\}. \tag{9}$$

The following proposition asserts that Ω fully characterizes the incomparability relation $\asymp$ and, consequently, the preference relation $\succ$ as well. It states that any line connecting two lotteries is an incomparability curve if it is projected from some source point $o \in \Omega$.

Proposition 3. *For every $p, q \in \Delta(X)$, $p \asymp q$ if there is $\tau > 0$ such that $o = \frac{p-\tau q}{1-\tau} \in \Omega$.*

For each p define $\Phi(p) = \{(\alpha_p, \tau_p) : \alpha_p \in A(p), \tau_p \in T(p, \zeta_{\alpha_p})\}$ as the collection of utility-weight pairs, each defining a source point $o_p = \frac{p - \tau_p\zeta_{\alpha_p}}{1-\tau_p} \in \Omega$. A utility function over X is given by a collection of utility weight pairs $\{(\alpha_i, \tau_i)\}_{i=1}^{n}$ corresponding to each of the degenerate lotteries $\{\delta_{x_i}\}_{i=1}^{n}$ such that $(\alpha_p, \tau_p) = \left(\frac{\sum_{i=1}^{n} p_i\tau_i\alpha_i}{\sum_{i=1}^{n} p_i\tau_i}, \sum_{i=1}^{n} p_i\tau_i\right) \in \Phi(p)$, for any lottery $p \in \Delta(X)$. Define

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10 To see this, suppose $p \asymp q$ then, by (A.4) and letting $r = q$, for every $\beta \in (0,1)$ there is $\gamma \in (0,1)$ such that $\beta p + (1-\beta)q \asymp \gamma q + (1-\gamma)q = q$. Likewise, as $q \asymp p$, let $r = p$ then for every $\beta \in (0,1)$ there is $\gamma \in (0,1)$ such that $\beta p + (1-\beta)q \asymp \gamma p + (1-\gamma)p = p$. Hence for every $\beta \in (0,1)$, we have that $p \asymp \beta p + (1-\beta)q \asymp q$. Suppose next that $\neg(p \asymp q)$, then by (A.3) either $p \succ \beta p + (1-\beta)q \succ q$ or $p \prec \beta p + (1-\beta)q \prec q$, for every $\beta \in (0,1)$. Hence, that $p \asymp q$ if and only if $p \asymp \beta p + (1-\beta)q \asymp q$ for every $\beta \in (0,1)$.

the set of all such collections as

$$\Psi = \left\{ \{(\alpha_i, \tau_i)\}_{i=1}^n : (\alpha_p, \tau_p) = \left(\frac{\sum_{i=1}^n p_i \tau_i \alpha_i}{\sum_{i=1}^n p_i \tau_i}, \sum_{i=1}^n p_i \tau_i \right) \in \Phi(p), \forall p \in \Delta(X) \right\}. \tag{10}$$

Every $\psi = \{(\alpha_i^\psi, \tau_i^\psi)\}_{i=1}^n \in \Psi$ defines a weighted linear utility function. Letting $o_i^\psi = \frac{\delta_{x_i} - \tau_i^\psi \zeta_{\alpha_i^\psi}}{1-\tau_i^\psi} \in \Omega$ denote the source point that ψ associates with outcome x_i, we have that for every $p \in \Delta(X)$,

$$o_p^\psi = \frac{p - \tau_p^\psi \zeta_{\alpha_p^\psi}}{1 - \tau_p^\psi} = \frac{\sum_{i=1}^n p_i (1 - \tau_i^\psi) o_i^\psi}{\sum_{i=1}^n p_i (1 - \tau_i^\psi)} \in \Omega. \tag{11}$$

That is, each source point associated with any lottery p is also a weighted convex combination of elements of $\{o_i^\psi\}_{i=1}^n$.[11] Collecting these points forms a subset of the source space, $O^\psi = \{o_p^\psi\}_{p \in \Delta(X)} \subseteq \Omega$ which characterizes a function pair (u^ψ, w^ψ). Therefore, to establish the representation theorem, we need to show that the collection of these subsets covers the source space $\bigcup_{\psi \in \Psi} O^\psi = \Omega$. In other words, the collection of function pairs Ψ fully characterizes the incomparability map and, by extension, the preference relation $\succ$ itself.

3. Representation

3.1. Existence

Before presenting the main theorem, we first establish some preliminary results. For any collection $P \subseteq \Delta(X)$, let $\mathcal{L}(P) = \{q = \sum_{p \in P} \pi_p p : \pi_p \in \mathbb{R}, \forall p \in P\}$ be the linear manifold spanned by the lotteries in P,[12] and define $\Delta(P) = \mathcal{L}(P) \cap \Delta(X)$. Any collection $P \subseteq \Delta(X)$ of lotteries constitutes an *incomparability set* if $p \asymp q$ for any $p, q \in \Delta(P)$, thus forming the natural higher dimensional analogue to the incomparability curves encountered so far. As $\delta_{x_n} \succ \delta_{x_1}$ implies that $\succ$ is non-empty, even the maximal incomparability set cannot span the entire simplex, so that $\Delta(P) \subsetneq \Delta(X)$ and is at most of dimension $n-2$.[13]

Suppose we start with any single lottery $p \in \Delta(X) \setminus \{\delta_{x_n}, \delta_{x_1}\}$, then we can find some other lottery q such that $p \asymp q$. Applying the partial substitution axiom, this relation implies the existence of a set of incomparability curves converging at a source point o, which is collinear with p and q. If $p \asymp q$, then the line defined by this pair of lotteries $\Delta(\{p, q\})$ is an incomparability curve originating from o. If $n = 3$, then each source point o characterizes a weighted utility function and we can construct our representation by locating all of them. However, if $n > 3$ then constructing a weighted utility requires stringing together a number of source points into a structure that projects higher dimensional incomparability sets. Lemma 2 shows that if $p \asymp q$ and $n > 3$, we can find some $r \in \Delta(X) \setminus \Delta(\{p, q\})$ that is incomparable to both p and q as well as any lottery in $\Delta(\{p, q\})$, so that $\Delta(\{p, q, r\})$ defines an incomparability plane.

Lemma 2. *If $\succ$ satisfies (A.1)–(A.4) and $n > 3$ then, for all $p, q \in \Delta(X)$, the following statements are equivalent:*

(i) $p \asymp q$.
(ii) There exists $r \in \Delta(X) \setminus \Delta(\{p, q\})$ such that $\lambda p + (1-\lambda) q \asymp r$ for all $\lambda \in \mathbb{R}$ such that $\lambda p + (1-\lambda) q \in \Delta(\{p, q\})$.
(iii) There exists $r \in \Delta(X) \setminus \Delta(\{p, q\})$ and $\tau_p, \tau_q > 0$ such that for all $\lambda \in \mathbb{R}$ such that $\lambda p + (1-\lambda) q \in \Delta(\{p, q\})$ and $\beta \in (0, 1)$, $\beta[\lambda p + (1-\lambda) q] + (1-\beta) s \asymp \frac{\beta[\lambda \tau_p + (1-\lambda)\tau_q] r + (1-\beta) s}{\beta[\lambda \tau_p + (1-\lambda)\tau_q] + (1-\beta)}$, for all $s \in \Delta(X)$.
(iv) There exists $r \in \Delta(X) \setminus \Delta(\{p, q\})$ such that $p' \asymp q'$, for all $p', q' \in \Delta(\{p, q, r\})$.

By Lemma 2, every source point that projects a set of incomparability curves itself lies on a line on which every point is a source point. Such a source line in turn projects a set of incomparability planes. The natural next step is to generalize this property, allowing us to construct a set of source points that will fully characterize a utility function.

Lemma 3. *If $\succ$ satisfies (A.1)–(A.4), then for all $P \subseteq \Delta(X)$ such that* $\dim \Delta(P) < n - 2$*, the following statements are equivalent:*

(i) $p \asymp q$ for all $p, q \in \Delta(P)$.
(ii) There exists $r \in \Delta(X) \setminus \Delta(P)$ such that $p \asymp r$, for every $p \in \Delta(P)$.
(iii) There exists $r \in \Delta(X) \setminus \Delta(P)$ and $\{\tau_p\}_{p \in P} \subseteq \mathbb{R}_{++}$ such that for all $\{\pi_p\}_{p \in P} \subseteq \mathbb{R}_+$ such that $\sum_{p \in P} \pi_p = 1$, and $\beta \in (0, 1)$, $\beta \left(\sum_{p \in P} \pi_p p\right) + (1-\beta) s \asymp \frac{\beta\left(\sum_{p \in P} \pi_p \tau_p\right) r + (1-\beta) s}{\beta\left(\sum_{p \in P} \pi_p \tau_p\right) + (1-\beta)}$, for all $s \in \Delta(X)$.
(iv) There exists $r \in \Delta(X) \setminus \Delta(P)$ such that $p' \asymp q'$, for all $p', q' \in \Delta(P \cup \{r\})$.

Starting with any pair of incomparable lotteries, we can repeatedly apply Lemma 3 to add incomparable lotteries until, after a finite number of repetitions, we obtain a maximal incomparability set P of dimension $n-2$. The following lemma shows that every such P maps to some collection $\psi \in \Psi$.

Lemma 4. *If $\succ$ satisfies (A.1)–(A.4), then for all $p, q \in \Delta(X)$, $p \asymp q$ if and only if there is $\psi \in \Psi$ such that $\frac{\sum_{i=1}^n p_i \tau_i^\psi \alpha_i^\psi}{\sum_{i=1}^n p_i \tau_i^\psi} = \frac{\sum_{i=1}^n q_i \tau_i^\psi \alpha_i^\psi}{\sum_{i=1}^n q_i \tau_i^\psi}$.*

Lemma 4 establishes $\Psi \neq \varnothing$ as long as the incomparability relation is itself nonempty. Furthermore, this result is only necessary to establish the representation in the case where $n > 3$.

11 By definition of ψ, we have that

$$\begin{aligned} o_p^\psi &= \frac{p - \tau_p^\psi \zeta_{\alpha_p^\psi}}{1 - \tau_p^\psi} \\ &= \frac{\sum_{i=1}^n p_i \delta_{x_i} - (\sum_{i=1}^n p_i \tau_i^\psi) \zeta_{\frac{\sum_{i=1}^n p_i \tau_i^\psi \alpha_i^\psi}{\sum_{i=1}^n p_i \tau_i^\psi}}}{1 - (\sum_{i=1}^n p_i \tau_i^\psi)} \\ &= \frac{\sum_{i=1}^n p_i \delta_{x_i} - (\sum_{i=1}^n p_i \tau_i^\psi) \left(\frac{\sum_{i=1}^n p_i \tau_i^\psi \zeta_{\alpha_i^\psi}}{\sum_{i=1}^n p_i \tau_i^\psi} \right)}{1 - (\sum_{i=1}^n p_i \tau_i^\psi)} \\ &= \frac{\sum_{i=1}^n p_i (\delta_{x_i} - \tau_i^\psi \zeta_{\alpha_i^\psi})}{\sum_{i=1}^n p_i (1 - \tau_i^\psi)} \\ &= \frac{\sum_{i=1}^n p_i (1 - \tau_i^\psi) \left(\frac{\delta_{x_i} - \tau_i^\psi \zeta_{\alpha_i^\psi}}{1 - \tau_i^\psi} \right)}{\sum_{i=1}^n p_i (1 - \tau_i^\psi)} \\ &= \frac{\sum_{i=1}^n p_i (1 - \tau_i^\psi) o_i^\psi}{\sum_{i=1}^n p_i (1 - \tau_i^\psi)} \in \Omega. \end{aligned}$$

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12 Recall that $p = (p(x_1), \ldots, p(x_n)) \in \mathbb{R}^n$.

13 As $\sum_{i=1}^n p(x_i) = 1$ for every $p \in \Delta(X)$, $\dim \Delta(X) = n - 1$, and if $\Delta(P) \subsetneq \Delta(X)$, $\dim \Delta(P) < \dim \Delta(X)$.

However, if $n = 3$, then the result still holds since $\Phi(\delta_{x_1})$ and $\Phi(\delta_{x_3})$ are singletons by construction and therefore every $\psi \in \Psi$ will assign $\alpha_1^\psi = 0$, $\alpha_3^\psi = 1$, and $\tau_1^\psi = \tau_3^\psi = 1$, so that each $(\alpha_2, \tau_2) \in \Phi(\delta_{x_2})$ maps to a weighted utility $\psi \in \Psi$. Thus constructing the set of utilities from Ψ, any two lotteries are incomparable if and only if there is some weighted utility that assigns them equal value, which leads directly into our central result.

Lemma 4 also implies that whenever p and q are incomparable there exists $\psi \in \Psi$ that assigns the same weighted utility to both. Consequently the elements of Ψ collectively gives us the representation in Theorem 1.

Theorem 1. *A binary relation $\succ$ over $\Delta(X)$ is bounded and satisfies (A.1)–(A.4) if and only if there is a closed and bounded set $\mathcal{V}$ of utility $u : X \mapsto \mathbb{R}$ and weight $w : X \mapsto \mathbb{R}_{++}$ function pairs (u, w) such that for every $p \in \Delta(X) \setminus \{\delta_{x_n}, \delta_{x_1}\}$,*

$$u(x_n) > \frac{\sum_{x\in X} p(x)w(x)u(x)}{\sum_{x\in X} p(x)w(x)} > u(x_1), \ \forall (u, w) \in \mathcal{V}. \tag{12}$$

And for every $p, q \in \Delta(X)$,

$$p \succ q \iff \frac{\sum_{x\in X} p(x)w(x)u(x)}{\sum_{x\in X} p(x)w(x)} > \frac{\sum_{x\in X} q(x)w(x)u(x)}{\sum_{x\in X} q(x)w(x)}, \quad \forall (u, w) \in \mathcal{V}. \tag{13}$$

Moreover the set of utilities $\mathcal{U} = \{u : (u, w) \in \mathcal{V}\}$ and weights $\mathcal{W} = \{w : (u, w) \in \mathcal{V}\}$ are closed and bounded.

3.2. Uniqueness

Having established the existence of a utility representation, we now turn our attention to the question of uniqueness. As our model lies at the convergence of weighted utility and expected multi-utility theory, our uniqueness result naturally incorporates elements of the uniqueness results found in both. From weighted utility theory, we know that taking an affine transformation of a utility function u will not preserve its weighted linearity, and we must instead apply a rational affine transformation to both u and the associated weight function w jointly.

Proposition 4. *For utility and weight function (u, w) and constants a, b, c, d such that $ad > bc$, define the rational affine transformation $(\tilde{u}, \tilde{w}) = \left(\frac{au+b}{cu+d}, w[cu + d]\right)$. Then for every $p, q \in \Delta(X)$,*

$$\frac{\sum_{x\in X} p(x)w(x)u(x)}{\sum_{x\in X} p(x)w(x)} > \frac{\sum_{x\in X} q(x)w(x)u(x)}{\sum_{x\in X} q(x)w(x)} \iff \frac{\sum_{x\in X} p(x)\tilde{w}(x)\tilde{u}(x)}{\sum_{x\in X} p(x)\tilde{w}(x)} > \frac{\sum_{x\in X} q(x)\tilde{w}(x)\tilde{u}(x)}{\sum_{x\in X} q(x)\tilde{w}(x)}. \tag{14}$$

The utility functions we construct from Ψ are normalized, with $(u^\psi(x_1), w^\psi(x_1)) = (0, 1)$ for the worst outcome and $(u^\psi(x_n), w^\psi(x_n)) = (1, 1)$ for the best, for every $\psi \in \Psi$. The following proposition shows that any function pair (u, w) has a rational affine transformation $(\hat{u}, \hat{w})$ that is similarly normalized, which in turn maps to some collection $\psi \in \Psi$.

Proposition 5. *Let the collection $\mathcal{V}$ represent $\succ$. Then for every $(u, w) \in \mathcal{V}$ there is a normalized rational affine transformation $(\hat{u}, \hat{w})$ such that $\{(\hat{u}(x_i), \hat{w}(x_i))\}_{i=1}^n \in \Psi$.*

For any $\mathcal{V}$, define the normalized set as $\hat{\mathcal{V}} = \{(\hat{u}, \hat{w}) : (u, w) \in \mathcal{V}\}$. In expected multi-utility theory, if we have a set $\hat{\mathcal{U}}$ of normalized utilities, then the preferences it represents are identical to those represented by taking the closure of the convex cone spanned by $\langle\hat{\mathcal{U}}\rangle$ and all the constant functions, (see Dubra et al. (2004)), as the incomparability curves generated by taking a linear combination $\lambda u_1 + (1 - \lambda)u_2$ of utilities $u_1, u_2 \in \hat{\mathcal{U}}$ will lie between those corresponding to u_1 and u_2 and will not alter the preference ordering.

A similar idea exists here, as for example in Fig. 4 the representation is identical whether we take as our source space only the four points $\{o_{11}, o_{12}, o_{21}, o_{22}\}$ or the entire shaded area Ω, as any incomparability curves projected from the interior of Ω will not form the boundary of the upper or lower contour sets $B(p)$ and $W(p)$ for any $p \in \Delta(X)$. However, we cannot speak of a convex hull as such because we cannot simply take convex combinations of weighted linear functions that simultaneously preserve weighted linearity while maintaining the preference ordering.[14]

To overcome this difficulty, we focus on a neighborhood around some $p \in \Delta(X)$ and invoke the notion of local utility functions. For every $(u^k, w^k) \in \hat{\mathcal{V}}$, let $\bar{u}_p^k := \frac{\sum_{x\in X} p(x)w^k(x)u^k(x)}{\sum_{x\in X} p(x)w^k(x)}$ and $\bar{w}_p^k := \sum_{x\in X} p(x)w^k(x)$, and, following Chew and Nishimura (1992), define the local utility function $\hat{u}_p^k : X \mapsto \mathbb{R}$ induced by the weighted utility functional[15] as

$$\hat{u}_p^k := \frac{w^k(u^k - \bar{u}_p^k)}{\bar{w}_p^k}. \tag{15}$$

As linear utilities are invariant up to a positive affine transformation, let

$$u_p^k = \bar{w}_p^k \hat{u}_p^k + \bar{u}_p^k = w^k u^k + (1 - w^k)\bar{u}_p^k. \tag{16}$$

Then u_p^k is a normalized linear approximation of (u^k, w^k) around p, so that taking the expectation of u_p^k at p gives $\bar{u}_p^k$.[16]

Collecting all such u_p^k gives a set $\hat{\mathcal{U}}_p = \{u_p^k : (u^k, w^k) \in \hat{\mathcal{V}}\}$ of local utility functions that form a local multi-utility representation at p, that is,

$$p \succ q \iff \sum_{x\in X} p(x)u_p(x) > \sum_{x\in X} q(x)u_p(x), \ \forall u_p \in \hat{\mathcal{U}}_p. \tag{17}$$

As betweenness ensures the incomparability curves extend linearly in every direction, the local utility representation is valid

14 Given $(u^1, w^1), (u^2, w^2) \in \hat{\mathcal{V}}$ and $\lambda \in (0, 1)$, we could take a direct convex combination by setting $u^\lambda = \lambda u^1 + (1 - \lambda)u^2$ and $w^\lambda = \lambda w^1 + (1 - \lambda)w^2$. Unless $w^1 = w^2$, this would not preserve weighted linearity. Alternatively, we could take the weighted convex combination and set $u^\lambda = \frac{\lambda w^1 u^1 + (1-\lambda)w^2 u^2}{\lambda w^1 + (1-\lambda)w^2}$ and $w^\lambda = \lambda w^1 + (1-\lambda)w^2$. This however would not necessarily preserve the ordering of lotteries, as $u^k(x_i) > u^k(x_j)$ for $k = 1, 2$ does not imply $u^\lambda(x_i) > u^\lambda(x_j)$.

