Weighted Utility Theory with Incomplete Preferences

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August 14, 2019

Abstract

This paper axiomatizes the representations of weighted utility theory with incomplete preferences. These include the general multiple weighted utility representation as well as special cases of multiple utilities or multiple weights only.

Keywords: Incomplete preferences, weighted utility theory, multiple weighted expected utility representation

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‡We are grateful to Zvi Safra and two anonymous referees for their useful comments and suggestions.
1 Introduction

1.1 Motivation and Literature Review

Two of the assumptions of expected utility theory seem less satisfactory than the others, those of completeness and independence. Completeness requires that decision makers are able to compare and express clear preferences between any two risky prospects, while independence requires that decision makers rank prospects only by their distinct characteristics, disregarding their common aspects.

That the completeness axiom may be too demanding was recognized from the outset by von Neumann and Morgenstern (1947) who say that “It is conceivable – and may even in a way be more realistic – to allow for cases where the individual is neither able to state which of two alternatives he prefers nor that they are equally desirable.” Aumann (1962), who was the first to study expected utility theory without the completeness axiom, claims that “Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from a normative viewpoint.” Later studies by Shapley and Bauccelles (1998), Dubra, Maчерonni and Ok (2004) and, most recently, Galaabaatar and Karni (2012) all conclude that the departure from completeness axiom leads to expected multi-utility representations.

Experimental evidence, such as the Allais paradox, motivated developments in the 1980s of theories of decision making under risk that depart from the independence axiom. These theories include Quiggin’s (1982) anticipated utility theory, Chew and MacCrimmon’s (1979) weighted utility theory, Yaari’s (1987) dual theory, Dekel’s (1986) implicit weighted utility, and Gul’s (1991) theory of disappointment aversion.¹

Thus far, the only works that simultaneously depart from both the completeness and independence axioms are Maccheroni (2004) and Safra (2014). Maccheroni (2004) showed that without the completeness axiom the representation of Yaari’s dual theory entails the existence of a set of probability transformation functions such that one risky prospect is preferred over another if and only if its rank-dependent expected value is larger according to every probability transformation function in that set. Safra (2014) studied a general model of decision making under risk that has the betweenness property.² Safra showed that without completeness, the representation entails the existence of a set of continuous functionals displaying betweenness such that one risky prospect is preferred over another if and only if it is assigned a higher value by every element in this set. Weighted utility theory, the subject of this work, also displays the betweenness property but is more structured and therefore calls for a different analysis.

The objective of this paper is to study weighted utility theory without the completeness axiom. Introduced by Chew and MacCrimmon (1979) and Chew (1983, 1989), weighted utility theory is based on a natural weakening of the independence axiom to a ratio substitution property, allowing the outcomes to hold different degrees of salience for the decision maker, captured in the representation by the namesake weight function. Incompleteness in weighted utility theory may thus be the result not only of indecisive tastes, captured by a set of utility functions that rank the outcomes differently, but also of conflicting perceptions of the alternatives presented, captured by a set of weight functions that represent different transformations of the probabilities, or some combination of both. We begin by analyzing the general multiple weighted expected utility model, and follow with the two special cases of multiple utilities paired with a single weight function, or a single utility paired with multiple weights.

1.2 An Informal Review

To set the stage and develop some intuition, we begin with an informal review. Let \( X = \{x_1, \ldots, x_n\} \) be the set of outcomes, and denote the set of lotteries over \( X \) by \( \Delta(X) = \{ p \in \mathbb{R}_+^n : \sum_{x \in X} p(x) = 1 \} \).³ Denote by \( \delta_x \) the degenerate lottery that assigns \( x \in X \) unit probability mass. Let \( \succ \) be a strict preference relation over \( \Delta(X) \), that is, an irreflexive and transitive binary relation which may or may not be negatively transitive. If \( \succ \) is negatively transitive, implying completeness, but violates the independence

¹See Karni and Schmeidler (1991) for a review of this literature.
²Models of decision making under risk with the betweenness property include Chew (1983), Dekel (1986) and Gul (1991).
³For each \( p, q \in \Delta(X) \) and \( \alpha \in [0, 1] \), define \( \alpha p + (1 - \alpha)q \in \Delta(X) \) by \( (\alpha p + (1 - \alpha)q)(x) = \alpha p(x) + (1 - \alpha)q(x) \) for all \( x \in X \). Then \( \Delta(X) \) is a convex subset of the linear space \( \mathbb{R}^n \).
axiom and instead satisfies only the weaker substitution axiom of Chew (1989), then there exists a utility function \( u \) and a positive valued weight function \( w \) mapping \( X \) to \( \mathbb{R} \), such that, for all \( p, q \in \Delta (X) \),

\[
p \succ q \iff \sum_{x \in X} p(x)w(x)u(x) > \sum_{x \in X} q(x)w(x)u(x).
\]

(1)

For example, if \( n = 3 \) and \( \delta_{x_3} \succ \delta_{x_2} \succ \delta_{x_1} \), the indifference map induced by (1) is depicted in Figure 1 below. The indifference curves all emanate from a source point \( o \) lying outside the simplex.