15 To grasp this, observe that since $\bar{u}_p^k$ is Gateaux differentiable, invoking the analysis of Machina (1982) we have

$$\left.\frac{d}{d\alpha}\bar{u}_p^k((1-\alpha)p + \alpha q)\right|_{\alpha=0} = \frac{\sum_{x\in X} [q(x) - p(x)]\, w^k(x) \left[u^k(x) - \bar{u}_p^k\right]}{\sum_{x\in X} p(x)\, w^k(x)}.$$

Then the local utility function at p is $\hat{u}_p^k$..

16 Since $w^k(x_1) = w^k(x_n) = 1$, we have that $u_p^k(x_1) = u^k(x_1) = 0$ and $u_p^k(x_n) = u^k(x_n) = 1$. Furthermore,

$$\sum_{x\in X} p(x)u_p^k(x) = \sum_{x\in X} p(x)w^k(x)u^k(x) + \left[1 - \sum_{x\in X} p(x)w^k(x)\right] \times \left[\frac{\sum_{x\in X} p(x)w^k(x)u^k(x)}{\sum_{x\in X} p(x)w^k(x)}\right] = \frac{\sum_{x\in X} p(x)w^k(x)u^k(x)}{\sum_{x\in X} p(x)w^k(x)}.$$

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for any $q \in \Delta(X)$, not just in an ϵ-neighborhood of p. Define $\langle \hat{\mathcal{U}}_p \rangle$ as the closure of the convex cone spanned by the set $\hat{\mathcal{U}}_p$ and the constant function.[17] As $\hat{\mathcal{U}}_p$ consists of linear utility functions, it is easy to verify that $\langle \hat{\mathcal{U}}_p \rangle$ is an identical local multi-utility representation at p.

Proposition 6. *For every $p \in \Delta(X)$ there is a closed convex set $\langle \hat{\mathcal{U}}_p \rangle$ of utilities $u_p : X \mapsto \mathbb{R}$ such that for every $q \in \Delta(X)$,*

$$p \succ q \Longleftrightarrow \sum_{x \in X} p(x) u_p(x) > \sum_{x \in X} q(x) u_p(x), \ \forall\, u_p \in \langle \hat{\mathcal{U}}_p \rangle. \tag{18}$$

As each $\langle \hat{\mathcal{U}}_p \rangle$ is a local multi-utility representation of $\succ$ at p, we can define the set of utility and weight pairs (u, w) that map to some $u_p \in \langle \hat{\mathcal{U}}_p \rangle$ around every p as $\langle \hat{\mathcal{V}} \rangle = \{(u^k, w^k) : u^k_p \in \langle \hat{\mathcal{U}}_p \rangle, \forall\, p \in \Delta(X)\}$. We refer to $\langle \hat{\mathcal{V}} \rangle$ as the exhaustive *set of normalized utility and weight pairs that agree everywhere with the ordering of lotteries prescribed by $\succ$. The uniqueness theorem presented below asserts that two utility representations are identical if and only if they are characterized by identical exhaustive sets.*

Theorem 2. *For $j = 1, 2$, let $\succ^j$ be a binary relation over $\Delta(X)$ that has a multiple weighted expected utility representation by a set $\mathcal{V}^j$ of utility $u : X \mapsto \mathbb{R}$ and weight $w : X \mapsto \mathbb{R}_{++}$ function pairs (u, w). The preference relations $\succ^1$ and $\succ^2$ are identical if and only if $\langle \hat{\mathcal{V}}^1 \rangle = \langle \hat{\mathcal{V}}^2 \rangle$.*

Fig. 5 illustrates the uniqueness theorem.

In Fig. 5(a), the two utility representations $\mathcal{V}^1$ and $\mathcal{V}^2$ have identical exhaustive sets $\langle \hat{\mathcal{V}}^1 \rangle = \langle \hat{\mathcal{V}}^2 \rangle$. This is demonstrated in the diagram by the corresponding source spaces Ω^1 in and Ω^2 having identical weighted convex hulls[18] so that the preferences they represent are everywhere identical $\succ^1 = \succ^2$. Projecting incomparability curves from source points $o^1 = \Omega^1 \setminus \Omega^2$ or $o^2 = \Omega^2 \setminus \Omega^1$ does not alter the ranking. On the other hand, in Fig. 5(b) the two representations are not identical, as the weighted convex hulls of the corresponding source spaces overlap only partially, and hence $\langle \hat{\mathcal{V}}^1 \rangle \neq \langle \hat{\mathcal{V}}^2 \rangle$ and $\succ^1 \neq \succ^2$. Indeed we can observe that $p \prec^1 q$ and $p \asymp^1 r$, but $p \asymp^2 q$ and $p \succ^2 r$.

4. Special cases

Weighted utility theory with incomplete preferences admits incompleteness arising either from conflicting perceptions, represented by multiple weight functions, or from indecisive tastes, represented by multiple utility functions. Thus, the general framework we have devised admits a pair of special cases, those of multiple utilities paired with a single weight function $\mathcal{V} = \mathcal{U} \times \{w\}$ or a single utility paired with multiple weights $\mathcal{V} = \{u\} \times \mathcal{W}$. Each of these may be regarded as a partial completion of an incomplete preference relation.

In non-expected utility, local risk attitudes are captured by the local utility functions and global risk attitude depends on the variations of the local risk attitudes, as in Machina (1982). In weighted utility, the utility and weight functions play distinct roles, with the shape of the utility function capturing the decision maker's risk attitude while the weight function captures the nature and degree of the variation in local attitudes. Specifically, the weight function reflects the extent to which the indifference map exhibits the fanning in or fanning out structure described by Machina (1982).[19]

A decision maker with multiple utility functions and a single weight function has incomplete preferences solely due to her indecisive risk attitude, and has no more difficulty evaluating a lottery than she would evaluating each of its possible outcomes. On the other hand, a decision maker with a single utility function and multiple weight functions is sure of her risk attitude, but is indecisive when comparing lotteries because she is unsure of how to perceive randomness, and thus cannot always rank lotteries properly even if she knows how she would rank their components.

4.1. Multiple utilities

A decision maker whose preference relation is represented by multiple utilities paired with a single weight function is indecisive about the valuation of each of the outcomes in X, but is confident of how much importance to attach to these outcomes when evaluating any lottery $p \in \Delta(X)$. For example, the decision maker may have several utilities exhibiting varying degrees of risk aversion, but is sure of how much attention she should pay to each of the possible payoffs. To ensure that a preference relation $\succ$ has such a representation, we adopt a stronger variant of the partial substitution axiom.

(A.5) **(Parallel Substitution)** For all $p, q \in \Delta(X)$, $p \asymp q$ if and only if for every $\beta \in (0, 1)$ there is a unique $\gamma \in (0, 1)$, such that $\beta p + (1 - \beta) r \asymp \gamma q + (1 - \gamma) r$ for all $r \in \Delta(X)$.

Under this assumption, for every $p \asymp q$ there must be a unique substitution ratio $T(p, q) = \{\tau_{p,q}\}$. The following lemma shows that for every p we can pair a unique weight τ_p with any of the utility values $\alpha_p \in A(p)$.

Lemma 5. *If $\succ$ satisfies (A.1)–(A.3), (A.5) then, for all $p \in \Delta(X)$, there is $\tau_p > 0$ such that $\Phi(p) = A(p) \times \{\tau_p\}$.*

This result leads directly into the following representation theorem.

Theorem 3. *A binary relation $\succ$ over $\Delta(X)$ is bounded and satisfies (A.1)–(A.3), (A.5) if and only if there is a closed and bounded set $\mathcal{U}$ of utility functions $u : X \mapsto \mathbb{R}$ and a weight function $w : X \mapsto \mathbb{R}_{++}$ such that for every $p \in \Delta(X) \setminus \{\delta_{x_n}, \delta_{x_1}\}$,*

$$u(x_n) > \frac{\sum_{x \in X} p(x) w(x) u(x)}{\sum_{x \in X} p(x) w(x)} > u(x_1), \ \forall\, u \in \mathcal{U}. \tag{19}$$

And for every $p, q \in \Delta(X)$,

$$p \succ q \Longleftrightarrow \frac{\sum_{x \in X} p(x) w(x) u(x)}{\sum_{x \in X} p(x) w(x)} > \frac{\sum_{x \in X} q(x) w(x) u(x)}{\sum_{x \in X} q(x) w(x)}, \ \forall\, u \in \mathcal{U}. \tag{20}$$

This representation is unique up to the same transformation as in Theorem 2.

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[17] The closure is with respect to the $\mathbb{R}^n$ topology.

[18] Given two source points $o^1 = \frac{\delta_{x_2} - \tau^1 \zeta_{\alpha^1}}{1 - \tau^1}$ and $o^2 = \frac{\delta_{x_2} - \tau^2 \zeta_{\alpha^2}}{1 - \tau^2}$ and $\kappa \in [0, 1]$, take their weighted convex combination as

$$\begin{aligned} o^\kappa &= \frac{\kappa(1 - \tau^1) o^1 + (1 - \kappa)(1 - \tau^2) o^2}{\kappa(1 - \tau^1) + (1 - \kappa)(1 - \tau^2)} \\ &= \frac{\kappa(\delta_{x_2} - \tau^1 \zeta_{\alpha^1}) + (1 - \kappa)(\delta_{x_2} - \tau^2 \zeta_{\alpha^2})}{\kappa(1 - \tau^1) + (1 - \kappa)(1 - \tau^2)} \\ &= \frac{\delta_{x_2} - [\kappa \tau^1 + (1 - \kappa) \tau^2] \zeta_{\frac{\kappa \tau^1 \alpha^1 + (1 - \kappa) \tau^2 \alpha^2}{\kappa \tau^1 + (1 - \kappa) \tau^2}}}{1 - [\kappa \tau^1 + (1 - \kappa) \tau^2]}. \end{aligned}$$

We can then define the weighted convex hull as $\langle \Omega \rangle = \{o^\kappa : o^1, o^2 \in \Omega, \kappa \in [0, 1]\}$. Note that this operation is only meaningful for $n = 3$, as in higher dimensions elements of $\mathcal{V}$ would map to subsets rather than points of Ω.

[19] Fanning out reflects a decision maker who underweights the median outcome relative to the extremes, corresponding to monotonically increasing local risk aversion with respect to first order stochastic dominance, whereas fanning in reflects an overweight of the median outcome and hence decreasing local risk aversion with respect to first order stochastic dominance.

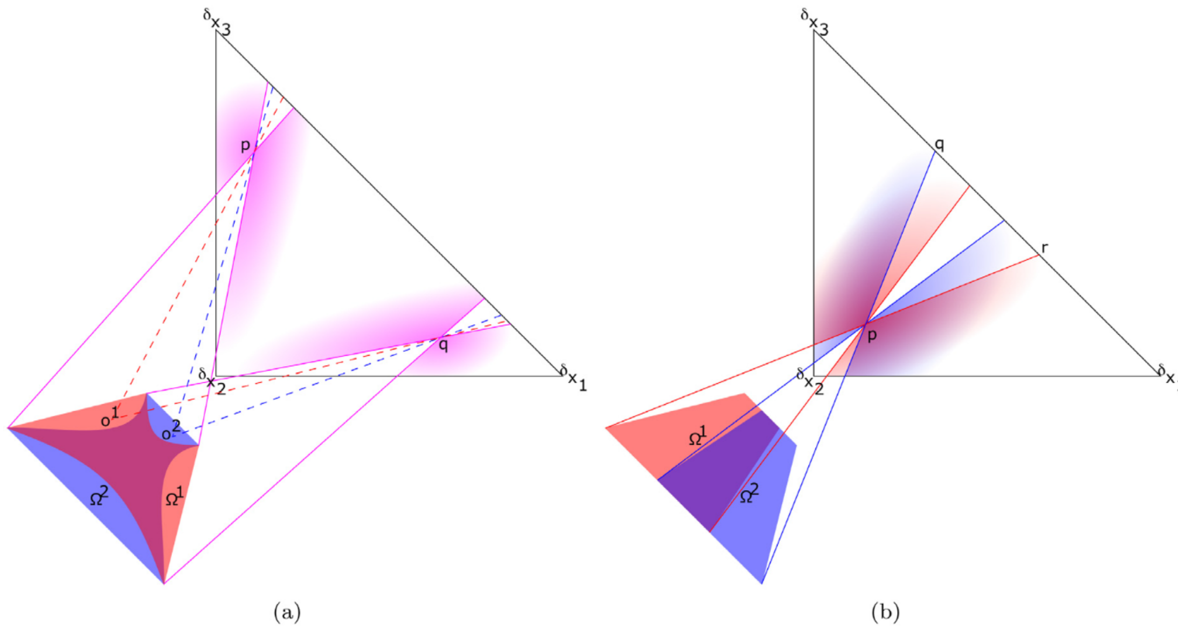

Fig. 5. Uniqueness.

4.2. Multiple weights

A decision maker whose preferences are represented by a single utility function paired with multiple weight functions is confident of how she would evaluate all of the outcomes in X, but is indecisive over how much importance each of these outcomes carries when evaluating a lottery $p \in \Delta(X)$. Such a decision maker is sure of her tastes, but when trying to compare alternative lotteries is unable to determine what aspects to focus on and attach more weight to. To ensure that a preference relation has such a representation, we impose the following axiom.

(A.6) **(Partial Completeness)** $\succ$ is negatively transitive over $\{\delta_x : x \in X\} \cup \{\zeta_\alpha : \alpha \in [0, 1]\}$.

It is immediate that (A.6) implies that for each $x \in X$ there is a unique $\alpha \in [0, 1]$ such that $\delta_x \asymp \zeta_\alpha$. Thus, every degenerate lottery has only a single utility value so that $A(\delta_{x_i}) = \{\alpha_i\}$ for every $i = 1, \ldots, n$, but may take multiple weight values so that $T(\delta_{x_i}, \zeta_{\alpha_i})$ need not be a singleton, and hence non-degenerate lotteries $p \in \Delta(X) \setminus X$ may still have multiple utility values. Imposing this assumption leads to a single utility, multiple weight representation.

Theorem 4. *A binary relation $\succ$ over $\Delta(X)$ is bounded and satisfies (A.1)–(A.4), (A.6) if and only if there is a utility function $u : X \mapsto \mathbb{R}$ and a closed and bounded set $\mathcal{W}$ of weight functions $w : X \mapsto \mathbb{R}_{++}$ such that for every $p \in \Delta(X) \setminus \{\delta_{x_n}, \delta_{x_1}\}$,*

$$u(x_n) > \frac{\sum_{x\in X} p(x)w(x)u(x)}{\sum_{x\in X} p(x)w(x)} > u(x_1), \ \forall w \in \mathcal{W}. \tag{21}$$

And for every $p, q \in \Delta(X)$,

$$p \succ q \Longleftrightarrow \frac{\sum_{x\in X} p(x)w(x)u(x)}{\sum_{x\in X} p(x)w(x)} > \frac{\sum_{x\in X} q(x)w(x)u(x)}{\sum_{x\in X} q(x)w(x)}, \ \forall w \in \mathcal{W}. \tag{22}$$

Again, this representation is unique up to the same transformation as in Theorem 2.

5. Behavioral implications

5.1. Inertia

To illustrate the differences between the two sources of indecisiveness, recall that the main empirical manifestation of incompleteness is *inertia*. Given an alternative a in some choice set, there is a range of non-comparable alternatives that will not be accepted if they were offered in exchange for a. In weighted utility theory with incomplete preferences, the nature of inertia depends on the source of indecisiveness. Specifically, if her indecisiveness is due to conflicting risk attitudes then the decision maker displays inertia everywhere. By contrast, if the source of her indecisiveness is incomplete perception then the decision maker displays inertia everywhere except at degenerate lotteries δ_x. These observations have testable implications. For example, the subject in an experiment may receive δ_x by default and be offered the opportunity to trade it for some lottery ζ_α. Using standard experimental methods it is possible to verify if the subject switches at a single α, thus indicating preferences that can be described by a multiple weight representation, or choose to hold on to δ_x over a range α, indicating that a multiple utility representation is more appropriate.

5.2. Portfolio selection

To further illustrate the applicability of multiple weighted utility models, consider a simple example of portfolio choice. Let there be two states $s = 1, 2$ whose probabilities are p and $1 - p$, respectively, and corresponding Arrow securities a_s each paying one dollar contingent on the realization of state s. Let r be the relative price of a_2 in terms of shares of a_1. Consider a weighted utility maximizing decision maker whose utility and weight functions are u and w respectively, and is endowed with z_0 shares of each security. Her problem is to choose the portfolio (z_1, z_2) to maximize $\frac{pw(z_1)u(z_1)+(1-p)w(z_2)u(z_2)}{pw(z_1)+(1-p)w(z_2)}$ subject to the budget constraint $z_1 + rz_2 = z_0 + rz_0$.

Denote by D_J the set of cumulative distribution functions with a compact support $J \subseteq \mathbb{R}$. A weighted utility maximizing decision maker evaluates the lottery $F \in D_J$ by the weighted utility functional $U(F) = \frac{\int_J u(t)w(t)\,dF(t)}{\int_J w(t)\,dF(t)}$ and attaches weight $W(F) = \int_J w(t)\,dF(t)$. Following Chew and Nishimura (1992), the local utility function is the Gateaux derivative at F.

$$u_F^{(u,w)}(t) = \frac{w(t)[u(t) - U(F)]}{W(F)}. \tag{23}$$

The decision maker displays local risk aversion at F if the $u_F^{(u,w)}$ is monotonic increasing and concave, and displays global risk aversion if she displays local risk aversion at all $F \in D_J$. The decision maker's problem is to choose an optimal portfolio along the path

$$F(\cdot|z_1) = \left\{p\delta_{z_1} + (1-p)\delta_{z_0 + \frac{z_0 - z_1}{r}} : z_1 \in [0, z_0 + rz_0]\right\}. \tag{24}$$

If $r = \frac{1-p}{p}$, then the Arrow securities are fairly priced and all weighted-utility maximizers whose utility function u is monotonic increasing and concave, would choose an optimal portfolio position, $(z_1^*, z_2^*) = (z_0, z_0)$.[20] If $r > \frac{1-p}{p}$, then every weighted-utility maximizer would choose a portfolio position, (z_1^*, z_2^*) such that $z_1^* > z_0$ and $z_2^* < z_0$. Moreover, suppose there two weighted-utility maximizing decision makers characterized by utility-weight pairs (u, w) and $(\hat{u}, \hat{w})$, where everywhere along the path of portfolios the local utility $u_{F(\cdot|z_1)}^{(\hat{u},\hat{w})}$ displays greater risk aversion than $u_{F(\cdot|z_1)}^{(u,w)}$. Therefore, $z_1^{(u,w)} > z_1^{(\hat{u},\hat{w})}$ and $z_2^{(u,w)} < z_2^{(\hat{u},\hat{w})}$, so that the more risk averse decision maker takes a less risky position closer to the certainty line, that is, holding fewer shares of a_1.

Thus far we reviewed some results concerning the simple portfolio choice of weighted-utility maximizing decision makers whose preference relations are complete. When a weighted-utility maximizer's preference relation is incomplete, there is a set $\mathcal{V}$ of utility-weight pairs that must agree in order for one risky prospect to be strictly preferred over another. Let $\mathcal{Z} := \{(z_1^{(u,w)}, z_2^{(u,w)} : (u, w) \in \mathcal{V})\}$ be the set of optimal portfolio positions corresponding to each of the pairs of utility and weight functions $(u, w) \in \mathcal{V}$. Since the various utility-weight pairs represent distinct risk attitudes, the portfolio positions in $\mathcal{Z}$ are non-comparable. Hence, the decision maker exhibits indecisiveness with regard to the portfolio positions in $\mathcal{Z}$, which may be resolved by random choice. However, once the decision maker chooses a portfolio composition it becomes the status quo or default position, and she displays inertia by avoiding making portfolio adjustments for some range of variations in the relative price r. Inertia was suggested by Bewley (2002) and may be summarized by the dictum "if in doubt do nothing."

This situation is depicted in Fig. 6 where the set of optimal portfolios corresponds to the interval $\mathcal{Z} = [(z_1^L, z_2^L), (z_1^H, z_2^H)]$. On this interval, the decision maker has randomly chosen (z_1^M, z_2^M), which is now the status quo position. As the upper contour set at (z_1^M, z_2^M) has a kink, some variations in r will not induce portfolio adjustment, because the distinct pairs $(u, w) \in \mathcal{V}$ disagree on the desirable adjustments. On the other hand, the decision maker would exhibit inertia only with respect to increasing r at (z_1^H, z_2^H) and inertia only with respect to decreasing r at (z_1^L, z_2^L). By contrast to the expected multi-utility model, where the set of local utilities is the same everywhere and hence the decision maker displays equal inertia everywhere, in the weighted multi-utility model the set of local utilities $\mathcal{U}_F = \{u_F^{(u,w)} : (u, w) \in \mathcal{V}\}$ varies along the portfolio path $F(\cdot|z_1)$. As each of the local utilities $u_F^{(u,w)} \in \mathcal{U}_F$ may show distinct patterns of variation in risk attitude depending on the reference point F, these differences may result in variations along the portfolio path both in the range of indecisiveness and the degree of inertia with respect to changes in r.

6. Experimental design

Consider an experiment designed to test whether the pattern of choices displayed by subjects is consistent with the predictions of the model. Of particular interest is the consistency of the choices with the partial substitution axiom (A.4), which is the critical assumption that provides the weighted utility structure.

6.1. Challenges

If we were to assume that the subjects' preferences were complete, it is relatively straightforward to design an experiment to test if the substitution axiom holds, since the incomparability relation $\asymp$ would manifest an equivalence relation $\sim$ which can be elicited from participants as follows. Let $x_1 < x_2 < x_3$, where x_i, $i = 1, 2, 3$ are dollar amounts and suppose that $p_2 = \delta_{x_2} \sim \zeta_{\alpha_2}$ and $p_3 = \frac{1}{2}\delta_{x_2} + \frac{1}{2}\delta_{x_3} \sim \zeta_{\alpha_3}$. Then by Lemma 1 there is $\tau_2 \geq 0$ such that

$$p_3 \sim \frac{\frac{1}{2}\tau_2\zeta_{\alpha_2} + \frac{1}{2}\delta_{x_3}}{\frac{1}{2}\tau_2 + \frac{1}{2}} = \zeta_{\frac{\frac{1}{2}\tau_2\alpha_2 + \frac{1}{2}}{\frac{1}{2}\tau_2 + \frac{1}{2}}} = \zeta_{\alpha_3}. \tag{25}$$

Therefore $\tau_2 = \frac{1-\alpha_3}{\alpha_3 - \alpha_2}$. Letting $p_1 = \frac{1}{2}\delta_{x_2} + \frac{1}{3}\delta_{x_1}$, we have that

$$p_1 \sim \frac{\frac{1}{2}\tau_2\zeta_{\alpha_2} + \frac{1}{2}\delta_{x_1}}{\frac{1}{2}\tau_2 + \frac{1}{2}} = \zeta_{\frac{\frac{1}{2}\tau_2\alpha_2}{\frac{1}{2}\tau_2 + \frac{1}{2}}} = \zeta_{\alpha_1}. \tag{26}$$

Hence,

$$\alpha_1 = \frac{\frac{1}{2}\tau_2\alpha_2}{\frac{1}{2}\tau_2 + \frac{1}{2}} = \frac{\frac{1}{2}\left(\frac{1-\alpha_3}{\alpha_3 - \alpha_2}\right)\alpha_2}{\frac{1}{2}\left(\frac{1-\alpha_3}{\alpha_3-\alpha_2}\right) + \frac{1}{2}} = \frac{(1-\alpha_3)\alpha_2}{1-\alpha_2}. \tag{27}$$

Therefore, according to weighted utility it must be the case that $p_1 \sim \zeta_{\alpha_1}$. For example, if the subject expresses that $\delta_{x_2} \sim \frac{1}{2}\delta_{x_3} + \frac{1}{2}\delta_{x_1}$ and $\frac{1}{2}\delta_{x_2} + \frac{1}{2}\delta_{x_3} \sim \frac{5}{6}\delta_{x_3} + \frac{1}{6}\delta_{x_1}$, then she must also set $\frac{1}{2}\delta_{x_2} + \frac{1}{2}\delta_{x_1} \sim \frac{1}{6}\delta_{x_3} + \frac{5}{6}\delta_{x_1}$.[21] Any other preference constitutes a violation of weighted utility theory. In other words, if the incomparability curves fan out in the lower half of the simplex then they must also fan out in the upper half as well, so that the decision maker maintains the underweighting of the median outcome x_2 relative to the extremes regardless of whether it is being mixed with something desirable or not.

When the subject's preference relation is incomplete, however, there are some practical difficulties associated with eliciting $\asymp$ in an experimental setting. We describe below an elicitation scheme and an experiment designed to test the model of weighted utility with incomplete preferences.

6.2. Elicitation

Let $0 < x_1 < x_2 < x_3$, where x_i, $i = 1, 2, 3$ are dollar amounts. Consider the following scheme designed to elicit the range of utility values $A(p) = \{\alpha \in [0, 1] : p \asymp \zeta_\alpha\}$ of a lottery $p = \sum_{i=1}^{3} p_i\delta_{x_i}$. The mechanism involves three stages. At time $t = 0$ the subject is required to report two numbers, $\underline{a}, \overline{a} \in [0, 1]$ such that $\underline{a} \leq \overline{a}$, and a random number, a, is drawn from a uniform distribution on $[0, 1]$. In the interim period, $t = 1$, the subject is awarded the lottery p if $a < \underline{a}$ and if $a > \overline{a}$ the subject is

20 Note that on the certainty line, the weight function plays no role in defining risk attitude.

21 Here $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{5}{6}$, and therefore $\alpha_1 = \frac{\left(1-\frac{5}{6}\right)\left(\frac{1}{2}\right)}{1-\frac{1}{2}} = \frac{1}{6}$.

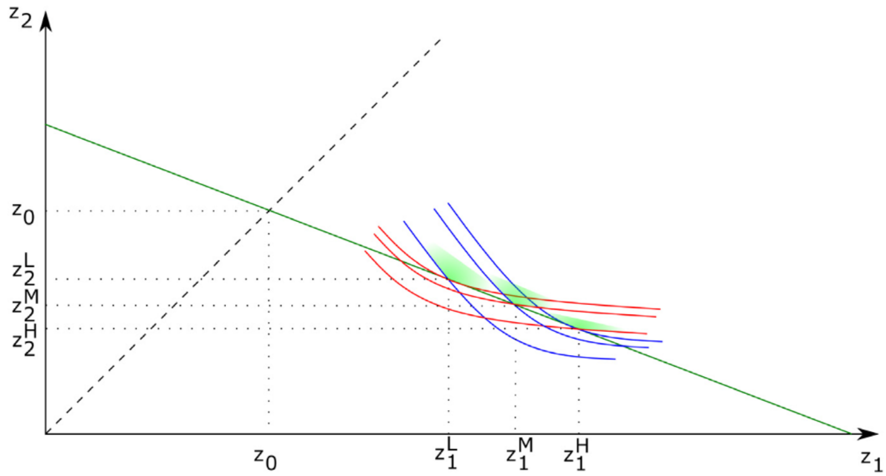

Fig. 6. Portfolio selection.

awarded the lottery $\zeta_a = a\delta_{x_3} + (1-a)\,\delta_{x_1}$. If $\underline{a} < \overline{a}$ and $a \in [\underline{a}, \overline{a}]$, then the subject is allowed to choose between the lottery $p(\theta) = \sum_{i=1}^{3} p_i \delta_{x_i - \theta}$ and the lottery $\zeta_a(\theta) = a\delta_{x_3-\theta} + (1-a)\,\delta_{x_1-\theta}$, where $\theta > 0$. In the last period, $t = 2$, the outcome of the lotteries is revealed, and all payments are made.

At time $t = 1$ the subject who considers p and ζ_a to be incomparable must nevertheless make a choice. We propose that costly procrastinating is justified by the subject's expectation that she will receive a signal that would resolve her indecision. While the signal generating stochastic process is not specified, the subject behaves as if some $(u, w) \in \mathcal{V}$ was selected and this utility-weight pair governs the subject's choice. Thus, we may regard the set $\mathcal{V}$ as the canonical signal space and suppose that the signal (u, w) is drawn according to some unspecified probability measure μ on $\mathcal{V}$.[22] In the interim period the subject chooses the alternative that maximizes the weighted utility functional corresponding to signal (u, w).

The following theorem asserts that, under this mechanism, it is incentive compatible for the subject to truthfully report her range $A(p)$.

Theorem 5. *Suppose that the subject's preference relation $\succ$ satisfies (A.1)–(A.4) then, given the aforementioned mechanism and $p \in \Delta(X)$, there is $\varepsilon > 0$ such that, for all $\theta \in [0, \varepsilon)$, the subject's unique dominant strategy is to report $\underline{a} = \underline{\alpha}(p)$ and $\overline{a} = \overline{\alpha}(p)$.*

6.3. Consistency

Let $p_1 = \frac{1}{2}\delta_{x_2} + \frac{1}{2}\delta_{x_1}$, $p_2 = \delta_{x_2}$, and $p_3 = \frac{1}{2}\delta_{x_2} + \frac{1}{2}\delta_{x_3}$. For every $\alpha_3 \in A(p_3)$, there is $\alpha_2 \in A(p_2)$ and $\tau_2 \in T(p_2, \zeta_{\alpha_2})$ such that $p_2 \asymp \zeta_{\alpha_2}$ and

$$p_3 \asymp \frac{\frac{1}{2}\tau_2\zeta_{\alpha_2} + \frac{1}{2}\delta_{x_3}}{\frac{1}{2}\tau_2 + \frac{1}{2}} = \zeta_{\frac{\frac{1}{2}\tau_2\alpha_2 + \frac{1}{2}}{\frac{1}{2}\tau_2 + \frac{1}{2}}} = \zeta_{\alpha_3}. \tag{28}$$

As before, we have $\tau_2 = \frac{1-\alpha_3}{\alpha_3 - \alpha_2}$. However, without knowledge of the structure of the source space we cannot be sure of which α_3 corresponds to the same utility function that assigns α_2 to p_2 and therefore the possible range of weights associated with this value is

$$T(p_2, \zeta_{\alpha_2}) \subseteq \left[\frac{1-\overline{\alpha}_3}{\overline{\alpha}_3 - \alpha_2}, \frac{1-\underline{\alpha}_3}{\underline{\alpha}_3 - \alpha_2}\right]. \tag{29}$$

Likewise, for every $\alpha_1 \in A(p_1)$ there are $\alpha_2 \in A(p_2)$ and $\tau_2 \in T(p_2, \zeta_{\alpha_2})$ such that

$$p_1 \asymp \frac{\frac{1}{2}\tau_2\zeta_{\alpha_2} + \frac{1}{2}\delta_{x_1}}{\frac{1}{2}\tau_2 + \frac{1}{2}} = \zeta_{\frac{\frac{1}{2}\tau_2\alpha_2}{\frac{1}{2}\tau_2 + \frac{1}{2}}} = \zeta_{\alpha_1}. \tag{30}$$

If $\alpha_3 \in A(p_3)$ and $\alpha_2 \in A(p_2)$ correspond to the same utility function, then $\alpha_1 = \frac{(1-\alpha_3)\alpha_2}{1-\alpha_2} \in A(p_1)$.

For $i = 1, 2, 3$, apply the elicitation scheme to determine the range and note that by Theorem 5, $[\underline{a}_i, \overline{a}_i] = [\underline{\alpha}_i, \overline{\alpha}_i]$. The subject's responses for $[\underline{a}_3, \overline{a}_3]$ to those of $[\underline{a}_2, \overline{a}_2]$ restrict the range that $[\underline{a}_1, \overline{a}_1]$ may take while still adhering to the model. However, the compatible range for $[\underline{a}_1, \overline{a}_1]$ depends on how we match the elements of $[\underline{a}_3, \overline{a}_3]$ to those of $[\underline{a}_2, \overline{a}_2]$, so that

$$\begin{aligned} \overline{a}_1 &\in \left[\frac{(1-\overline{a}_3)\overline{a}_2}{1-\overline{a}_2}, \frac{(1-\underline{a}_3)\overline{a}_2}{1-\overline{a}_2}\right], \\ \underline{a}_1 &\in \left[\frac{(1-\overline{a}_3)\underline{a}_2}{1-\underline{a}_2}, \frac{(1-\underline{a}_3)\underline{a}_2}{1-\underline{a}_2}\right]. \end{aligned} \tag{31}$$

Therefore, in order for the subject to adhere to multiple weighted utility, her range for p_1 must adhere to the bounds

$$\begin{aligned} \left[\frac{(1-\underline{a}_3)\underline{a}_2}{1-\underline{a}_2}, \frac{(1-\overline{a}_3)\overline{a}_2}{1-\overline{a}_2}\right] &\subseteq [\underline{a}_1, \overline{a}_1] \\ &\subseteq \left[\frac{(1-\overline{a}_3)\underline{a}_2}{1-\underline{a}_2}, \frac{(1-\underline{a}_3)\overline{a}_2}{1-\overline{a}_2}\right]. \end{aligned} \tag{32}$$

Fig. 7 depicts a possible result of this experiment.

Let $[\underline{a}_2, \overline{a}_2]$ and $[\underline{a}_3, \overline{a}_3]$ be the ranges elicited for p_2 and p_3, respectively. Observe that this is insufficient to determine the utility representation, as there are several possible source spaces that would produce the same ranges for p_2 and p_3, but different ranges at p_1. Choosing the pair of source points $\{o^1, o^2\}$ gives us the narrow range $[\underline{a}_1^S, \overline{a}_1^S]$ at p_1, while choosing $\{o^3, o^4\}$ produces identical behavior at p_2 and p_3 but yields the wider range $[\underline{a}_1^B, \overline{a}_1^B]$ at p_1. Indeed we may choose a number of other configurations for the source space, up to the entire shaded area Ω that would be compatible with the ranges $[\underline{a}_2, \overline{a}_2]$ and $[\underline{a}_3, \overline{a}_3]$ and yield a range at p_1 bounded by the two extreme cases. Therefore, the range $[\underline{a}_1, \overline{a}_1]$ we elicit at p_1 must obey $[\underline{a}_1^S, \overline{a}_1^S] \subseteq [\underline{a}_1, \overline{a}_1] \subseteq [\underline{a}_1^B, \overline{a}_1^B]$. For example, suppose that we elicit $[\underline{a}_2, \overline{a}_2] = [\frac{2}{5}, \frac{3}{5}]$ and $[\underline{a}_3, \overline{a}_3] = [\frac{4}{5}, \frac{13}{15}]$. Then, the range of utility values we elicit for p_1 must lie

22 Karni and Safra (2016) axiomatized a general random choice behavior model of this nature.

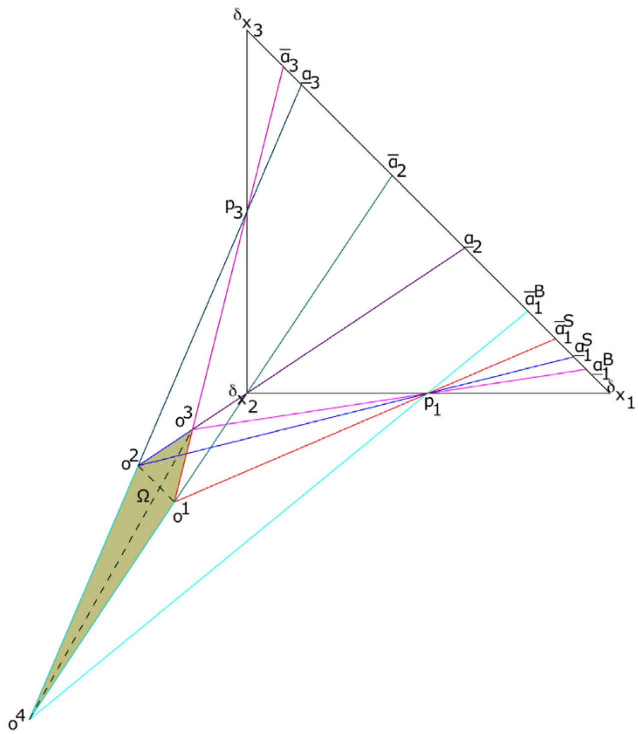

Fig. 7. Eliciting utility ranges.

within the bounds $\left[\frac{2}{15}, \frac{1}{5}\right] \subseteq [\underline{a}_1, \bar{a}_1] \subseteq \left[\frac{4}{45}, \frac{3}{10}\right]$.[23] Otherwise the multi-weight multi-utility model is falsified.