Figure 1: Weighted Utility

Figure 1 depicts a decision maker that attaches greater weight to the extreme outcomes \( x_1 \) and \( x_3 \) than to the median outcome \( x_2 \), indicating that the former have more influence on his evaluation of any particular lottery than their probability would justify. The degree of risk aversion would vary across the simplex and thus the decision maker would exhibit Allais-type behavior, being willing to take risks when he feels he has nothing to lose that he would otherwise avoid if his alternatives were more attractive. The extent of this distortion depends on the proximity of the source point \( o \) to the simplex and as it is moved farther away from the diagram, approaches the parallel indifference map of expected utility.

Now suppose that \( \succ \) is also incomplete. As in multiple expected utility models with independence such as those of Dubra, Maccheroni, and Ok (2004), and Galaabaatar and Karni (2012), the preference relation cannot be meaningfully characterized with indifference curves, as two lotteries that are not strictly comparable are not necessarily equivalent. Consider the lottery \( p \) in Figure 2 below, let \( B(p) = \{ r \in \Delta (X) : r \succ p \} \) and \( W(p) = \{ r \in \Delta (X) : r \preceq p \} \) respectively denote the upper and lower contour sets of \( p \), and observe that they are demarcated by rays emanating from a pair of distinct source points \( o^1 \) and \( o^2 \). Unlike in classic weighted utility theory, these rays are not indifference curves, but indicate only that no two lotteries lying on a single ray are strictly comparable, a relation which is not transitive and hence not an equivalence relation.

Nevertheless these incomparability curves do inherit many of the properties of indifference curves from the weighted utility setup. As each set of such curves converges at a source point, each in turn has a weighted linear utility representation as in (1), with the two sources \( o^1 \) and \( o^2 \) respectively corresponding to utility and weight pairs \( (u^1, w) \) and \( (u^2, w) \). As the diagram indicates, for any lottery to be strictly
preferred to \( p \) it must lie above both of the incomparability curves intersecting \( p \), and thus the preference relation has a multiple weighted expected utility representation, with a set of utilities \( U = \{ u^1, u^2 \} \).

\[
P \succ q \iff \sum_{x \in X} p(x)w(x)u(x) > \sum_{x \in X} q(x)w(x)u(x), \ \forall u \in U. \tag{2}
\]

Two other aspects of this setup are noteworthy. Firstly, every point on the line segment connecting \( o^1 \) and \( o^2 \) also projects a set of incomparability curves, always lying between the rays projected by the two endpoints. Any such point \( o^\kappa \) would thus also be a source point and correspond to some utility \( u^\kappa \), which could be included within \( U \) without altering the preference relation it represents. The location of \( o^\kappa \) between \( o^1 \) and \( o^2 \) implies that \( u^\kappa \) would be some convex combination of \( u^1 \) and \( u^2 \), and thus could not contradict any ordering jointly established by these utilities. This leads us to conclude that, just as in multiple expected utility models with independence, the representation will only be unique up to some closed convex hull, though as the utilities here are not linear we will need to adopt a slightly different approach to establish this result.

Secondly, the line segment connecting the source points \( o^1 \) and \( o^2 \) is parallel to that connecting the best and worst outcomes \( \delta_x^2 \) and \( \delta_x^1 \). Hence these sources are equidistant from the simplex and represent different utilities paired with the same weight function. As the incomparability curves projected from \( o^1 \) are everywhere steeper than those projected from \( o^2 \), \( u^1 \) is uniformly more risk averse than \( u^2 \). This naturally leads us to consider the dual case, where a single utility function might be paired with multiple weight functions.

Figure 3(a) depicts such a case, where there are a pair of source points \( o^1 \) and \( o^2 \) corresponding to utility-weight pairs \((u, w^1)\) and \((u, w^2)\). Here the utility functions are identical, as the incomparability curves drawn from both sources through \( \delta_x^2 \) coincide and thus rank the median outcome identically, but as \( o^2 \) is closer to the simplex, \( w^2 \) represents a greater deviation from the uniform weights of expected utility theory. Figure 3(b) depicts a similar case, where there are again two sources \( o^1 \) and \( o^2 \), and two corresponding utility-weight pairs \((u, w^1)\) and \((u, w^2)\), but here \( o^2 \) is located on the other side of the simplex. This produces incomparability curves that fan in rather than out, and indicating that \( x_2 \) is weighted more heavily than the extreme outcomes, rather than less. The preferences depicted in either incomparability map would have a representation consisting of the single utility \( u \) and multiple weights.
$W = \{w^1, w^2\}$.

$$p \succ q \iff \frac{\sum_{x \in X} p(x)w(x)u(x)}{\sum_{x \in X} p(x)w(x)} > \frac{\sum_{x \in X} q(x)w(x)u(x)}{\sum_{x \in X} q(x)w(x)}, \forall w \in W. \quad (3)$$

Note that in the multiple weight case depicted in Figure 3(a), analogously to the multiple utility case depicted in Figure 2, we may include in $W$ the weight function $w^\kappa$ corresponding to any point $o^\kappa$ on the line segment connecting $o^1$ and $o^2$ without altering the preferences. However, attempting the same in Figure 3(b) would be invalid, as it would produce source points lying within the simplex. In this case, we can instead include any source points lying on the line defined by $o^1$ and $o^2$ but not on the segment connecting them, effectively connecting $o^1$ to $o^2$ through the point at infinity, as any of these would produce incomparability curves that lie between those projected from the endpoints and hence their inclusion would not alter the representation.