6.4. Special cases

The ranges of $[\underline{a}_i, \bar{a}_i]$ for $i = 1, 2, 3$ may also be used to test the hypothesis that the subject's preference relation falls under one of the special cases of our model.

If the subject's preference relation has single weight multi-utility representation then the upper limits of the ranges all correspond to one utility and the lower limits of ranges to another, both of which assign the same weight to δ_{x_2}. Therefore, the subject has a single weight, multi-utility representation if there is a single weight $\tau_2 \geq 0$ that satisfies

$$\bar{a}_3 = \frac{\frac{1}{2}\tau_2\bar{a}_2 + \frac{1}{2}}{\frac{1}{2}\tau_2 + \frac{1}{2}}, \quad \underline{a}_3 = \frac{\frac{1}{2}\tau_2\underline{a}_2 + \frac{1}{2}}{\frac{1}{2}\tau_2 + \frac{1}{2}}, \quad \bar{a}_1 = \frac{\frac{1}{2}\tau_2\bar{a}_2}{\frac{1}{2}\tau_2 + \frac{1}{2}}, \quad \underline{a}_1 = \frac{\frac{1}{2}\tau_2\underline{a}_2}{\frac{1}{2}\tau_2 + \frac{1}{2}}. \tag{33}$$

This implies that

$$\tau_2 = \frac{1 - \bar{a}_3}{\bar{a}_3 - \bar{a}_2} = \frac{1 - \underline{a}_3}{\underline{a}_3 - \underline{a}_2} = \frac{\bar{a}_1}{\bar{a}_2 - \bar{a}_1} = \frac{\underline{a}_1}{\underline{a}_2 - \underline{a}_1}. \tag{34}$$

If any of these qualities fails to hold then the hypothesis that the subject's preference relation has a single-weight multi-utility representation is falsified.

If the subject's preference relation has a multi-weight single utility representation, the middle outcome p_2 has only a single utility value $a_2 = \bar{a}_2 = \underline{a}_2$ while $\bar{a}_3 > \underline{a}_3$ and $\bar{a}_1 > \underline{a}_1$, and there is a range of weights $[\underline{\tau}_2, \bar{\tau}_2]$ such that

$$\bar{a}_3 = \frac{\frac{1}{2}\bar{\tau}_2 a_2 + \frac{1}{2}}{\frac{1}{2}\bar{\tau}_2 + \frac{1}{2}}, \quad \underline{a}_3 = \frac{\frac{1}{2}\underline{\tau}_2 a_2 + \frac{1}{2}}{\frac{1}{2}\underline{\tau}_2 + \frac{1}{2}}, \quad \bar{a}_1 = \frac{\frac{1}{2}\underline{\tau}_2 a_2}{\frac{1}{2}\underline{\tau}_2 + \frac{1}{2}}, \quad \underline{a}_1 = \frac{\frac{1}{2}\bar{\tau}_2 a_2}{\frac{1}{2}\bar{\tau}_2 + \frac{1}{2}}. \tag{35}$$

Thus,

$$\bar{\tau}_2 = \frac{1 - \bar{a}_3}{\bar{a}_3 - a_2} = \frac{\underline{a}_1}{a_2 - \underline{a}_1}, \quad \underline{\tau}_2 = \frac{1 - \underline{a}_3}{\underline{a}_3 - a_2} = \frac{\bar{a}_1}{a_2 - \bar{a}_1}. \tag{36}$$

If this implication fails to hold then the hypothesis that subject's preference relation has a multi-weight single utility representation is falsified.

Furthermore, if $\bar{\tau}_2 = \underline{\tau}_2 = 1$, so that $\bar{a}_3 = \frac{1}{2}\bar{a}_2 + \frac{1}{2}$, $\underline{a}_3 = \frac{1}{2}\underline{a}_2 + \frac{1}{2}$, $\bar{a}_1 = \frac{1}{2}\bar{a}_2$, and $\underline{a}_1 = \frac{1}{2}\underline{a}_2$, the subject's preferences obey the independence axiom and thus have an expected multi-utility representation. If there is $a_i = \bar{a}_i = \underline{a}_i$ for $i = 1, 2, 3$ and $\tau_2 = \frac{1-a_3}{a_3-a_2} = \frac{a_1}{a_2-a_1}$, then the preferences are complete and have a weighted utility representation. If both of these conditions hold, so that $a_3 = \frac{1}{2}a_2 + \frac{1}{2}$ and $a_1 = \frac{1}{2}a_2$, then the subject adheres to expected utility theory.

7. Concluding remarks

In this paper, we consider a model of decision making under risk for preferences that satisfy neither independence nor completeness. Specifically we characterize a utility representation by the agreement of a set of utilities, as in multiple utility theory, each of which is weighted linear in the probabilities, as in weighted utility theory, thus uniting these separate strands in the literature under a unified framework. This multi-weight, multi-utility representation admits two special cases with distinct interpretations due ambivalent risk attitudes or incognizance of the relative salience of the possible outcomes. By directly imposing additional axioms that eliminate either of these possibilities, we obtain special cases where the multiplicity in the representation is restricted to either the utility or weight functions alone. The general framework we have devised thus serves as a useful foundation for studying decision making under risk from a variety of different perspectives.

8. Proofs

8.1. Proofs of propositions

8.1.1. Proof of Proposition 1

Fix $p \in \Delta(X)$, then as $\succ$ is bounded, $\{\alpha : p \prec \zeta_\alpha\}$ is bounded and, as $p \prec \zeta_1 = \delta_{\bar{x}}$, non-empty, so that $\bar{\alpha} = \inf\{\alpha : p \prec \zeta_\alpha\}$ exists. Suppose that for $\alpha' \leq \bar{\alpha}$ we have $p \prec \zeta_{\alpha'}$, then by (A.2) there is $\beta \in (0, 1)$ such that $p \prec \beta\zeta_{\alpha'} + (1 - \beta)\delta_{\underline{x}} = \zeta_{\alpha''}$. Since $\alpha'' = \beta\alpha' < \bar{\alpha}$, this contradicts the definition of $\bar{\alpha}$. Now suppose that some for $\alpha' > \bar{\alpha}$ we have $\neg(p \prec \zeta_{\alpha'})$, then for all $\alpha'' < \alpha'$ we have $\neg(p \prec \zeta_{\alpha''})$ or else $p \prec \zeta_{\alpha''} \prec \zeta_{\alpha'}$, which implies that $\alpha' \leq \inf\{\alpha : p \prec \zeta_\alpha\} = \bar{\alpha}$, a contradiction. Thus $\neg(p \prec \zeta_{\alpha'})$ if and only if $\alpha' \leq \bar{\alpha}$. Likewise, $\{\alpha : p \succ \zeta_\alpha\}$ is bounded and, as $p \succ \zeta_0 = \delta_{\underline{x}}$, non-empty, so that $\underline{\alpha} = \sup\{\alpha : p \succ \zeta_\alpha\}$ exists. By a similar argument, $\neg(p \succ \zeta_{\alpha'})$ if and only if $\alpha' \geq \underline{\alpha}$. Therefore, we have that $p \asymp \zeta_{\alpha'}$ if and only if $\alpha' \in [\underline{\alpha}, \bar{\alpha}]$. □

23 $[\underline{a}_1^B, \bar{a}_1^B] = \left[\frac{\left(1-\frac{13}{15}\right)\left(\frac{2}{5}\right)}{1-\frac{2}{5}}, \frac{\left(1-\frac{4}{5}\right)\left(\frac{3}{5}\right)}{1-\frac{3}{5}}\right] = \left[\frac{4}{45}, \frac{3}{10}\right]$ and $[\underline{a}_1^B, \bar{a}_1^B] = \left[\frac{\left(1-\frac{4}{5}\right)\left(\frac{2}{5}\right)}{1-\frac{2}{5}}, \frac{\left(1-\frac{13}{15}\right)\left(\frac{3}{5}\right)}{1-\frac{3}{5}}\right] = \left[\frac{2}{15}, \frac{1}{5}\right]$.

8.1.2. Proof of Proposition 2

Fix $p, q \in \Delta(X)$ such that $p \asymp q$. We first show that $T(p,q)$ is bounded. Suppose not, then letting $r = \delta_{\overline{x}}$ we have for every $\beta \in (0,1)$ that $\beta p + (1-\beta)r \asymp \lim_{\tau\to\infty} \frac{\beta\tau q+(1-\beta)r}{\beta\tau+(1-\beta)} = q$, but as $r \succ q$ and thus, by (A.2), $\beta p+(1-\beta)r \succ q$ for β sufficiently small, this is a contradiction. Hence τ cannot approach ∞ and therefore $T(p,q)$ is bounded. Now, for every $r \in \Delta(X)$, define

$$T^L(p,q,r) = \left\{\tau > 0 : \exists\,\beta,\ \beta p + (1-\beta)r \succ \frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)}\right\},$$

$$T^R(p,q,r) = \left\{\tau > 0 : \exists\,\beta,\ \beta p + (1-\beta)r \prec \frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)}\right\},$$

$$T(p,q,r) = \left\{\tau > 0 : \forall\,\beta,\ \beta p + (1-\beta)r \asymp \frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)}\right\}.$$

Let $R = \{r \in \Delta(X) : \neg(r \asymp p) \vee \neg(r \asymp q)\}$ denote the set of all lotteries that are comparable with either p or q. We will establish that for every $r \in R$, $T(p,q,r)$ is a closed interval $[\underline{\tau}_r, \overline{\tau}_r]$, and for every $r \notin R$, there is $s \in R$ such that $T(p,q,r) \supseteq T(p,q,s)$. Taken together these will allow us to conclude that $T(p,q)$ is given by the intersection of closed intervals and is hence itself a closed, and bounded, interval.

Suppose $r \in R$. Then if $r \succ q$, define $\overline{\tau}_r = \inf T^L(p,q,r)$ and $\underline{\tau}_r = \sup T^R(p,q,r)$.[24] Suppose that for $\tau' \le \overline{\tau}_r$ we have $\tau' \in T^L(p,q,r)$, then there is $\tau'' \in T^L(p,q,r)$ such that $\tau'' < \tau' \le \overline{\tau}_r$, contradicting the definition of $\overline{\tau}_r$.[25] Now suppose that for $\tau' > \overline{\tau}_r$ we have $\tau' \notin T^L(p,q,r)$, then we must have $\tau'' \notin T^L(p,q,r)$ for every $\tau'' < \tau'$,[26] and therefore $\tau' \le \inf T^L(p,q,r) = \overline{\tau}_r$, a contradiction. Thus $\tau' \in T^L(p,q,r)$ if and only if $\tau' > \overline{\tau}_r$, and by a similar argument $\tau' \in T^R(p,q,r)$ if and only if $\tau' < \underline{\tau}_r$. This implies that $\tau' \in T(p,q,r)$ if and only if $\tau' \in [\underline{\tau}_r, \overline{\tau}_r]$.

If $r \prec q$, then we can define $\overline{\tau}_r = \inf T^R(p,q,r)$ and $\underline{\tau}_r = \sup T^L(p,q,r)$[27] and apply a similar argument to the above. If $r \asymp q$ and either $r \succ p$ or $r \prec p$, we can again repeat the argument above by switching p and q and noting that $\tau' \in T(p,q,r)$ if and only if $\frac{1}{\tau'} \in T(q,p,r)$. Thus for every $r \in R$, $T(p,q,r)$ is a closed interval $[\underline{\tau}_r, \overline{\tau}_r]$.

Now suppose $r \notin R$. Then if there is $r' \in \Delta(\{p,q,r\})$ such that $r' \succ q$, there are $\lambda, \alpha \in [0,1]$ such that $r' = \lambda[\alpha p + (1-\alpha)r] + (1-\lambda)q \succ q$, so that by betweenness we have $s = \alpha p + (1-\alpha)r \succ q$. This implies that there is $s \in R$ such that $T(p,q,r) \supseteq T(p,q,s)$.[28] A similar result follows if we have $s \in \Delta(\{p,q,r\})$ such that $s \prec q$. Likewise, if there is $s \in \Delta(\{p,q,r\})$ such that $s \succ p$ or $s \prec p$, we repeat the argument again noting that $\tau \in T(p,q,r)$ if and only if $\frac{1}{\tau} \in T(q,p,r)$, so that $T(p,q,r) \supseteq T(p,q,s)$ if and only if $T(q,p,r) \supseteq T(q,p,s)$.

Now suppose that for $s \in \Delta(\{p,q,r\})$ we have $s \asymp p$ and $s \asymp q$. If for some $\theta \in (0,1)$ we have $s \succ \theta p + (1-\theta)q$, then by betweenness $p \asymp \theta p + (1-\theta)q$ and by the argument above, $T(p, \theta p + (1-\theta)q, r) \supseteq T(p, \theta p + (1-\theta)q, s)$, which in turn implies $T(p,q,r) \supseteq T(p,q,s)$.[29] A similar result follows if for some $\theta \in (0,1)$ we have $s \prec \theta p + (1-\theta)q$. Finally, if for all $s \in \Delta(\{p,q,r\})$ and $\theta \in (0,1)$ we have $s \asymp \theta p + (1-\theta)q$, then $T(p,q,r) = \mathbb{R}_{++}$,[30] so that $T(p,q,r) \supseteq T(p,q,s)$ for all $s \in R$.

By definition we have that $T(p,q) = \bigcap_{r\in\Delta(X)} T(p,q,r)$. Note that $T(p,q)$ is bounded if any $T(p,q,r)$ is bounded, which we can establish by setting $r = \delta_{\overline{x}}$.[31] Since for every $r \notin R$ there is $s \in R$ such that $T(p,q,r) \supseteq T(p,q,s)$, we have that $T(p,q) = \bigcap_{r\in R} T(p,q,r) = \bigcap_{r\in R} [\underline{\tau}_r, \overline{\tau}_r]$. Letting $\underline{\tau} = \sup_{r\in R} \underline{\tau}_r$ and $\overline{\tau} = \inf_{r\in R} \overline{\tau}_r$, we have that $T(p,q) = [\underline{\tau}, \overline{\tau}]$. □

8.1.3. Proof of Proposition 3

Suppose that for $p, q \in \Delta(X)$ there is $\tau > 0$ such that $o = \frac{p-\tau q}{1-\tau} \in \Omega$. Since $o \in \Omega$, there are $p', q' \in \Delta(X)$ such that $p' \asymp q'$ and $\tau' \in T(p',q')$ such that

$$o = \frac{p - \tau q}{1 - \tau} = \frac{p' - \tau' q'}{1 - \tau'}.$$

Suppose that the line defined by p and q intersects the interior of the simplex so that there are λ_u, λ_v, assuming without loss of generality that $\lambda_u > \lambda_v$, such that

$$u := \lambda_u p + (1-\lambda_u)q \in \text{int}\Delta(X),$$
$$v := \lambda_v p + (1-\lambda_v)q \in \text{int}\Delta(X),$$
$$o = \frac{u - \left[\frac{\lambda_u\tau + (1-\lambda_u)}{\lambda_v\tau + (1-\lambda_v)}\right] v}{1 - \left[\frac{\lambda_u\tau + (1-\lambda_u)}{\lambda_v\tau + (1-\lambda_v)}\right]} := \frac{u - \hat{\tau} v}{1 - \hat{\tau}}.$$

Pick $\lambda_u \approx \lambda_v$ so that $\hat{\tau} \approx 1$. Define s' to be the intersection of the lines $p'u$ and $q'v$. Then as u and v are close together, we have

$$s' := \frac{(1-\tau')u - (1-\hat{\tau})p'}{\hat{\tau} - \tau'} = \frac{(1-\tau')\hat{\tau}q - (1-\hat{\tau})\tau' q'}{\hat{\tau} - \tau'} \in \text{int}\Delta(X),$$

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24 Let $\overline{\tau}_r = \infty$ if $T^L(p,q,r) = \varnothing$ and $\underline{\tau}_r = 0$ if $T^R(p,q,r) = \varnothing$.

25 If $\tau' \in T^L(p,q,r)$ then there is $\beta \in (0,1)$ such that $\beta p + (1-\beta)r \succ \frac{\beta\tau' q+(1-\beta)r}{\beta\tau'+(1-\beta)}$. This implies by (A.2) that there is $\lambda \in (0,1)$ such that

$$\beta p + (1-\beta)r \succ \frac{\lambda[\beta\tau' + (1-\beta)]\left[\frac{\beta\tau' q+(1-\beta)r}{\beta\tau'+(1-\beta)}\right] + (1-\lambda)r}{\lambda[\beta\tau' + (1-\beta)] + (1-\lambda)} = \frac{\lambda\beta\tau' q + (1-\lambda\beta)r}{\lambda\beta\tau' + (1-\lambda\beta)}.$$

Letting $\tau'' = \frac{\lambda\tau'\beta/(1-\lambda\beta)}{\beta/(1-\beta)} < \tau'$ completes the argument.

26 Otherwise if some $\tau'' \in T^L(p,q,r)$, then for some $\beta \in (0,1)$ we have $\beta p + (1-\beta)r \succ \frac{\beta\tau'' q+(1-\beta)r}{\beta\tau''+(1-\beta)} \succ \frac{\beta\tau' q+(1-\beta)r}{\beta\tau'+(1-\beta)}$, implying $\tau' \in T^L(p,q,r)$ as well.

27 As before, let $\overline{\tau}_r = \infty$ if $T^R(p,q,r) = \varnothing$ and $\underline{\tau}_r = 0$ if $T^L(p,q,r) = \varnothing$.

28 Pick $\tau \in T(p,q,s)$ and for any $\beta \in (0,1)$, let $\beta' = \beta + (1-\beta)\alpha$ so that $p' = \beta p + (1-\beta)s = \beta' p + (1-\beta')r$ and let $q' = \frac{\beta'\tau q+(1-\beta')r}{\beta'\tau+(1-\beta')}$. Then we have that

$$p' \asymp \frac{\beta\tau q + (1-\beta)s}{\beta\tau + (1-\beta)} = \frac{\beta\tau q + (1-\beta)\alpha p + (1-\beta)(1-\alpha)r}{\beta\tau + (1-\beta)} = \frac{(1-\beta)\alpha p' + \beta[\beta'\tau + (1-\beta')]q'}{(1-\beta)\alpha + \beta[\beta'\tau + (1-\beta')]}.$$

By betweenness, the above implies that $p' \asymp q'$, and taking the odds ratio gives us $\tau \in T(p,q,r)$.

29 We can show that for any $t \in \Delta(X)$ there is a one to one mapping from $T(p,q,t)$ to $T(p, \theta p + (1-\theta)q, t)$ by noting that, again by betweenness,

$$\beta p + (1-\beta)t \asymp \frac{\theta\tau[\beta p + (1-\beta)t] + (1-\theta)[\beta\tau + (1-\beta)]\left[\frac{\beta\tau q+(1-\beta)t}{\beta\tau+(1-\beta)}\right]}{\theta\tau + (1-\theta)[\beta\tau + (1-\beta)]} = \frac{\beta\tau[\theta p + (1-\theta)q] + (1-\beta)[\theta\tau + (1-\theta)]t}{\beta\tau + (1-\beta)[\theta\tau + (1-\theta)]}.$$

Taking the odds ratio of the above, we conclude that $\tau \in T(p,q,t)$ if and only if $\frac{\tau}{\theta\tau+(1-\theta)} \in T(p, \theta p+(1-\theta)q, t)$. Thus $T(p, \theta p+(1-\theta)q, r) \supseteq T(p, \theta p+(1-\theta)q, s)$ if and only if $T(p,q,r) \supseteq T(p,q,s)$.

30 Pick $\beta \in (0,1)$ and let $s = \beta p + (1-\beta)r$, and for any $\tau > 0$ let $\theta = \frac{1}{1-\tau}$. Then, by assumption, $\beta p + (1-\beta)r \asymp \frac{p-\tau q}{1-\tau}$ and invoking betweenness yet again we have that

$$\beta p + (1-\beta)r \asymp \frac{[\beta p + (1-\beta)r] - \beta(1-\tau)\left[\frac{p-\tau q}{1-\tau}\right]}{1 - \beta(1-\tau)} = \frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)}.$$

This implies that $\tau \in T(p,q,r)$ for every $\tau > 0$.

31 Since for any $\overline{\beta} \in (0,1)$ we have $r \succ \overline{\beta}p + (1-\overline{\beta})r$, by (A.2) there is $\underline{\gamma} \in (0,1)$ such that $\underline{\gamma}q + (1-\underline{\gamma})r \succ \overline{\beta}p + (1-\overline{\beta})r$, implying $\underline{\tau}_r > \frac{\underline{\gamma}/(1-\underline{\gamma})}{\overline{\beta}/(1-\overline{\beta})}$. Likewise for any $\overline{\gamma} \in (0,1)$ we have $r \succ \overline{\gamma}q + (1-\overline{\gamma})r$, so there is $\underline{\beta} \in (0,1)$ such that $\underline{\beta}p + (1-\underline{\beta})r \succ \overline{\gamma}q + (1-\overline{\gamma})r$ so that $\overline{\tau}_r < \frac{\overline{\gamma}/(1-\overline{\gamma})}{\underline{\beta}/(1-\underline{\beta})}$.

$$u = \frac{1-\hat{\tau}}{1-\tau'}p' + \frac{\hat{\tau}-\tau'}{1-\tau'}s' := \hat{\beta}p' + (1-\hat{\beta})s',$$
$$v = \frac{(1-\hat{\tau})\tau'}{(1-\tau')\hat{\tau}}q' + \frac{\hat{\tau}-\tau'}{(1-\tau')\hat{\tau}}s' := \hat{\gamma}q' + (1-\hat{\gamma})s'.$$

Since $p' \asymp q'$ and $\tau' = \frac{\hat{\gamma}/(1-\hat{\gamma})}{\hat{\beta}/(1-\hat{\beta})} \in T(p', q')$, we have

$$u = \lambda_u p + (1-\lambda_u)q = \hat{\beta}p' + (1-\hat{\beta})s' \asymp \frac{\hat{\beta}\tau' q' + (1-\hat{\beta})s'}{\hat{\beta}\tau' + (1-\hat{\beta})} = \lambda_v p + (1-\lambda_v)q = v.$$

Therefore since $\lambda_u > \lambda_v$, if $p \succ q$ then by betweenness $u \succ v$, a contradiction, so $\neg(p \succ q)$ and by a similar argument $\neg(p \prec q)$, hence $p \asymp q$.

Now suppose that the line defined by p and q lies entirely on the boundary of the simplex. Then pick $t \in \text{int}\Delta(X)$ such that $t \succ p$ and $t \succ q$ and, for every n, let

$$p^n = \frac{1}{n}t + \left(1-\frac{1}{n}\right)p,$$
$$q^n = \frac{\frac{1}{n}t + \left(1-\frac{1}{n}\right)\tau q}{\frac{1}{n} + \left(1-\frac{1}{n}\right)\tau}.$$

Thus $\lim_{n\to\infty} p^n = p$ and $\lim_{n\to\infty} q^n = q$. For each n, we have that the line $p^n q^n$ intersects $\Delta(X)^o$ and can verify that

$$o = \frac{p^n - [\frac{1}{n} + \left(1-\frac{1}{n}\right)\tau]q^n}{1 - [\frac{1}{n} + \left(1-\frac{1}{n}\right)\tau]}.$$

Hence, by the argument above $p^n \asymp q^n$ for every $n = 1, 2, \ldots$. Suppose that $p \succ q$, then there is $s \in \Delta(X)$ such that $p \succ s \succ q$. By betweenness, since $t \succ p$ every $p^n \succ p \succ s$. By the strong Archimedean, since $s \succ q$ there is $\alpha \in (0, 1)$ such that $s \succ \alpha q + (1-\alpha)t$. Hence, for any $n \geq \frac{1-(1-\alpha)(1-\tau)}{1-\alpha\tau}$ then $\frac{\left(1-\frac{1}{n}\right)\tau}{\frac{1}{n}+\left(1-\frac{1}{n}\right)\tau} \geq \alpha$ we have $s \succ \alpha q + (1-\alpha)t \succ q^n$. By transitivity, $p^n \succ q^n$, a contradiction. Thus $\neg(p \succ q)$ and, by a similar argument, $\neg(p \prec q)$ so that $p \asymp q$. □

8.1.4. Proof of Proposition 4

Define $v = wu$, so that for $p \in \Delta(X)$ we may write

$$\begin{bmatrix} V(p) \\ W(p) \end{bmatrix} = \begin{bmatrix} v(x_1) & \cdots & v(x_n) \\ w(x_1) & \cdots & w(x_n) \end{bmatrix} \begin{bmatrix} p(x_1) \\ \vdots \\ p(x_n) \end{bmatrix} = \mathbf{Vp},$$
$$U(p) = \frac{\sum_{i=1}^n p(x_i)v(x_i)}{\sum_{i=1}^n p(x_i)w(x_i)} = \frac{V(p)}{W(p)}.$$

For $p, q \in \Delta(X)$, we have that $U(p) > U(q)$ if and only if

$$W(p)W(q)[U(p) - U(q)] = V(p)W(q) - V(q)W(p) = \begin{vmatrix} V(p) & V(q) \\ W(p) & W(q) \end{vmatrix} = \begin{vmatrix} \mathbf{Vp} & \mathbf{Vq} \end{vmatrix} = |\mathbf{VP}| > 0.$$

Now consider a positive affine transformation

$$\tilde{\mathbf{V}} = \begin{bmatrix} \tilde{v}(x_1) & \cdots & \tilde{v}(x_n) \\ \tilde{w}(x_1) & \cdots & \tilde{w}(x_n) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v(x_1) & \cdots & v(x_n) \\ w(x_1) & \cdots & w(x_n) \end{bmatrix} = \mathbf{AV}.$$

This implies that $\tilde{U}(p) > \tilde{U}(q)$ if and only if $|\tilde{\mathbf{V}}\mathbf{P}| = |\mathbf{A}||\mathbf{VP}| > 0$, so that the ranking of lotteries is unchanged as long as $|\mathbf{A}| > 0$, or $ad - bc > 0$. □

8.1.5. Proof of Proposition 5

Let $\mathcal{V}$ represent $\succ$, pick any pair $(u, w) \in \mathcal{V}$, and let $v = wu$. We begin by showing that there exists a normalized function pair $(\hat{u}, \hat{w})$ for which (u, w) is a rational affine transformation, so that there are a, b, c, d such that

$$\begin{bmatrix} v(x_1) & v(x_n) \\ w(x_1) & w(x_n) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \hat{v}(x_1) & \hat{v}(x_n) \\ \hat{w}(x_1) & \hat{w}(x_n) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Solving for these constants, we see that we indeed have a positive rational affine transformation as long as u ranks the best element x_n above the worst x_1. To grasp this claim, observe that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} v(x_1) & v(x_n) \\ w(x_1) & w(x_n) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} v(x_n) - v(x_1) & v(x_1) \\ w(x_n) - w(x_1) & w(x_1) \end{bmatrix}.$$

Hence,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = w(x_1)[v(x_n) - v(x_1)] - v(x_1)[w(x_n) - w(x_1)] = w(x_1)w(x_n)[u(x_n) - u(x_1)] > 0.$$

Inverting this matrix, we transform (u, w) back to the normalized $(\hat{u}, \hat{w})$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{\begin{bmatrix} w(x_1) & -v(x_1) \\ -[w(x_n) - w(x_1)] & v(x_n) - v(x_1) \end{bmatrix}}{w(x_1)[v(x_n) - v(x_1)] - v(x_1)[w(x_n) - w(x_1)]} = \frac{\begin{bmatrix} w(x_1) & -w(x_1)u(x_1) \\ -[w(x_n) - w(x_1)] & w(x_n)u(x_n) - w(x_1)u(x_1) \end{bmatrix}}{w(x_1)w(x_n)[u(x_n) - u(x_1)]}.$$

For any $x \in X$, we have that

$$\begin{bmatrix} \hat{v}(x) \\ \hat{w}(x) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} v(x) \\ w(x) \end{bmatrix} = \frac{\begin{bmatrix} w(x_1) & -w(x_1)u(x_1) \\ -[w(x_n) - w(x_1)] & w(x_n)u(x_n) - w(x_1)u(x_1) \end{bmatrix} \begin{bmatrix} w(x)u(x) \\ w(x) \end{bmatrix}}{w(x_1)w(x_n)[u(x_n) - u(x_1)]} = \frac{\begin{bmatrix} w(x_1)w(x)[u(x) - u(x_1)] \\ w(x_n)w(x)[u(x_n) - u(x)] + w(x_1)w(x)[u(x) - u(x_1)] \end{bmatrix}}{w(x_1)w(x_n)[u(x_n) - u(x_1)]}.$$

This gives us the utility and weight functions

$$\hat{u}(x) = \frac{\hat{v}(x)}{\hat{w}(x)} = \frac{w(x_1)w(x)[u(x) - u(x_1)]}{w(x_n)w(x)[u(x_n) - u(x)] + w(x_1)w(x)[u(x) - u(x_1)]},$$
$$\hat{w}(x) = \frac{w(x_n)w(x)[u(x_n) - u(x)] + w(x_1)w(x)[u(x) - u(x_1)]}{w(x_1)w(x_n)[u(x_n) - u(x_1)]}.$$

It is easily verified that $(\hat{u}(x_1), \hat{w}(x_1)) = (0, 1)$ and $(\hat{u}(x_n), \hat{w}(x_n)) = (1, 1)$. Now for every $p \in \Delta(X)$, define

$$\hat{U}(p) = \frac{\sum_{i=1}^n p(x_i)\hat{w}(x_i)\hat{u}(x_i)}{\sum_{i=1}^n p(x_i)\hat{w}(x_i)} = \frac{\sum_{i=1}^n p_i\tau_i\alpha_i}{\sum_{i=1}^n p_i\tau_i} = \alpha_p,$$
$$\hat{W}(p) = \sum_{i=1}^n p(x_i)\hat{w}(x_i) = \sum_{i=1}^n p_i\tau_i = \tau_p.$$

Since the utility function is normalized by assumption, we have, for any α,

$$\hat{U}(\zeta_\alpha) = \frac{\alpha\hat{w}(x_n)\hat{u}(x_n) + (1-\alpha)\hat{w}(x_1)\hat{u}(x_1)}{\alpha\hat{w}(x_n) + (1-\alpha)\hat{w}(x_1)} = \alpha,$$
$$\hat{W}(\zeta_\alpha) = \alpha\hat{w}(x_n) + (1-\alpha)\hat{w}(x_1) = 1.$$

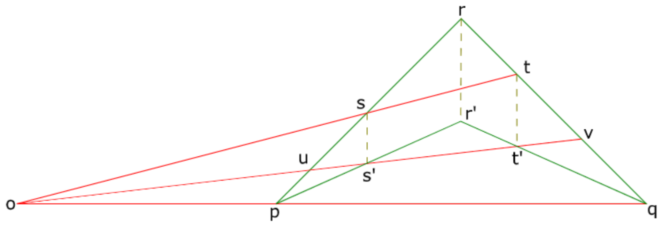

Fig. 8. Proof of Lemma 1.

As $\hat{U}(p) = \hat{U}(\zeta_{\alpha_p})$ and $\hat{W}(p) = \tau_p \hat{W}(\zeta_{\alpha_p})$, we have for every $\beta \in (0,1)$ and $r \in \Delta(X)$ that

$$\begin{aligned}\hat{U}(\beta p + (1-\beta) r) &= \frac{\beta \hat{W}(p)\hat{U}(p) + (1-\beta)\hat{W}(r)\hat{U}(r)}{\beta \hat{W}(p) + (1-\beta)\hat{W}(r)} \\ &= \frac{\beta \tau_p \hat{W}(\zeta_{\alpha_p})\hat{U}(\zeta_{\alpha_p}) + (1-\beta)\hat{W}(r)\hat{U}(r)}{\beta \tau_p \hat{W}(\zeta_{\alpha_p}) + (1-\beta)\hat{W}(r)} \\ &= \hat{U}\left(\frac{\beta \tau_p \zeta_{\alpha_p} + (1-\beta) r}{\beta \tau_p + (1-\beta)}\right).\end{aligned}$$

This implies that every $\beta p + (1-\beta) r \asymp \frac{\beta \tau_p \zeta_{\alpha_p} + (1-\beta) r}{\beta \tau_p + (1-\beta)}$, so that $(\alpha_p, \tau_p) \in \Phi(p)$ for every $p \in \Delta(X)$. Hence, $\{(\hat{u}(x_i), \hat{w}(x_i))\}_{i=1}^n = \{(\alpha_i, \tau_i)\}_{i=1}^n \in \Psi$. $\square$

8.1.6. Proof of Proposition 6

Let $\hat{\mathcal{V}}$ be a normalized set of utilities that represents $\succ$, and for every $(u^k, w^k) \in \mathcal{V}$, define

$$U^k(p) = \frac{\sum_{x \in X} p(x) w^k(x) u^k(x)}{\sum_{x \in X} p(x) w^k(x)}, \quad W^k(p) = \sum_{x \in X} p(x) w^k(x).$$

Around any lottery $p \in \Delta(X)$, let $\bar{u}_p^k = U^k(p)$ and $\hat{\mathcal{U}}_p = \{u_p^k = w^k u^k + (1-w^k)\bar{u}_p^k : (u^k, w^k) \in \mathcal{V}\}$ be the set of normalized local utilities. For every $q \in \Delta(X)$, let

$$\begin{aligned}U_p^k(q) &= \sum_{x \in X} q(x) u_p^k(x) = \sum_{x \in X} q(x)[w^k(x) u^k(x) + [1 - w^k(x)] U^k(p)] \\ &= W^k(q) U^k(q) + [1 - W^k(q)] U^k(p).\end{aligned}$$

Since $U_p^k(p) = U^k(p)$, we have $U_p^k(p) - U_p^k(q) = W^k(q)[U^k(p) - U^k(q)]$. Thus $p \succ q$ if and only if $U^k(p) > U^k(q)$ for every $(u^k, w^k) \in \hat{\mathcal{V}}$, which in turn holds if and only if $U_p^k(p) > U_p^k(q)$ for every $u_p^k \in \hat{\mathcal{U}}_p$. Denote the closure of the convex hull of $\hat{\mathcal{U}}_p$ by $\langle \hat{\mathcal{U}}_p \rangle = \mathrm{cl}\{u_p^\pi = \sum_{u_p^k \in \hat{\mathcal{U}}_p} \pi^k u_p^k : \sum_{u_p^k \in \hat{\mathcal{U}}_p} \pi^k = 1\}$, then

$$\begin{aligned}&U_p^k(p) > U_p^k(q),\ \forall u_p^k \in \mathcal{U}_p \iff U_p^\pi(p) \\ &= \sum_{u_p^k \in \hat{\mathcal{U}}_p} \pi^k U_p^k(p) > \sum_{u_p^k \in \hat{\mathcal{U}}_p} \pi^k U_p^k(q) = U_p^\pi(q),\ \forall u_p^\pi \in \langle \hat{\mathcal{U}}_p \rangle.\end{aligned}$$

Therefore, $p \succ q$ if and only if every $U_p^\pi(p) > U_p^\pi(q)$, completing the proof. $\square$

8.2. Proofs of lemmas

8.2.1. Proof of Lemma 1

Fix $p, q \in \Delta(X)$ such that $p \asymp q$. Fix $r \in \Delta(X)$ and pick $\beta, \gamma \in (0,1)$ that satisfy partial substitution so that $s := \beta p + (1-\beta) r \asymp \gamma q + (1-\gamma) r := t$. Now pick $\beta', \gamma' \in (0,1)$ such that $\tau := \frac{\gamma'/(1-\gamma')}{\beta'/(1-\beta')} = \frac{\gamma/(1-\gamma)}{\beta/(1-\beta)}$, proving the proposition requires showing that $u := \beta' p + (1-\beta') r \asymp \gamma' q + (1-\gamma') r := v$.