Finally, we consider the general case that incorporates both multiple utilities and multiple weights, as depicted in Figure 4. Here the four source points $\Omega = \{o^{1,1}, o^{1,2}, o^{2,1}, o^{2,2}\}$ correspond to pairs of utility and weight functions $\mathcal{V} = \{(u^1, w^1), (u^1, w^2), (u^2, w^1), (u^2, w^2)\}$ and the preferences depicted have the representation

$$p \succ q \iff \frac{\sum_{x \in X} p(x)w(x)u(x)}{\sum_{x \in X} p(x)w(x)} > \frac{\sum_{x \in X} q(x)w(x)u(x)}{\sum_{x \in X} q(x)w(x)}, \forall (u, w) \in \mathcal{V}. \quad (4)$$

Here the set of utility-weight pairs is separable, as $\mathcal{V} = \mathcal{U} \times \mathcal{W} = \{u^1, w^2\} \times \{w^1, w^2\}$, though this need not be the case generally. Any point lying in the convex hull of $\Omega$ would map to a utility-weight pair that could be included in $\mathcal{V}$ without altering the preferences represented. Therefore, this representation admits any of the models considered so far as special cases, with the single utility or single weight cases in (2) and (3) if respectively $\mathcal{U}$ or $\mathcal{W}$ are singletons, weighted utility if $\mathcal{V}$ is a singleton, multiple expected utility if every element of $\mathcal{W}$ is a constant function, and finally expected utility if all of these hold.

The next section introduces the basic model. Section 3 details the general multiple weighted expected utility model, with the special cases of a single utility or single weight covered in section 4. Concluding remarks appear in section 5 and the proofs are collected in section 6.
2 Analytical Framework

2.1 Preference Structure

Let $X = \{x_1, \ldots, x_n\}$ be a set of outcomes and $\Delta(X) = \{p \in \mathbb{R}_+^n : \sum_{x \in X} p(x) = 1\}$ the set of lotteries over $X$. Let $\succ$ be a binary relation on $\Delta(X)$, which we refer to as a strict preference relation. The set $\Delta(X)$ is said to be $\succ$-bounded if there are best and worst outcomes $\hat{x}, \underline{x} \in X$ such that $\hat{x} \succ p \succ \underline{x}$, for all $p \in \Delta(X) \setminus \{\hat{x}, \underline{x}\}$, which we assume throughout.\footnote{Generally speaking, $\Delta(X)$ is $\succ$-bounded if there are $\hat{p}, \underline{p} \in \Delta(X)$ such that $\hat{p} \succ p \succ \underline{p}$ for all $p \in \Delta(X) \setminus \{\hat{p}, \underline{p}\}$. However, anticipating the monotonicity of the strict preference relation described below, there is no essential loss in our definition.}

Number the elements in $X$ in nondecreasing order of preference, so that $x_1 = x$ and $x_n = \underline{x}$.\footnote{We may still define weak preference and indifference relations that have the usual properties by following Galabaaatar and Karni (2013) and letting $p \succeq q$ if $r \succ p$ implies $r \succ q$, and $p \sim q$ if $p \succeq q$ and $q \succeq p$.}

If the strict preference relation $\succ$ is negatively transitive, then its negation $\neg(p \succ q)$ defines the complete and transitive weak preference relation $p \preceq q$.\footnote{See Dekel (1986) for an example and Chew (1989) for a review of this class of models.}

Mathematically, the inability to rank a pair of alternatives does not necessarily mean that the decision maker considers them to be identical, but rather may imply that he evaluates them by multiple criteria that disagree on their ranking.

We assume throughout that $\succ$ is a continuous strict partial order.

(A.1) (Strict Partial Order) The preference relation $\succ$ is irreflexive and transitive.

The following axiom is a slight strengthening of the usual Archimedean axiom, disposing with the requirement that a strict ranking $p \succ q \succ r$ exists, anticipating the possibility that two of these lotteries may be incomparable instead.

(A.2) (Strong Archimedean) For all $p, q, r \in \Delta(X)$ if $p \succ q$ then there is $\alpha \in (0, 1)$ such that $\alpha p + (1 - \alpha) r \succ q$ for all and if $p \prec q$, there is $\alpha' \in (0, 1)$ such that $\alpha' p + (1 - \alpha') r \prec q$.

The next axiom asserts that a probability mixture of two lotteries must rank between them. It characterizes a class of models including expected, weighted, and implicit weighted utility theory.\footnote{We may still define weak preference and indifference relations that have the usual properties by following Galabaaatar and Karni (2013) and letting $p \succeq q$ if $r \succ p$ implies $r \succ q$, and $p \sim q$ if $p \succeq q$ and $q \succeq p$.}
(A.3) (Betweenness) For all \( p, q \in \Delta(X) \) and \( \alpha \in (0,1) \), \( p \succ q \) if and only if \( p \succ \alpha p + (1-\alpha)q \succ q \).

For every \( \alpha \in [0,1] \), let \( \varsigma_\alpha := \alpha \varsigma_x + (1-\alpha)\delta_x \). For every \( p \in \Delta(X) \), let \( A(p) = \{ \alpha \in [0,1] : p \succ \varsigma_\alpha \} \) denote the range of utility values assigned to \( p \), measured along the line connecting the best and worst outcomes. The following proposition establishes that each of these utility ranges is a closed interval.

**Proposition 1** For all \( p \in \Delta(X) \), there are \( \underline{\alpha}, \overline{\alpha} \in [0,1] \) such that \( A(p) = [\underline{\alpha}, \overline{\alpha}] \).

In the standard expected utility and multi-utility models, applying the independence axiom at this step produces the desired utility representation.

### 2.2 Partial Substitution

At the core of weighted utility theory is the weak substitution axiom that replaces the independence axiom,\(^7\) which can be equivalently expressed as a ratio substitution property.

#### (Weak Substitution)

For all \( p, q \in \Delta(X) \), \( p \sim q \) if and only if for every \( \beta \in (0,1) \) there is \( \gamma \in (0,1) \) such that \( \beta p + (1-\beta) \gamma q \sim (1-\gamma) p + \gamma q \) for all \( r \in \Delta(X) \).

#### (Ratio Substitution)

For all \( p, q \in \Delta(X) \), \( p \sim q \) if and only if there is \( \tau > 0 \) such that for every \( \beta \in (0,1) \), \( \beta p + (1-\beta) r \sim \frac{\beta r + (1-\beta) \tau}{\beta \tau + (1-\beta)} \) for all \( r \in \Delta(X) \).

That these are equivalent can be shown by setting \( \tau = \frac{1-\gamma}{\beta(1-\gamma)} \), and interpreting this odds ratio as the weight of \( p \) relative to that of \( q \). If \( \succ \) is complete, a weighted linear utility function can thus be obtained by finding, for each \( x \in X \), the unique \( \alpha_\beta \) such that \( \delta_x \sim \varsigma_\alpha \), and \( \gamma \), satisfying ratio substitution between these two lotteries, and for any \( p \in \Delta(X) \) repeatedly applying weak substitution to obtain

\[
p := \sum_{i=1}^n \frac{p_i \delta_{x_i} p_{i1} \varsigma_{\alpha_{1i}} + \sum_{j=2}^n \frac{p_i \delta_{x_i} p_{ij} \varsigma_{\alpha_{ji}}}{p_{i1} \tau_1 + \sum_{j=2}^n p_{ij} \tau_j} \sim \ldots \sim \sum_{i=1}^n \frac{p_i \delta_{x_i} p_{i1} \varsigma_{\alpha_{1i}}}{p_{i1} \tau_1 + \sum_{j=1}^n p_{ij} \tau_j} = \frac{\sum_{i=1}^n p_i \delta_{x_i} p_{i1} \varsigma_{\alpha_{1i}}}{\sum_{i=1}^n p_i \tau_i} := \varsigma_{\alpha_p}.
\] (5)

By betweenness, the above implies that for any \( p, q \in \Delta(X) \), \( p \succ q \Leftrightarrow \alpha_p > \alpha_q \), so that we obtain a weighted utility representation by setting \( u(x_i) = \alpha_i \) and \( u(x_i) = \tau_i \) for \( i = 1, \ldots, n \). The critical step in this construction lies in exploiting the transitivity of the indiscernibility relation \( \sim \).

For preferences \( \succ \) that are not necessarily complete, we consider a modification that replaces the intransitivity relation \( \succ \) with the incomparability relation \( \succsim \).

#### (Partial Substitution)

For all \( p, q \in \Delta(X) \), \( p \succsim q \) if and only if for every \( \beta \in (0,1) \) there is \( \gamma \in (0,1) \) such that \( \beta p + (1-\beta) r \succeq \gamma q + (1-\gamma) r \) for all \( r \in \Delta(X) \).

The next lemma establishes the analogous ratio substitution property in our setup.

**Lemma 1** If \( \succsim \) satisfies (A.1)-(A.4), then for all \( p, q \in \Delta(X) \), \( p \succsim q \) if and only if there is \( \tau > 0 \) such that for every \( \beta \in (0,1) \), \( \beta p + (1-\beta) r \succeq \frac{\beta \tau + (1-\beta) \gamma}{\beta \tau + (1-\beta)} r \) for all \( r \in \Delta(X) \).