As depicted in Fig. 8, the extensions of the lines st and uv intersect at some source point lying outside of the simplex on the extended line pq, located at $o = \frac{p - \tau q}{1-\tau}$. Draw parallel lines from s and r such that the line from s intersects uv at some point s', and extending ps' intersects the line from r at some r', and let t' denote the intersection of uv and qr'. By Desargues' theorem, the triangles rst and $r's't'$ are perspective from the line opq, and hence the lines rr', ss', and tt' are parallel. This implies that $s' = \beta p + (1-\beta) r'$ and $t' = \gamma q + (1-\gamma) r'$, and hence by weak substitution that $s' \asymp t'$. Since both s' and t' lie on uv, we have by betweenness that $u \asymp v$ as well, completing the proof. $\square$

8.2.2. Proof of Lemma 2

We will show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$ Pick $p, q \in \Delta(X)$ and suppose that $p \asymp q$, and pick $\hat{r} \in \Delta(X) \setminus \Delta(\{p, q\})$.[32] Letting $\bar{r} = \theta \delta_{x_n} + (1-\theta)\hat{r}$ and $\underline{r} = \theta \delta_{x_1} + (1-\theta)\hat{r}$, we have for θ sufficiently large that $\bar{r} \succ p, q \succ \underline{r}$. As we are now working in a space where $\bar{r}$ and $\underline{r}$ are the best and worst elements, we can analogously define $\zeta_\alpha^r = \alpha \bar{r} + (1-\alpha)\underline{r}$ for every $\alpha \in (0,1)$ and $A_\lambda = \{\alpha \in [0,1] : \lambda p + (1-\lambda) q \asymp \alpha \zeta_\alpha^r\}$ for every $\lambda \in [0,1]$. By Proposition 1, we have that each A_λ is a closed interval $[\underline{\alpha}_\lambda, \overline{\alpha}_\lambda]$. If $A^* = \bigcap_\lambda A_\lambda = \varnothing$, then there are λ_1, λ_2 and $\alpha' \in (0,1)$ such that $\underline{\alpha}_{\lambda_1} > \alpha' > \overline{\alpha}_{\lambda_2}$, so that $\lambda_1 p + (1-\lambda_1) q \succ \zeta_{\alpha'}^r \succ \lambda_2 p + (1-\lambda_2) q$. By betweenness, if $\lambda_1 > \lambda_2$ then $p \succ q$, and if $\lambda_1 < \lambda_2$ then $p \prec q$. As either would contradict $p \asymp q$, we must have that $A^* \neq \varnothing$, so letting $r = \zeta_\alpha^r$ for any $\alpha \in A^*$ establishes the result.

$(ii) \Rightarrow (iii)$ Pick $p, q \in \Delta(X)$ such that $p \asymp q$ and pick any $\tau^* \in T(p, q)$, so that there is a source point $o^* = \frac{p - \tau^* q}{1-\tau^*} \in \Omega$. Pick $\bar{r}, \underline{r} \in \Delta(X)$ such that $\bar{r} \succ p, q \succ \underline{r}$, then by the preceding argument, there is $r = \alpha \bar{r} + (1-\alpha)\underline{r} \in \Delta(X)$ such that every $\lambda p + (1-\lambda) q \asymp r$. We need to show that there are $\tau_p, \tau_q > 0$ such that every $\lambda \tau_p + (1-\lambda)\tau_q \in T(\lambda p + (1-\lambda) q)$.

For every λ, let $p_\lambda = \lambda p + (1-\lambda) q$ and $\tau_\lambda^* = \lambda \tau^* + (1-\lambda)$, then since $\tau^* \in T(p, q)$ we have that

$$\begin{aligned}o^* &= \frac{p - \tau^* q}{1-\tau^*} = \frac{[\lambda p + (1-\lambda) q] - [\lambda \tau^* + (1-\lambda)] q}{1 - [\lambda \tau^* + (1-\lambda)]} \\ &= \frac{p_\lambda - \tau_\lambda^* q}{1 - \tau_\lambda^*} \in \Omega.\end{aligned}$$

This implies by Proposition 3 that $\tau_\lambda^* \in T(p_\lambda, q)$. We now claim that there is some $\tau_q \in T(q, r)$ such that for every λ, $\tau_\lambda^* \tau_q \in T(p_\lambda, r)$. Suppose not, then since by Proposition 2 $T(p_\lambda, r) = [\underline{\tau}_\lambda, \overline{\tau}_\lambda]$ and $T(q, r) = [\underline{\tau}_q, \overline{\tau}_q]$, there is λ such that $\tau_\lambda^* > \frac{\overline{\tau}_\lambda}{\underline{\tau}_q}$ or $\tau_\lambda^* < \frac{\underline{\tau}_\lambda}{\overline{\tau}_q}$. Assume the former without loss of generality, then there are $\tau_q' < \underline{\tau}_q$ and $\tau_\lambda' > \overline{\tau}_\lambda$ such that $\tau_\lambda^* = \frac{\tau_\lambda'}{\tau_q'}$. Let $s = \bar{r} \succ r$, then we have that since $\tau_\lambda' > \overline{\tau}_\lambda$, $o_\lambda' = \frac{p_\lambda - \tau_\lambda' r}{1 - \tau_\lambda'} \notin \Omega$, so that for some $\beta \in (0,1)$,

$$p_\lambda' = \beta p_\lambda + (1-\beta) s \succ \frac{\beta \tau_\lambda' r + (1-\beta) s}{\beta \tau_\lambda' + (1-\beta)} := r'.$$

Now let $\gamma = \frac{\beta \tau_\lambda^*}{\beta \tau_\lambda^* + (1-\beta)}$ so that $\tau_\lambda^* = \frac{\gamma/(1-\gamma)}{\beta/(1-\beta)}$. Since $\tau_q' < \underline{\tau}_q$, $o_q' = \frac{q - \tau_q' r}{1 - \tau_q'} \notin \Omega$ so that

$$q' = \gamma q + (1-\gamma) s \prec \frac{\gamma \tau_q' r + (1-\gamma) s}{\gamma \tau_q' + (1-\gamma)} := r''$$

Since by construction $\tau_\lambda' = \tau_\lambda^* \tau_q'$, taking the above together we have that $p_\lambda' \succ r' = r'' \succ q'$. But $\tau_\lambda^* \in T(p_\lambda, q)$ implies

[32] That such $\hat{r}$ exists is implied by the dimensionality of $\Delta(X)$.

$p'_\lambda \asymp q'$. This is a contradiction. Thus we must have that $\tau^*_\lambda \leq \frac{\overline{\tau}_\lambda}{\underline{\tau}_q}$. a similar argument shows $\tau^*_\lambda \geq \frac{\underline{\tau}_\lambda}{\overline{\tau}_q}$. Therefore, there is $\tau_q \in [\underline{\tau}_q, \overline{\tau}_q]$ such that $\tau^*_\lambda \tau_q \in T(p_\lambda, r)$ for every λ. Letting $\tau_p = \tau^* \tau_q \in T(p, r)$, this implies $[\lambda\tau^* + (1-\lambda)]\tau_q = \lambda\tau_p + (1-\lambda)\tau_q \in T(\lambda p + (1-\lambda)q, r)$ for every λ, completing the proof.[33]

$(iii) \Rightarrow (iv)$ Pick $p, q \in \Delta(X)$ such that $p \asymp q$, and by (iii) there is $r \in \Delta(X)$ and $\tau_p, \tau_q > 0$ such that every $\lambda\tau_p + (1-\lambda)\tau_q \in T(\lambda p + (1-\lambda)q, r)$. This defines a line of source points

$$O(p,q) = \left\{ o_\lambda = \frac{\lambda p + (1-\lambda)q - [\lambda\tau_p + (1-\lambda)\tau_q]r}{1 - [\lambda\tau_p + (1-\lambda)\tau_q]} : \lambda \in [0,1] \right\} \subseteq \Omega.$$

Now pick $p', q' \in \Delta(\{p, q, r\})$, then there are $\tau'_p, \tau'_q > 0$ such that we can define source points $o'_p = \frac{p' - \tau'_p r}{1-\tau'_p}$ and $o'_q = \frac{q' - \tau'_q r}{1-\tau'_q}$.[34] Now let $\tau'_t = \frac{\tau'_p}{\tau'_q}$ and $\lambda'_t = \frac{1}{1-\tau'_t}$, then we have that

$$o'_t = \frac{\lambda'_t(1-\tau'_p)o'_p + (1-\lambda'_t)(1-\tau'_q)o'_q}{\lambda'_t(1-\tau'_p) + (1-\lambda'_t)(1-\tau'_q)}$$
$$= \frac{(p' - \tau'_p r) - \frac{\tau'_p}{\tau'_q}(q' - \tau'_q r)}{(1-\tau'_p) - \frac{\tau'_p}{\tau'_q}(1-\tau'_q)} = \frac{p' - \tau'_t q'}{1-\tau'_t} \in O(p,q) \subseteq \Omega.$$

By Proposition 3 this implies that $p' \asymp q'$ and $\tau'_t \in T(p', q')$.[35]

$(iv) \Rightarrow (i)$ This is immediate. □

8.2.3. Proof of Lemma 3

Again, we will show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$ Suppose that P is an incomparability set, then if dim $\Delta(P) < n-2$ we can pick $\hat{r} \in \Delta(X) \setminus \Delta(P)$[36] and let $\overline{r} = \theta\delta_{x_n} + (1-\theta)\hat{r}$ and $\underline{r} = \theta\delta_{x_1} + (1-\theta)\hat{r}$, such that for θ sufficiently close to 1, $\overline{r} \succ p \succ \underline{r}$, for every $p \in \Delta(P)$. For every $p \in \Delta(P)$ let $A_p = \{\alpha : p \asymp \alpha\overline{r} + (1-\alpha)\underline{r}\} = [\underline{\alpha}_p, \overline{\alpha}_p]$. If $A^* = \bigcap_{p\in\Delta(P)} A_p = \varnothing$, then there are $p_1, p_2 \in \Delta(P)$ and α' such that $\underline{\alpha}_{p_1} > \alpha' > \overline{\alpha}_{p_2}$. But this implies that $p_1 \succ p_2$, a contradiction. Picking any $\alpha \in A^* \neq \varnothing$ and letting $r = \alpha\overline{r} + (1-\alpha)\underline{r}$ establish the result.

$(ii) \Rightarrow (iii)$ Fix $r \in \Delta(X) \setminus \Delta(P)$ satisfying (ii). Pick $P_k = \{p_1, \ldots, p_k\} \subseteq P$ such that $\Delta(P_k) = \Delta(P)$, then it will be sufficient to show that there are $(\tau_1, \ldots, \tau_k) \in \mathbb{R}^k_{++}$ such that for every $q = \sum_{j=1}^k \pi_j p_j \in \Delta(P_k)$, $\tau_q = \sum_{j=1}^k \pi_j \tau_j \in T(q, r)$.

Claim For any $\ell \leq k$, let $P_\ell = \{p_1, \ldots, p_\ell\}$, then there is $(\tau_1, \ldots, \tau_\ell) \in \mathbb{R}^\ell_{++}$ such that for every $(\pi_1, \ldots, \pi_\ell) \in \mathbb{R}^\ell_+$

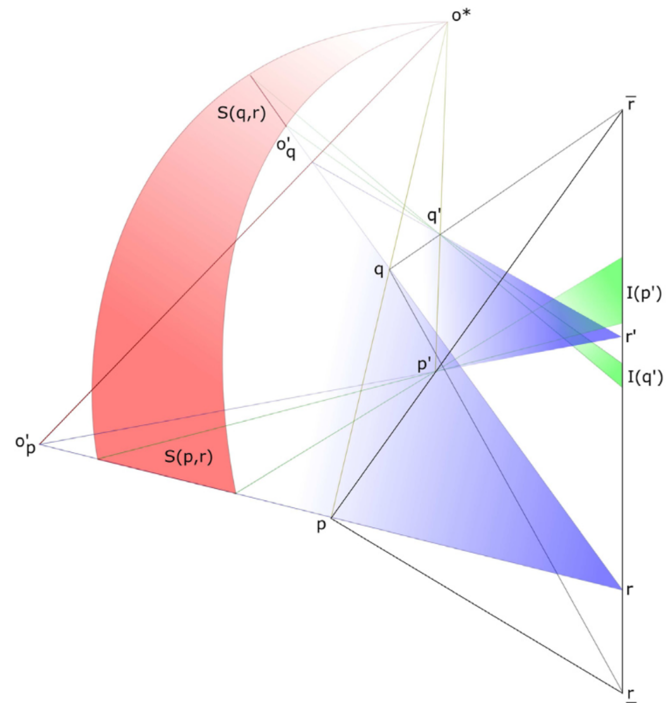

Fig. 9. Proof of Lemma 2 $[(ii) \Rightarrow (iii)]$.

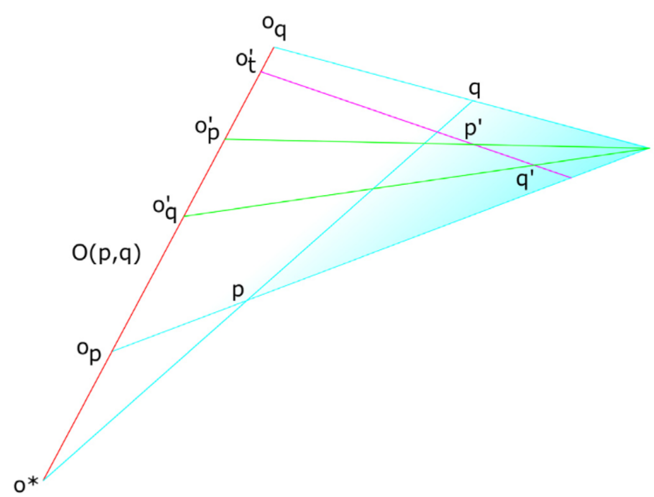

Fig. 10. Proof of Lemma 2 $[(iii) \Rightarrow (iv)]$.

33 This argument is illustrated in Fig. 9, showing the contradiction when the claim is violated for $\lambda = 1$, so that $\tau^*\tau_q \notin T(p, r)$. For every $p, q \in \Delta(X)$ let $S(p,q) = \{o = \frac{p-\tau q}{1-\tau} : \tau \in T(p,q)\}$ the range of source points on the line defined by p and q. For any $p \in \Delta(X)$, let $I(p) = \{r' = \alpha\overline{r} + (1-\alpha)\underline{r} \asymp p : \alpha \in [0,1]\}$ the range of mixtures of $\overline{r}$ and $\underline{r}$ to which p is incomparable, then $r \in I(p)$ and $\tau \in T(p,r)$ implies that $r' = \frac{\beta\tau r + (1-\beta)\overline{r}}{\beta\tau + (1-\beta)} \in I(\beta p + (1-\beta)\overline{r})$. As shown in Fig. 9, if $o^* \in S(p,q)$, then we must be able to draw a line from it that intersects both $S(p,r)$ and $S(q,r)$, or else there are $o'_p, o'_q \notin \Omega$ that indicate $I(p')$ and $I(q')$ are disjoint, so that $p' \succ q'$ which in turn would imply $o^* \notin \Omega$.

34 Let $p' := \mu_p p + \mu_q q + (1-\mu_p-\mu_q)r$ and $q' := \nu_p p + \nu_q q + (1-\nu_p-\nu_q)r$ and $\tau'_p = \mu_p\tau_p + \mu_q\tau_q + (1-\mu_p-\mu_q)$ and $\tau'_q = \nu_p\tau_p + \nu_q\tau_q + (1-\nu_p-\nu_q)$. Then,

$$o'_p = \frac{p' - \tau'_p r}{1-\tau'_p}$$
$$= \frac{[\mu_p p + \mu_q q + (1-\mu_p-\mu_q)r] - [\mu_p\tau_p + \mu_q\tau_q + (1-\mu_p-\mu_q)]r}{1 - [\mu_p\tau_p + \mu_q\tau_q + (1-\mu_p-\mu_q)]}$$
$$= \frac{\left[\frac{\mu_p p + \mu_q q}{\mu_p+\mu_q}\right] - \left[\frac{\mu_p\tau_p + \mu_q\tau_q}{\mu_p+\mu_q}\right] r}{1 - \left[\frac{\mu_p\tau_p + \mu_q\tau_q}{\mu_p+\mu_q}\right]}.$$

Letting $\lambda'_p = \frac{\mu_p}{\mu_p+\mu_q}$ show that $o'_p \in O(p,q) \subseteq \Omega$, and by a similar argument $o'_q \in \Omega$.

35 As Fig. 10 shows, for any $p', q' \in \Delta(\{p,q,r\})$ we can draw a line connecting these two lotteries that intersects the source line $O(p,q)$ at some o'_t.

36 Again, $\hat{r}$ exists by the dimensionality of $\Delta(X)$.

satisfying $\sum_{j=1}^{\ell}\pi_j = 1$ and $q = \sum_{j=1}^{\ell}\pi_j p_j \in \Delta(P_\ell)$, we have that $\tau_q = \sum_{j=1}^{\ell}\pi_j\tau_j \in T(q,r)$.

Proof We establish this property by induction. If $\ell = 1$ then the simplex is a singleton $\Delta(\{p_1\}) = \{p_1\}$ so the property is trivially satisfied by picking any $\tau_1 \in T(p_1, r)$. For the inductive step, suppose $\ell < k$ and we have such $(\tau_1,\dots,\tau_\ell) \in \mathbb{R}^{\ell}_{++}$ and let $\tau_q = \sum_{j=1}^{\ell}\pi_j\tau_j$ for every $q \in \Delta(P_\ell)$.

By Lemma 2, for every $q_1, q_2 \in \Delta(P_\ell)$ we have that $q_1 \asymp q_2$ and $\frac{\tau_{q_1}}{\tau_{q_2}} \in T(q_1, q_2)$.[37] Pick any $p \in P \setminus \Delta(P_\ell)$. Since P is an incomparability set, $p \asymp q$ for any $q \in \Delta(P_\ell)$. By Lemma 2 we have that

$$Z_p^q := \{\tau_p \in T(p,r) : \lambda\tau_p + (1-\lambda)\tau_q \in T(\lambda p + (1-\lambda)q, r), \forall\lambda \in [0,1]\} \neq \varnothing.$$

Suppose that $Z_p^* := \bigcap_{q\in\Delta(P_\ell)} Z_p^q = \varnothing$, then there are $q_1, q_2 \in \Delta(P_\ell)$ and τ_p' such that $\min Z_p^{q_1} > \tau_p' > \max Z_p^{q_2}$. Since by Lemma 2, $\tau_p \in Z_p^q$ if and only if $\frac{\tau_p}{\tau_q} \in T(p,q)$, $\tau_p' < \min Z_p^{q_1}$ implies that $\frac{\tau_p'}{\tau_{q_1}} < \min T(p, q_1)$, and therefore that $\frac{\tau_{q_1}}{\tau_p'} > \max T(q_1, p)$,[38] and $\tau_p' > \max Z_p^{q_2}$ implies $\frac{\tau_p'}{\tau_{q_2}} > \max T(p, q_2)$. Letting $s = \bar{r} \succ p$, for $\beta \in (0,1)$ we have that

$$\beta q_1 + (1-\beta)s \succ \frac{\beta\left(\frac{\tau_{q_1}}{\tau_p'}\right)p + (1-\beta)s}{\beta\left(\frac{\tau_{q_1}}{\tau_p'}\right) + (1-\beta)} \succ \frac{\beta\left(\frac{\tau_{q_1}}{\tau_p'}\right)\left(\frac{\tau_p'}{\tau_{q_2}}\right)q_2 + (1-\beta)s}{\beta\left(\frac{\tau_{q_1}}{\tau_p'}\right)\left(\frac{\tau_p'}{\tau_{q_2}}\right) + (1-\beta)} = \frac{\beta\left(\frac{\tau_{q_1}}{\tau_{q_2}}\right)q_2 + (1-\beta)s}{\beta\left(\frac{\tau_{q_1}}{\tau_{q_2}}\right) + (1-\beta)}.$$

This contradicts $\frac{\tau_{q_1}}{\tau_{q_2}} \in T(q_1, q_2)$. Hence $Z_p^* \neq \varnothing$. Let $p = p_{\ell+1}$ and pick any $\tau_{\ell+1} \in Z_{p_{\ell+1}}^*$, then we have that the set $(\tau_1,\dots,\tau_{\ell+1}) \in \mathbb{R}^{\ell+1}_{++}$ has the desired property.[39] This completes the proof of the claim. □

Returning to the proof of the lemma, set $\ell = k$ and choose $(\tau_1,\dots,\tau_k) in \mathbb{R}^k_{++}$ that satisfies the claim. Then for every $p := \sum_{j=1}^{k}\pi_j p_j \in P$, letting $\tau_p = \sum_{j=1}^{k}\pi_j\tau_j$ establish the result.