For any pair of incomparable lotteries, define the set of substitution odds ratios as

\[
T(p,q) = \left\{ \tau > 0 : \beta p + (1-\beta) r \succeq \frac{\beta \tau + (1-\beta) r}{\beta \tau + (1-\beta)} r, \quad \forall \beta \in (0,1), r \in \Delta(X) \right\}.
\]

By Lemma 1, \( T(p,q) \neq \emptyset \) if and only if \( \succsim \). Under completeness, weak substitution implies that for every \( \beta \in (0,1) \) we have a unique \( \gamma \in (0,1) \), which can be seen by picking any \( r \succ q \) and applying betweenness. Therefore the odds ratio \( \tau \) must be unique as well. This is not the case here however, as \( \succsim \) is intransitive and hence \( T(p,q) \) is not necessarily a singleton. Consequently, lotteries may have a range of weights in addition to a range of utility values.

**Proposition 2** For all \( p, q \in \Delta(X) \) such that \( p \succsim q \), there are \( \underline{\tau}, \overline{\tau} > 0 \) such that \( T(p,q) = [\underline{\tau}, \overline{\tau}] \).

We can now attempt to replicate the construction of the utility representation as in (5). For every \( i = 1, \ldots, n \), consider picking some \( \alpha_i \in A(\delta_{x_i}) \) and \( \tau_i \in T(\delta_{x_i}, \varsigma_{\alpha_i}) \), and then repeatedly applying

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\(^7\)See Chew (1989).
partial substitution to yield
\[
p := \sum_{i=1}^{n} p_i \delta_{x_i} = \frac{p_1 \tau_1 \alpha_1 + \sum_{i=2}^{n} p_i \delta_{x_i}}{p_1 \tau_1} + \sum_{i=1}^{n} \frac{p_i \tau_i \alpha_i}{\sum_{i=1}^{n} p_i \tau_i} = \frac{\sum_{i=1}^{n} p_i \tau_i \alpha_i}{\sum_{i=1}^{n} p_i \tau_i} := \alpha_p.
\]
However, as \( \succ \) is intransitive, (6) does not necessarily imply that \( p \succ \alpha_p \). Intuitively, if \( \succ \) is negatively transitive, then every \( \alpha_i \) and \( \tau_i \) is unique, so that we can obtain for any \( p \) the unique \( \alpha_p \) by simply taking the weighted convex combination as in (5). Under incompleteness, while we know that each \( x_i \in X \) has utility range \( A(\delta_{x_i}) \), if we arbitrarily select \( \{\alpha_1, \ldots, \alpha_n\} \in \prod_{i=1}^{n} A(\delta_{x_i}) \), these values need not be assigned by the same utility function, and hence the \( \alpha_p \) produced by (6) need not belong to \( A(p) \). It is the converse, that every \( \alpha_p \in A(p) \) can be constructed from some \( \{\alpha_1, \ldots, \alpha_n\} \in \prod_{i=1}^{n} A(\delta_{x_i}) \), that we need to show to ensure that the preference relation can be represented by a set of weighted linear utility functions.

2.3 Source Space

As discussed in the introduction, a preference relation with a multiple weighted expected utility representation can be visualized as a set of lotteries with incomparability curves projected from a set of source points lying outside the simplex. Suppose we have \( p, q \in \Delta(X) \) such that \( p \succeq q \), and some \( \tau \in T(p, q) \). By definition, for every \( \beta \in (0, 1) \) and \( r \in \Delta(X) \), the line defined by \( \beta p + (1 - \beta) r \) and \( \frac{\beta p +(1-\beta) r }{\beta + (1-\beta)} \) is an incomparability curve. All of these curves converge at some source point \( o \). As its location can depend on neither \( \beta \) nor \( r \), we have that
\[
o = \frac{1}{\beta(1-\tau)}[\beta p + (1-\beta) r] - \frac{\beta \tau + (1-\beta) r}{\beta(1-\tau)} = \frac{p - \tau q}{1 - \tau}.
\]
Define the source space \( \Omega \) as the collection of all such source points.
\[
\Omega = \left\{ o = \frac{p - \tau q}{1 - \tau} : p \succeq q, \tau \in T(p, q) \right\}
\]
The following proposition asserts that \( \Omega \) fully characterizes the incomparability relation \( \succ \) and, consequently, the preference relation \( \succ \) as well. It states that any line connecting two lotteries is an incomparability curve if and only if it is projected from some source point \( o \in \Omega \).

**Proposition 3** For every \( p, q \in \Delta(X) \), \( p \succeq q \) if and only if there is \( \tau \in T(p, q) \) such that \( o = \frac{p - \tau q}{1 - \tau} \in \Omega \).

For each \( p \) define \( \Phi(p) = \{ (\alpha_p, \tau_p) : \alpha_p \in A(p), \tau_p \in T(p, \alpha_p) \} \) as the collection of utility-weight pairs, each defining a source point \( o_p = \frac{p - \tau_p \alpha_p}{1 - \tau_p} \in \Omega \).