(iii) ⇒ (iv) Fix $r \in \Delta(X) \setminus \Delta(P)$ and $\{\tau_p\}_{p\in P} \subseteq \mathbb{R}_{++}$ satisfying (iii). Pick $q_1, q_2 \in \Delta(P \cup \{r\})$, then for $i = 1, 2$, $q_i = \theta_i r + (1-\theta_i)q_i'$ for some $q_i' := \sum_{p\in\Delta(P)}\pi_{p,i}p \in \Delta(P)$. Let $\tau_i' = \sum_{p\in P}\pi_{p,i}\tau_p \in T(q_i', r)$ and $\tau_i = \theta_i + (1-\theta_i)\tau_i'$, then there is a source point at $o_i = \frac{q_i' - \tau_i' r}{1-\tau_i'} = \frac{q_i - \tau_i r}{1-\tau_i} \in \Omega$. By (iii), we have that $\lambda\tau_1' + (1-\lambda)\tau_2' \in T(\lambda q_1' + (1-\lambda)q_2', r)$. Thus, letting $\tau^* = \frac{\tau_1}{\tau_2}$ and $\lambda^* = \frac{1}{1-\tau^*}$ we have that

$$o^* = \frac{\lambda^*(1-\tau_1)o_1 + (1-\lambda^*)(1-\tau_2)o_2}{\lambda^*(1-\tau_1) + (1-\lambda^*)(1-\tau_2)} = \frac{(q_1 - \tau_1 r) - \frac{\tau_1}{\tau_2}(q_2 - \tau_2 r)}{(1-\tau_1) - \frac{\tau_1}{\tau_2}(1-\tau_2)} = \frac{q_1 - \tau^* q_2}{1-\tau^*} \in \Omega.$$

By Proposition 3, this implies that $q_1 \asymp q_2$ and $\tau^* \in T(q_1, q_2)$.

(iv) ⇒ (i) This is immediate. □

8.2.4. Proof of Lemma 4

Fix $p, q \in \Delta(X)$. By repeated application of Lemma 3, we have that $p \asymp q$ if and only if they both belong to the same maximal incomparability set $P \subseteq \Delta(X)$ with $\dim\Delta(P) = n-2$. As boundedness of $\succ$ implies $\delta_{x_n} \succ p \succ \delta_{x_1}$ for every $p \in \Delta(X)$, by betweenness there is a unique $\alpha \in (0,1)$ such that $\zeta_\alpha \in \Delta(P)$. Hence, for every $i = 2,\dots,n-1$ there exists some $p_i = \lambda_i\delta_{x_i} + (1-\lambda_i)\zeta_{\theta_i} \in \Delta(P) \cap \Delta(\{x_i, x_1, x_n\})$. By Lemma 3, this implies that there exist $\{\tau_2',\dots,\tau_{n-1}'\}$ such that every $\tau_i' \in T(p_i, \zeta_\alpha)$ and hence $(\alpha, \tau_i') \in \Phi(p_i)$. Let $\tau_i = \frac{1}{\lambda_i}\tau_i' + \left(1 - \frac{1}{\lambda_i}\right)$ and $\alpha_i = \frac{\tau_i'\alpha - (1-\lambda_i)\theta_i}{\tau_i' - (1-\lambda_i)} = \frac{[\lambda_i\tau_i + (1-\lambda_i)]\alpha - (1-\lambda_i)\theta_i}{\lambda_i\tau_i}$. Then we have

$$o_i = \frac{\delta_{x_i} - \tau_i\zeta_{\alpha_i}}{1-\tau_i} = \frac{\lambda_i\delta_{x_i} + (1-\lambda_i)\zeta_{\theta_i} - [\lambda_i\tau_i + (1-\lambda_i)]\zeta_\alpha}{1 - [\lambda_i\tau_i + (1-\lambda_i)]} = \frac{p_i - \tau_i'\zeta_\alpha}{1-\tau_i'} \in \Omega.$$

This implies that every $(\alpha_i, \tau_i) \in \Phi(\delta_{x_i})$.

This argument that follows is shown in Fig. 11.[40] For every $p' := \sum_{i=1}^{n}\pi_i\delta_{x_i} \in \Delta(X)$, there are linearly independent $\{p_2,\dots,p_{n-1}\} \subseteq P$ $q' = \sum_{i=2}^{n-1}\pi_i' p_i \in \Delta(P)$ such that $p' = \lambda' q' + (1-\lambda')\zeta_{\theta'}$. This implies that

$$(\alpha, \tau_q') = \left(\frac{\sum_{i=2}^{n-1}\pi_i'\tau_i'\alpha}{\sum_{i=2}^{n-1}\pi_i'\tau_i'}, \sum_{i=2}^{n-1}\pi_i'\tau_i'\right) = \left(\frac{\sum_{i=2}^{n-1}\pi_i'[\lambda_i\tau_i\alpha_i + (1-\lambda_i)\theta_i]}{\sum_{i=2}^{n-1}\pi_i'[\lambda_i\tau_i + (1-\lambda_i)]}, \sum_{i=2}^{n-1}\pi_i'[\lambda_i\tau_i + (1-\lambda_i)]\right) \in \Phi(q').$$

Letting $\tau_p' = \lambda'\tau_q' + (1-\lambda')$ and $\alpha_p' = \frac{\lambda'\tau_q'\alpha + (1-\lambda')\theta'}{\lambda'\tau_q' + (1-\lambda')}$ we have

$$o_p' = \frac{p' - \tau_p'\zeta_{\alpha_p'}}{1-\tau_p'} = \frac{[\lambda' q' + (1-\lambda')\zeta_{\theta'}] - [\lambda'\tau_q'\zeta_\alpha + (1-\lambda')\zeta_{\theta'}]}{1 - [\lambda'\tau_q' + (1-\lambda')]} = \frac{q' - \tau_q'\zeta_\alpha}{1-\tau_q'} \in \Omega.$$

This implies $(\alpha_p', \tau_p') \in \Phi(p')$. Furthermore, letting $(\alpha_1, \tau_1) = (0, 1)$ and $(\alpha_n, \tau_n) = (1, 1)$, we have that

$$p' = \lambda'\sum_{i=2}^{n-1}\pi_i'[\lambda_i\delta_{x_i} + (1-\lambda_i)\zeta_{\theta_i}] + (1-\lambda')\zeta_{\theta'} := \sum_{i=1}^{n}\pi_i\delta_{x_i},$$

37 Following the proof of Lemma 2, this is seen by invoking (iii) for $\lambda = \frac{1}{1-\frac{\tau_{q_1}}{\tau_{q_2}}}$.

38 Note that $\tau \in T(p,q)$ if and only if $\frac{1}{\tau} \in T(q,p)$, as $o = \frac{p-\tau q}{1-\tau} = \frac{q - \frac{1}{\tau}p}{1-\frac{1}{\tau}}$.

39 For any $q = \sum_{j=1}^{\ell+1}\pi_j p_j$, let $\lambda = \pi_{\ell+1}$ and $q' = \frac{\sum_{j=1}^{\ell}\pi_j p_j}{\sum_{j=1}^{\ell}\pi_j}$. Then since $\tau_{\ell+1} \in Z_{p_{\ell+1}}^* \subseteq Z_{p_{\ell+1}}^{q'}$, we have that $\sum_{j=1}^{\ell+1}\pi_j\tau_j = \lambda\tau_{\ell+1} + (1-\lambda)\tau_{q'} \in T(\lambda p_{\ell+1} + (1-\lambda)q', r) = T(q, r)$.

40 The existence of a maximal incomparability set P implies that $\Delta(P)$ crosses every triangle $\Delta(\{x_1, x_i, x_n\})$, formed by the best and worst outcomes along with some third outcome $x_i \in X$, at some p_i'. Furthermore, as betweenness implies that we may have at most one $\zeta_\alpha \in \Delta(P)$, every $p_i \asymp \zeta_\alpha$, so that we may find a source point o_i in the usual manner. Drawing a line from o_i through δ_{x_i} allows us to find the utility weight pair (α_i, τ_i) for x_i. By the result of Lemma 3, every point on the line connecting two source points is itself a source point o_p', and drawing this through any lottery $p' \in \Delta(X)$ produces the pair (α_p', τ_p') which is in turn a linear combination of the (α_i, τ_i).

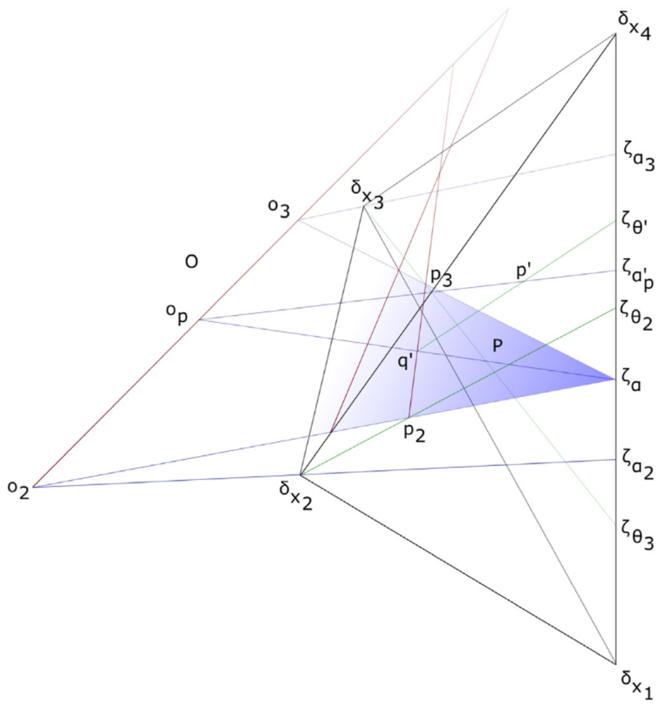

Fig. 11. Proof of Lemma 4.

$$(\alpha_p', \tau_p') = \left(\frac{\lambda' \sum_{i=2}^{n-1} \pi_i'[\lambda_i\tau_i\alpha_i + (1-\lambda_i)\theta_i] + (1-\lambda')\theta'}{\lambda' \sum_{i=2}^{n-1} \pi_i'[\lambda_i\tau_i + (1-\lambda_i)] + (1-\lambda')}, \lambda' \sum_{i=2}^{n-1} \pi_i'[\lambda_i\tau_i + (1-\lambda_i)] + (1-\lambda')\right)$$
$$= \left(\frac{\sum_{i=1}^{n} \pi_i\tau_i\alpha_i}{\sum_{i=1}^{n} \pi_i\tau_i}, \sum_{i=1}^{n} \pi_i\tau_i\right) \in \Phi(p').$$

Since the above holds for any $p \in \Delta(X)$, the collection $\{(\alpha_i, \tau_i)\}_{i=1}^n \in \Psi$.

Returning to the proof, we have that for every $p, q \in \Delta(X)$ that $p \asymp q$ if and only if they lie on some maximal incomparability set P, which defines $\{(\alpha_i, \tau_i)\}_{i=1}^n \in \Psi$. Since there is a unique $\alpha \in [0, 1]$ for which $p, q, \zeta_\alpha \in P$, we must have $\alpha_p = \alpha_q = \alpha$, which completes the proof. □

8.2.5. Proof of Lemma 5

Fix $p \in \Delta(X)$ and $(\alpha^1, \tau^1), (\alpha^2, \tau^2) \in \Phi(p)$. If $\tau^1 \neq \tau^2$, let

$$\beta^* = \frac{\tau^2\alpha^2 - \tau^1\alpha^1}{(1-\tau^1)\tau^2\alpha^2 - (1-\tau^2)\tau^1\alpha^1},$$
$$\alpha^* = \frac{\beta^*\tau^1\alpha^1}{\beta^*\tau^1 + (1-\beta^*)} = \frac{\beta^*\tau^2\alpha^2}{\beta^*\tau^2 + (1-\beta^*)} = \frac{\tau^2\alpha^2 - \tau^1\alpha^1}{\tau^2 - \tau^1},$$
$$\tau^{1*} = \beta^*\tau^1 + (1-\beta^*) = \frac{(\tau^2 - \tau^1)\tau^1\alpha^1}{(1-\tau^1)\tau^2\alpha^2 - (1-\tau^2)\tau^1\alpha^1},$$
$$\tau^{2*} = \beta^*\tau^2 + (1-\beta^*) = \frac{(\tau^2 - \tau^1)\tau^2\alpha^2}{(1-\tau^1)\tau^2\alpha^2 - (1-\tau^2)\tau^1\alpha^1}.$$

Then for $j = 1, 2$, we have

$$o^j = \frac{p - \tau^j\zeta_{\alpha^j}}{1-\tau^j} = \frac{[\beta^*p + (1-\beta^*)\delta_{x_1}] - [\beta^*\tau^j\zeta_{\alpha^j} + (1-\beta^*)\delta_{x_1}]}{1 - [\beta^*\tau^j + (1-\beta^*)]}$$
$$= \frac{[\beta^*p + (1-\beta^*)\delta_{x_1}] - \tau^{j*}\zeta_{\alpha^*}}{1-\tau^{j*}} \in \Omega.$$

This implies that $(\alpha^*, \tau^{j*}) \in \Phi(\beta^*p + (1-\beta^*)\delta_{x_1})$ for $j = 1, 2$, but as $\tau^1 \neq \tau^2$ implies $\tau^{1*} \neq \tau^{2*}$, this would violate (A.4), so we must have $\tau^1 = \tau^2$. Hence, there is a unique τ_p such that $\alpha_p \in A(p)$ implies $(\alpha_p, \tau_p) = \Phi(p) = A(p) \times \{\tau_p\}$. □

8.3. Proof of Theorem 1

(Necessity) Suppose that there is such a closed $\mathcal{V}$ that represents $\succ$. Then for any $(u^k, w^k) \in \mathcal{V}$, define

$$U^k(p) = \frac{\sum_{x\in X} p(x)w^k(x)u^k(x)}{\sum_{x\in X} p(x)w^k(x)}, \quad W^k(p) = \sum_{x\in X} p(x)w^k(x).$$

Hence $p \succ q$ if and only if $U^k(p) > U^k(q)$ for every $(u^k, w^k) \in \mathcal{V}$. It is easily verified that U^k is weighted linear.[41] To show that $\succ$ is bounded, let $\overline{x} = x_n$ and $\underline{x} = x_1$ and noting that for every $p \in \Delta(X)$, $u^k(x_n) > U^k(p) > u^k(x_1)$ for every $(u^k, w^k) \in \mathcal{V}$ implies $\delta_{x_n} \succ p \succ \delta_{x_1}$.

To show that $\succ$ satisfies (A.1), note that for every $p \in \Delta(X)$, $\neg(U^k(p) > U^k(p))$, so $\succ$ is irreflexive, and that for every $p, q, r \in \Delta(X)$, $U^k(p) > U^k(q) > U^k(r)$ implies $U^k(p) > U^k(r)$, so $\succ$ is transitive.

To show that $\succ$ satisfies (A.2), pick $p, q, r \in \Delta(X)$ such that $p \succ q$, then $U^k(p) > U^k(q)$ for every $(u^k, w^k) \in \mathcal{V}$. If $U^k(q) > U^k(r)$, define $\alpha^k \in (0, 1)$ such that

$$U^k(\alpha^k p + (1-\alpha^k)r) = \frac{\alpha^k W^k(p)U^k(p) + (1-\alpha^k)W^k(r)U^k(r)}{\alpha^k W^k(p) + (1-\alpha^k)W^k(r)} = U^k(q),$$
$$\alpha^k = \frac{W^k(r)[U^k(q) - U^k(r)]}{W^k(p)[U^k(p) - U^k(q)] + W^k(r)[U^k(q) - U^k(r)]}.$$

If $U^k(q) \le U^k(r)$, then pick any $\alpha^k \in (0, 1)$. Pick any $\alpha > \inf_{(u^k,w^k)\in\mathcal{V}} \alpha^k$, then $U^k(\alpha p + (1-\alpha)r) > U^k(q)$ for every $(u^k, w^k) \in \mathcal{V}$ and hence $\alpha p + (1-\alpha)r \succ q$. By a similar argument, there is $\alpha' \in (0, 1)$ such that $q \succ \alpha' p + (1-\alpha')r$.

To show that $\succ$ satisfies (A.3), pick $p, q \in \Delta(X)$ such that $p \succ q$, then $U^k(p) > U^k(q)$ for every $(u^k, w^k) \in \mathcal{V}$, so that for any $\alpha \in (0, 1)$,

$$U^k(p) > \frac{\alpha W^k(p)U^k(p) + (1-\alpha)W^k(q)U^k(q)}{\alpha W^k(p) + (1-\alpha)W^k(q)} = U^k(\alpha p + (1-\alpha)q) > U^k(q).$$

This implies that $p \succ \alpha p + (1-\alpha)q \succ q$.

To show that $\succ$ satisfies (A.4), pick $p, q \in \Delta(X)$ such that $p \asymp q$. This implies that there are $(u^1, w^1), (u^2, w^2) \in \mathcal{V}$ such that $U^1(p) \ge U^1(q)$ and $U^2(p) \le U^2(q)$. Define

$$U^\lambda(p) = \frac{\lambda W^1(p)U^1(p) + (1-\lambda)W^2(p)U^2(p)}{\lambda W^1(p) + (1-\lambda)W^2(p)},$$
$$W^\lambda(p) = \lambda W^1(p) + (1-\lambda)W^2(p).$$

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41 For every $p, q \in \Delta(X)$ and $\lambda \in (0, 1)$, we have that

$$U^k(\lambda p + (1-\lambda)q) = \frac{\sum_{x\in X}(\lambda p + (1-\lambda)q)(x)w^k(x)u^k(x)}{\sum_{x\in X}(\lambda p + (1-\lambda)q)(x)w^k(x)}$$
$$= \frac{\lambda\left[\sum_{x\in X} p(x)w^k(x)u^k(x)\right] + (1-\lambda)\left[\sum_{x\in X} q(x)w^k(x)u^k(x)\right]}{\lambda\left[\sum_{x\in X} p(x)w^k(x)\right] + (1-\lambda)\left[\sum_{x\in X} q(x)w^k(x)\right]}$$
$$= \frac{\lambda W^k(p)U^k(p) + (1-\lambda)W^k(q)U^k(q)}{\lambda W^k(p) + (1-\lambda)W^k(q)}.$$

Then there is some $\lambda \in [0,1]$ such that $U^{\lambda}(p) = U^{\lambda}(q)$. For every $\beta \in (0,1)$, fix $\gamma \in (0,1)$ such that the odds ratio $\frac{\gamma/(1-\gamma)}{\beta/(1-\beta)} = \frac{W^{\lambda}(p)}{W^{\lambda}(q)}$. Then, for every $r \in \Delta(X)$, we have

$$\begin{aligned} U^{\lambda}(\beta p + (1-\beta) r) &= \frac{\beta W^{\lambda}(p)U^{\lambda}(p) + (1-\beta)W^{\lambda}(r)U^{\lambda}(r)}{\beta W^{\lambda}(p) + (1-\beta)W^{\lambda}(r)} \\ &= \frac{\beta \frac{W^{\lambda}(p)}{W^{\lambda}(q)} W^{\lambda}(q)U^{\lambda}(q) + (1-\beta)W^{\lambda}(r)U^{\lambda}(r)}{\beta \frac{W^{\lambda}(p)}{W^{\lambda}(q)} W^{\lambda}(q) + (1-\beta)W^{\lambda}(r)} \\ &= \frac{\gamma W^{\lambda}(q)U^{\lambda}(q) + (1-\gamma)W^{\lambda}(r)U^{\lambda}(r)}{\gamma W^{\lambda}(q) + (1-\gamma)W^{\lambda}(r)} \\ &= U^{\lambda}(\gamma q + (1-\gamma) r). \end{aligned}$$

Thus we can have neither that $\beta p+(1-\beta)r \succ \gamma q+(1-\gamma)r$ nor that $\beta p+(1-\beta)r \prec \gamma q+(1-\gamma)r$. Hence, $\beta p+(1-\beta)r \asymp \gamma q + (1-\gamma)r$.

(Sufficiency) Suppose $\succ$ is bounded and satisfies (A.1)–(A.4). Then, for every $\psi \in \Psi$ we can construct utility and weight functions by letting $u^{\psi}(x_i) = \alpha_i^{\psi}$ and $w^{\psi}(x_i) = \tau_i^{\psi}$ for $i = 1, \dots, n$. By construction and Propositions 1 and 2, $\alpha_i^{\psi} \in [\underline{\alpha}_i, \overline{\alpha}_i]$ and $\tau_i^{\psi} \in [\underline{\tau}_i, \overline{\tau}_i]$ for every $i = 1, \dots, n$. Therefore, each u^{ψ} and w^{ψ} is bounded, and thus $\mathcal{U} = \{u^{\psi} : \psi \in \Psi\}$ and $\mathcal{W} = \{w^{\psi} : \psi \in \Psi\}$ are closed and bounded. For every $p \in \Delta(X)$, set

$$U^{\psi}(p) = \alpha_p = \frac{\sum_{i=1}^n p_i \tau_i^{\psi} \alpha_i^{\psi}}{\sum_{i=1}^n p_i \tau_i^{\psi}} = \frac{\sum_{i=1}^n p(x_i) w(x_i) u(x_i)}{\sum_{i=1}^n p(x_i) w(x_i)},$$

$$W^{\psi}(p) = \tau_p^{\psi} = \sum_{i=1}^n p_i \tau_i^{\psi} = \sum_{i=1}^n p(x_i) w(x_i).$$

Now suppose that $p \succ q$ then, by Lemma 4, we have that for every $\psi \in \Psi$, $U^{\psi}(p) \neq U^{\psi}(q)$. Suppose that, for some $\psi \in \Psi$, $U^{\psi}(p) < U^{\psi}(q)$, then we can pick some $r \succ p \succ q$ and $\beta \in (0,1)$ such that $U^{\psi}(\beta p + (1-\beta) r) = U^{\psi}(q)$. By Lemma 4, this would imply that $\beta p + (1-\beta) r \asymp q$. But betweenness implies $r \succ \beta p + (1-\beta) r \succ p \succ q$. Thus we must have that $U^{\psi}(p) > U^{\psi}(q)$, for every $\psi \in \Psi$.

Now suppose $U^{\psi}(p) > U^{\psi}(q)$ for every $\psi \in \Psi$. Then, by Lemma 4 $\neg(p \asymp q)$ and by the argument above $\neg(p \prec q)$, so we conclude that $p \succ q$. Hence $p \succ q$ if and only if $U^{\psi}(p) > U^{\psi}(q)$ for every $\psi \in \Psi$. Furthermore, by construction, for every $\psi \in \Psi$, $\alpha_n^{\psi} = 1$ and $\alpha_1^{\psi} = 0$. Hence for every $p \in \Delta(X)$ we have that $u^{\psi}(x_n) > U^{\psi}(p) > u^{\psi}(x_1)$ for every $\psi \in \Psi$. Letting $\mathcal{V} = \{(u^{\psi}, w^{\psi}) : \psi \in \Psi\}$ establishes the representation, and since $\mathcal{U}$ and $\mathcal{W}$ are closed and bounded, $\mathcal{V}$ is as well. □

8.4. Proof of Theorem 2

(Necessity) Suppose that $\langle \hat{\mathcal{V}}^1 \rangle = \langle \hat{\mathcal{V}}^2 \rangle := \mathcal{V}^*$, then for every $p \in \Delta(X)$ we have that $\langle \hat{\mathcal{U}}_p^1 \rangle = \langle \hat{\mathcal{U}}_p^2 \rangle = \{u_p = wu + (1-w)\bar{u}_p : (u, w) \in \mathcal{V}^*\} := \mathcal{U}_p^*$. By Proposition 6, this implies that for any $q \in \Delta(X)$,

$$p \succ^1 q \iff \sum_{x\in X} p(x) u_p(x) > \sum_{x \in X} q(x) u_p(x), \;\; \forall u_p \in \mathcal{U}_p^* \iff p \succ^2 q.$$

This implies that $\succ^1 = \succ^2$.

(Sufficiency) Suppose, without loss of generality, that there is $(u^*, w^*) \in \langle \hat{\mathcal{V}}^1 \rangle \setminus \langle \hat{\mathcal{V}}^2 \rangle$. Then for some $p \in \Delta(X)$ we have that $u_p^* = w^* u^* + (1-w^*)\bar{u}_p^* \in \langle \hat{\mathcal{U}}_p^1 \rangle \setminus \langle \hat{\mathcal{U}}_p^2 \rangle$. Then by the separating hyperplane theorem, there is $q \in \Delta(X)$ such that

$$\sum_{x\in X} [p(x) - q(x)] u_p(x) > 0 \geq \sum_{x\in X} [p(x) - q(x)] u_p^*(x), \;\; \forall u_p \in \langle \hat{\mathcal{U}}_p^2 \rangle$$

This implies on one hand that $\sum_{x\in X} p(x) u_p(x) > \sum_{x\in X} q(x) u_p(x)$ for every $u_p \in \langle \hat{\mathcal{U}}_p^2 \rangle$, so that $p \succ^2 q$, but on the other hand that $\sum_{x\in X} p(x) u_p^*(x) \leq \sum_{x\in X} q(x) u_p^*(x)$, so that as $u_p^* \in \langle \hat{\mathcal{U}}_p^1 \rangle$, we have $\neg(p \succ^1 q)$. Hence, $\succ^1 \neq \succ^2$.

Therefore, we conclude that $\succ^1 = \succ^2$ if and only if $\langle \hat{\mathcal{V}}^1 \rangle = \langle \hat{\mathcal{V}}^2 \rangle$. □

8.5. Proof of Theorem 3

(Necessity) Suppose that $\mathcal{U}$ and w represent $\succ$. Then for every $u^k \in \mathcal{U}$, let

$$U^k(p) = \frac{\sum_{x\in X} p(x) w(x) u^k(x)}{\sum_{x\in X} p(x) w(x)}, \quad W(p) = \sum_{x\in X} p(x) w(x).$$

Letting $\mathcal{V} = \mathcal{U} \times \{w\}$, by Theorem 1 we have that $\succ$ is bounded and (A.1)–(A.3) are satisfied. To show that (A.5) is satisfied, pick $p, q \in \Delta(X)$ such that $p \asymp q$, then there are $u^1, u^2 \in \mathcal{U}$ such that $U^1(p) \geq U^1(q)$ and $U^2(p) \leq U^2(q)$. For every $\beta \in (0,1)$, fix $\tau = \frac{\gamma/(1-\gamma)}{\beta/(1-\beta)} = \frac{W(p)}{W(q)}$, then for every $r \in \Delta(X)$,

$$\begin{aligned} U^1(\beta p + (1-\beta) r) &= \frac{\beta W(p) U^1(p) + (1-\beta) W(r) U^1(r)}{\beta W(p) + (1-\beta) W(r)} \\ &\geq \frac{\beta \frac{W(p)}{W(q)} W(q) U^1(q) + (1-\beta) W(r) U^1(r)}{\beta \frac{W(p)}{W(q)} W(q) + (1-\beta) W(r)} \\ &= \frac{\gamma W(q) U^1(q) + (1-\gamma) W(r) U^1(r)}{\gamma W(q) + (1-\gamma) W(r)} \\ &= U^1(\gamma q + (1-\gamma) r). \end{aligned}$$

Likewise, $U^2(\beta p + (1-\beta) r) \leq U^2(\gamma q + (1-\gamma) r)$, which implies that $\beta p + (1-\beta) r \asymp \gamma q + (1-\gamma) r$. To show that the substitution ratio $\tau = \frac{W(p)}{W(q)}$ is unique let $r = \delta_{x_n}$, then for $\tau' < \frac{W(p)}{W(q)}$ there is $\beta \in (0,1)$ such that $U^k(\beta p + (1-\beta) r) < U^k\left(\frac{\beta \tau' q + (1-\beta) r}{\beta \tau' + (1-\beta)}\right)$ for all $u^k \in \mathcal{U}$, and likewise for $\tau' > \frac{W(p)}{W(q)}$ there is $\beta \in (0,1)$ such that $U^k(\beta p + (1-\beta) r) > U^k\left(\frac{\beta \tau' q + (1-\beta) r}{\beta \tau' + (1-\beta)}\right)$ for all $u^k \in \mathcal{U}$. This implies that τ, and therefore γ, is unique, so that (A.5) is satisfied.

(Sufficiency) Suppose that $\succ$ is bounded and satisfies (A.1)–(A.3), (A.5). Then by Theorem 1 we have a representation by $\mathcal{V} = \{(u^{\psi}, w^{\psi}) : \psi \in \Psi\}$. By Lemma 5, for every $x_i \in X$ there is $\tau_i > 0$ such that $\Phi(\delta_{x_i}) = A(\delta_{x_i}) \times \{\tau_i\}$, and thus $w^{\psi}(x_i) = \tau_i := w(x_i)$ for every $\psi \in \Psi$. Thus letting $\mathcal{U} = \{u^{\psi} : \psi \in \Psi\}$, we have that $\mathcal{V} = \mathcal{U} \times \{w\}$, so that $\succ$ has the desired representation.

This completes the proof. □

8.6. Proof of Theorem 4

(Necessity) Suppose we have u and $\mathcal{W}$ that represent $\succ$. Then for every $w^k \in \mathcal{W}$, let

$$U^k(p) = \frac{\sum_{x\in X} p(x) w^k(x) u(x)}{\sum_{x\in X} p(x) w^k(x)}, \quad W^k(p) = \sum_{x\in X} p(x) w^k(x).$$

Letting $\mathcal{V} = \{u\} \times \mathcal{W}$, by Theorem 1 we have that $\succ$ is bounded and (A.1)–(A.4) are satisfied. To show that (A.6) is satisfied, for every $x_i \in X$ set $\alpha_i = \frac{u(x_i) - u(x_1)}{u(x_n) - u(x_1)}$, so that for every $w^k \in \mathcal{W}$ we have $U^k(\delta_{x_i}) = u(x_i) = \alpha u(x_n) + (1-\alpha) u(x_1) = U(\zeta_\alpha)$ if and only if $\alpha = \alpha_i$.

(Sufficiency) Suppose that $\succ$ is bounded and satisfies (A.1)–(A.4), (A.6). Then by Theorem 1 we have a representation by $\mathcal{V} = \{(u^{\psi}, w^{\psi}) : \psi \in \Phi\}$. By (A.6), for every $x_i \in X$ there is a α_i such that $\Phi(\delta_{x_i}) = \{\alpha_i\} \times T(\delta_{x_i}, \zeta_{\alpha_i})$, and thus $u^{\psi}(x_i) = \alpha_i := u(x_i)$ for every $\psi \in \Psi$. Thus letting $\mathcal{W} = \{w^{\psi} : \psi \in \Psi\}$, we have that $\mathcal{V} = \{u\} \times \mathcal{W}$, so that $\succ$ has the desired representation.

This completes the proof. □

8.7. Proof of Theorem 5

Fix $p \in \Delta(X)$ and suppose that the subject reports $\bar{a} > \overline{\alpha}(p)$. If $a \leq \overline{\alpha}(p)$ or $a > \bar{a}$ then the subject's payoffs are p and ζ_a, respectively, regardless of whether he reports $\bar{a}$ or $\overline{\alpha}(p)$. If $a \in (\overline{\alpha}(p), \bar{a}]$, then the subject's payoff is a choice between the lotteries $p(\theta)$ and the lottery $\zeta_a(\theta)$. Had she reported $\overline{\alpha}(p)$ instead of $\bar{a}$ her payoff would have been ζ_a. By stochastic dominance, $\zeta_a \succ \zeta_a(\theta)$ and, since $a > \overline{\alpha}(p)$, $\zeta_a \succ \zeta_{\overline{\alpha}(p)}$. Thus, by definition of $\overline{\alpha}(p)$, $\zeta_a \succ p(\theta)$. Consequently, the subject is worse off reporting $\bar{a}$ instead of $\overline{\alpha}(p)$.

Suppose that the subject reports $\underline{a} < \underline{\alpha}(p)$. If $a < \underline{a}$ or $a \geq \underline{\alpha}(p)$ the subject's payoffs are p and ζ_a, respectively, regardless of whether she reports $\underline{a}$ or $\underline{\alpha}(p)$. If $a \in [\underline{a}, \underline{\alpha}(p))$, the subject's payoff is a choice between $p(\theta)$ and $\zeta_a(\theta)$. Had she reported $\underline{\alpha}(p)$ instead of $\underline{a}$ her payoff would have been p. By stochastic dominance, $p \succ p(\theta)$, and, by definition of $\underline{\alpha}(p)$, $a < \underline{\alpha}(p)$ implies that $\zeta_{\underline{\alpha}(p)} \succ \zeta_a$. Hence, by definition, $p \succ \zeta_a \succ \zeta_a(\theta)$. Thus, the subject is worse off reporting $\underline{a}$ instead of $\underline{\alpha}(p)$.

Suppose that the subject reports $\bar{a} \in (\underline{\alpha}(p), \overline{\alpha}(p))$. If $a \in (\bar{a}, \bar{\alpha}(p)]$, the subject's payoff is ζ_a, whereas had she reported $\overline{\alpha}(p)$ she would have the opportunity to choose between the lottery $p(\theta)$ and the lottery $\zeta_a(\theta)$. The subject chooses $p(\theta)$ if the signal $(u, w) \in \mathcal{V}$ indicates that $p(\theta) \succ \zeta_a(\theta)$ so that

$$\frac{\sum_{i=1}^{3} p_i u(x_i - \theta) w(x_i - \theta)}{\sum_{i=1}^{3} p_i w(x_i - \theta)} > \frac{au(x_3 - \theta) w(x_3 - \theta) + (1-a) u(x_1 - \theta) w(x_1 - \theta)}{aw(x_3 - \theta) + (1-a) w(x_1 - \theta)}.$$

Otherwise the subject chooses $\zeta_a(\theta)$.

Let $\mathcal{B} = \{(u, w) \in \mathcal{V} \mid p(\theta) \succ \zeta_a(\theta)\}$ then the subject's payoff is

$$\Upsilon(\theta) := \int_{\mathcal{B}} \frac{\sum_{i=1}^{3} p_i u(x_i - \theta) w(x_i - \theta)}{\sum_{i=1}^{3} p_i w(x_i - \theta)} d\mu(u, w) + \int_{\mathcal{V} \setminus \mathcal{B}} \frac{au(x_3 - \theta) w(x_3 - \theta) + (1-a) u(x_1 - \theta) w(x_1 - \theta)}{aw(x_3 - \theta) + (1-a) w(x_1 - \theta)} \times d\mu(u, w).$$

But by definition of $\mathcal{B}$,

$$\Upsilon(0) = \int_{\mathcal{B}} \frac{\sum_{i=1}^{3} p_i u(x_i) w(x_i)}{\sum_{i=1}^{3} p_i w(x_i)} d\mu(u, w) + \int_{\mathcal{V} \setminus \mathcal{B}} \frac{au(x_3) w(x_3) + (1-a) u(x_1) w(x_1)}{aw(x_3) + (1-a) w(x_1)} d\mu(u, w)$$
$$> \int_{\mathcal{B}} \frac{au(x_3) w(x_3) + (1-a) u(x_1) w(x_1)}{aw(x_3) + (1-a) w(x_1)} d\mu(u, w) + \int_{\mathcal{V} \setminus \mathcal{B}} \frac{au(x_3) w(x_3) + (1-a) u(x_1) w(x_1)}{aw(x_3) + (1-a) w(x_1)} d\mu(u, w)$$
$$= \int_{\mathcal{V}} \frac{au(x_3) w(x_3) + (1-a) u(x_1) w(x_1)}{aw(x_3) + (1-a) w(x_1)} d\mu(u, w).$$

Hence, by continuity of $\Upsilon(\theta)$, there is $\varepsilon > 0$ such that, for all $\theta \in [0, \varepsilon)$,

$$\Upsilon(\theta) > \int_{\mathcal{V}} \frac{au(x_3) w(x_3) + (1-a) u(x_1) w(x_1)}{aw(x_3) + (1-a) w(x_1)} d\mu(u, w).$$

Thus, reporting $\bar{a} < u\bar{\alpha}(p)$ is dominated by reporting truthfully, $\bar{a} = \overline{\alpha}(p)$. By similar argument reporting $\underline{a} > \underline{\alpha}(p)$ is dominating by reporting truthfully, $\underline{a} = \underline{\alpha}(p)$. □

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