A utility function over \( X \) is given by a collection of utility weight pairs \( \{(\alpha_i, \tau_i)\}_{i=1}^{n} \) corresponding to each of the degenerate lotteries \( \{\delta_{x_i}\}_{i=1}^{n} \) such that \( (\alpha_p, \tau_p) = \left( \sum_{i=1}^{n} p_i \tau_i \alpha_i, \sum_{i=1}^{n} p_i \tau_i \right) \in \Phi(p) \) for any lottery \( p \in \Delta(X) \). Define the set of all such collections as
\[
\Psi = \left\{ \{(\alpha_i, \tau_i)\}_{i=1}^{n} : (\alpha_p, \tau_p) = \left( \frac{\sum_{i=1}^{n} p_i \tau_i \alpha_i, \sum_{i=1}^{n} p_i \tau_i}{\sum_{i=1}^{n} p_i \tau_i \alpha_i}, \sum_{i=1}^{n} p_i \tau_i \right) \in \Phi(p), \forall p \in \Delta(X) \right\}.
\]

Every \( \psi = \{(\alpha_i, \tau_i^\psi)\}_{i=1}^{n} \in \Psi \) defines a weighted linear utility function. Letting \( o_i^\psi = \frac{\delta_{x_i} - \tau_i^\psi \delta_{x_i}}{1 - \tau_i^\psi} \in \Omega \) denote the source point that \( \psi \) associates with outcome \( x_i \), we have that for every \( p \in \Delta(X) \),
\[
o_i^\psi = \frac{p - \tau_i^\psi \delta_{x_i}}{1 - \tau_i^\psi} = \frac{\sum_{i=1}^{n} p_i (1 - \tau_i^\psi) o_i^\psi}{\sum_{i=1}^{n} p_i (1 - \tau_i^\psi)} \in \Omega.
\]
That is, each source point associated with any lottery \( p \) is also a weighted convex combination of elements.
of \( \{ a_p \}^n_{i=1}. \) Collecting these points forms a subset of the source space, \( O_\psi = \{ a_p \}^n_{p \in \Delta(X)} \subseteq \Omega \) which characterizes a function pair \( (u^p, w^p) \). Therefore, to establish the representation theorem, we need to show that the collection of these subsets covers the source space \( \bigcup_{p \in \psi} O_\psi = \Omega \). In other words, the collection of function pairs \( \Psi \) fully characterizes the incomparability map and, by extension, the preference relation \( \succ \) itself.

3 Representation

3.1 Existence

Before presenting the main theorem, we first establish some preliminary results. For any collection \( P \subseteq \Delta(X) \), let \( L(P) \) be the linear space spanned by the lotteries in \( P \), and define \( \Delta(P) = L(P) \cap \Delta(X) = \{ g = \sum_{p \in P} \pi_p p : \sum_{p \in P} \pi_p = 1 \} \) as the set of reduced compound lotteries over \( P \). Any collection \( P \subseteq \Delta(X) \) of lotteries constitutes an incomparability set if \( p \succ q \) for any \( p, q \in \Delta(P) \), thus forming the natural higher dimensional analogue to the incomparability curves encountered so far. As \( \delta_{x_r} > \delta_x \) implies that \( \succ \) is non-empty, even the maximal incomparability set cannot span the entire simplex, so that \( \Delta(P) \subseteq \Delta(X) \) and is at most of dimension \( n - 2. \)

Suppose we start with any single lottery \( p \), then we can find some other lottery \( q \) to which it is incomparable \( p \approx q \). Applying the partial substitution axiom, this relation implies the existence of a set of incomparability curves converging at a source point \( o \), which is collinear with \( p \) and \( q \). If \( p \approx q \), then the line defined by this pair of lotteries \( \Delta(\{ p, q \}) \) is an incomparability curve originating from \( o \). If \( n = 3 \), then each source point \( o \) characterizes a weighted utility function and we can construct our representation by locating all of them. However if \( n > 3 \), then constructing a weighted utility requires stringing together a number of source points to form a structure that projects higher dimensional incomparability sets. Lemma 2 shows that if \( p \approx q \) and \( n > 3 \), we can find some \( r \in \Delta(X) \setminus \Delta(\{ p, q \}) \) that is incomparable to both \( p \) and \( q \) as well as any lottery in \( \Delta(\{ p, q \}) \), so that \( \Delta((p, q, r)) \) defines an incomparability plane.

Lemma 2 If \( \succ \) satisfies (A.1)-(A.4) and \( n > 3 \), then for all \( p, q \in \Delta(X) \), the following statements are equivalent:

(i) \( p \approx q \).
(ii) There exists \( r \in \Delta(X) \setminus \Delta(\{ p, q \}) \) such that \( \lambda p + (1 - \lambda)q \equiv r \) for all \( \lambda \in [0, 1] \).
(iii) There exists \( r \in \Delta(X) \setminus \Delta(\{ p, q \}) \) and \( \tau_p, \tau_q > 0 \) such that for all \( \lambda \in [0, 1] \) and \( \beta \in (0, 1) \),

\[
\beta[\lambda p + (1 - \lambda)q] + (1 - \beta)s \equiv \frac{\beta[\lambda p + (1 - \lambda)q]r + (1 - \beta)s}{\beta[\lambda p + (1 - \lambda)q]r + (1 - \beta)s}
\]

for all \( s \in \Delta(X) \).
(iv) There exists \( r \in \Delta(X) \setminus \Delta(\{ p, q \}) \) such that \( p' \approx q' \) for all \( p', q' \in \Delta((p, q, r)) \).

By Lemma 2, every source point that projects a set of incomparability curves lies on a line on which every point is a source point. Such a source line in turn projects a set of incomparability planes. The natural next step is to generalize this property, allowing us to construct a set of source points that will fully characterize a utility function.

Lemma 3 If \( \succ \) satisfies (A.1)-(A.4), then for all \( P \subseteq \Delta(X) \) such that \( \dim \Delta(P) < n - 2 \), the following statements are equivalent:

By definition of \( \psi \), we have that:

\[
o_\psi = \frac{p - \tau_x^y \psi}{1 - \tau_y^x} = \frac{\sum_{i=1}^n \pi_i \delta_{x_i} - (\sum_{i=1}^n \pi_i \tau^y_{\psi}) \psi}{\sum_{i=1}^n \pi_i \tau^y_{\psi}} = \frac{\sum_{i=1}^n \pi_i \delta_{x_i} - (\sum_{i=1}^n \pi_i \tau^y_{\psi}) \left( \frac{\sum_{i=1}^n \pi_i \tau^y_{\psi}}{\sum_{i=1}^n \pi_i \tau^y_{\psi}} \right)}{\left( \sum_{i=1}^n \pi_i \tau^y_{\psi} \right)} = \frac{\sum_{i=1}^n \pi_i \delta_{x_i} - (\sum_{i=1}^n \pi_i \tau^y_{\psi}) \psi}{\sum_{i=1}^n \pi_i \delta_{x_i} - (\sum_{i=1}^n \pi_i \tau^y_{\psi}) \left( \frac{\sum_{i=1}^n \pi_i \tau^y_{\psi}}{\sum_{i=1}^n \pi_i \tau^y_{\psi}} \right)} = \frac{\sum_{i=1}^n \pi_i \delta_{x_i} - (\sum_{i=1}^n \pi_i \tau^y_{\psi}) \psi}{\sum_{i=1}^n \pi_i \delta_{x_i} - (\sum_{i=1}^n \pi_i \tau^y_{\psi}) \left( \frac{\sum_{i=1}^n \pi_i \tau^y_{\psi}}{\sum_{i=1}^n \pi_i \tau^y_{\psi}} \right)} = \frac{\sum_{i=1}^n \pi_i \delta_{x_i} - (\sum_{i=1}^n \pi_i \tau^y_{\psi}) \psi}{\sum_{i=1}^n \pi_i \delta_{x_i} - (\sum_{i=1}^n \pi_i \tau^y_{\psi}) \left( \frac{\sum_{i=1}^n \pi_i \tau^y_{\psi}}{\sum_{i=1}^n \pi_i \tau^y_{\psi}} \right)} = \frac{\sum_{i=1}^n \pi_i (1 - \tau^y_{\psi}) \psi}{\sum_{i=1}^n \pi_i (1 - \tau^y_{\psi}) \left( \frac{\sum_{i=1}^n \pi_i \tau^y_{\psi}}{\sum_{i=1}^n \pi_i \tau^y_{\psi}} \right)} \in \Omega.
\]

Recall that \( p = (p(x_1), \ldots, p(x_n)) \in \mathbb{R}^n \).

As \( \sum_{i=1}^n \pi_i (x_i) = 1 \) for every \( p \in \Delta(X) \), \( \dim \Delta(X) = n - 1 \), and if \( \Delta(P) \subseteq \Delta(X) \), \( \dim \Delta(P) < \dim \Delta(X) \).

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(i) \( p \succ q \) for all \( p, q \in \Delta(P) \).

(ii) There exists \( r \in \Delta(X) \setminus \Delta(P) \) such that \( p \succ r \), for every \( p \in \Delta(P) \).

(iii) There exists \( r \in \Delta(X) \setminus \Delta(P) \) and \( \{\tau_p\}_{p \in P} \subseteq \mathbb{R}_{++} \) such that for all \( \{\tau_p\}_{p \in P} \subseteq \mathbb{R}_+ \) such that 

\[
\sum_{p \in P} \tau_p = 1, \text{ and } \beta \in (0, 1), \beta \left( \sum_{p \in P} \tau_pp \right) + (1 - \beta)s = \frac{\beta(\sum_{p \in P} \tau_pp) + (1 - \beta)s}{\beta(\sum_{p \in P} \tau_pp) + (1 - \beta)s}, \text{ for all } s \in \Delta(X).
\]

(iv) There exists \( r \in \Delta(X) \setminus \Delta(P) \) such that \( p' \succ q' \), for all \( p', q' \in \Delta(P \cup \{r\}) \).

Starting with any pair of incomparable lotteries, we can repeatedly apply Lemma 3 to add incomparable lotteries until, after a finite number of repetitions, we obtain a maximal incomparability set \( P \) of dimension \( n - 2 \). The following lemma shows that every such \( P \) maps to some utility function generated by a \( \psi \in \Psi \).

**Lemma 4** If \( \succ \) satisfies (A.1)-(A.4), then for all \( p, q \in \Delta(X) \), \( p \succ q \) if and only if there is \( \psi \in \Psi \) such that 

\[
\sum_{x \in X} \frac{p(x)w(x)u(x)}{\sum_{x \in X} p(x)w(x)} > \sum_{x \in X} \frac{q(x)w(x)u(x)}{\sum_{x \in X} q(x)w(x)}, \quad \forall (u, w) \in \mathcal{V}.
\]

3.2 Uniqueness

Having established the existence of a utility representation, we now turn our attention to the question of uniqueness. As our model lies at the convergence of weighted utility and expected multi-utility theory, our uniqueness result naturally incorporates elements of the uniqueness results found in both. From weighted utility theory, we know that taking an affine transformation of a utility function \( u \) will not preserve its weighted linearity, and we must instead apply a rational affine transformation to both \( u \) and the associated weight function \( w \) jointly.

**Proposition 4** For utility and weight function \((u, w)\) and constants \(a, b, c, d\) such that \(ad > bc\), define the rational affine transformation \((\tilde{u}, \tilde{w}) = \left(\frac{au + b}{cu + d}, w[au + d]\right)\). Then for every \( p, q \in \Delta(X) \),

\[
\sum_{x \in X} p(x)w(x)u(x) > \sum_{x \in X} q(x)w(x)u(x) \iff \sum_{x \in X} p(x)\tilde{w}(x)\tilde{u}(x) > \sum_{x \in X} q(x)\tilde{w}(x)\tilde{u}(x).
\]

The utility functions we construct from \( \Psi \) are normalized, with \((u^v(x_1), w^v(x_1)) = (0, 1)\) for the worst outcome and \((u^v(x_a), w^v(x_a)) = (1, 1)\) for the best, for every \( \psi \in \Psi \). The following proposition shows that any function pair \((u, w)\) has a rational affine transformation \((\tilde{u}, \tilde{w})\) that is similarly normalized, which in turn maps to some collection \( \psi \in \Psi \).

**Proposition 5** Let the collection \( \mathcal{V} \) represent \( \succ \). Then for every \((u, w) \in \mathcal{V}\) there is a normalized rational affine transformation \((\tilde{u}, \tilde{w})\) such that \(\{(\tilde{u}(x_i), \tilde{w}(x_i))\}_{i=1}^n \in \Psi\).

For any \( \mathcal{V} \), define the normalized set as \( \hat{\mathcal{V}} = \{ (\tilde{u}, \tilde{w}) : (u, w) \in \mathcal{V} \} \). In expected multi-utility theory, if we have a set \( \hat{U} \) of normalized utilities, some elements may merely be convex combinations of others and may be excluded without altering the preferences that \( \hat{U} \) represents. A similar idea exists here, though we cannot simply take convex combinations of weighted linear functions that simultaneously preserve
weighted linearity while maintaining the preference ordering.\textsuperscript{11} To overcome this difficulty, we focus on a neighborhood around some \( p \in \Delta(X) \) and invoke the notion of local utility functions. For every \((u^k, w^k) \in \hat{V}\), let \( \hat{u}_p^k := \frac{\sum_{x \in X} p(x)u^k(x)w^k(x)}{\hat{w}_p^k} \) and \( \bar{u}_p^k := \sum_{x \in X} p(x)w^k(x) \), and, following Machina (1982), define the local utility function induced \( \hat{u}_p^k : X \mapsto \mathbb{R} \) by the weighted utility functional as
\[
\hat{u}_p^k := \frac{w^k(u^k - \bar{u}_p^k)}{\hat{w}_p^k}.
\]
As linear utilities are invariant up to a positive affine transformation, let
\[
u^k_p = \hat{u}_p^k u_p^k + \bar{u}_p^k = w^k u^k + (1 - w^k) \bar{u}_p^k.
\]
Then \( \nu^k_p \) is a normalized linear approximation of \((u^k, w^k)\) around \( p \), so that taking the expectation of \( \nu^k_p \) at \( p \) gives \( \nu^k_p \).\textsuperscript{12}

Collecting all such \( \nu^k_p \) gives a set \( \hat{U}_p = \{ \nu^k_p : (u^k, w^k) \in \hat{V}\} \) of local utility functions that form a multiple expected utility representation that allows us to rank \( p \) against any other lottery. Furthermore, following Dubra, Maccheroni, and Ok (2004), define \( \bar{U}_p = \text{cl}\{\nu^k_p = \sum_{u^k \in \hat{U}_p} \pi^k u_p^k : \sum_{u^k \in \hat{U}_p} \pi^k = 1\} \) as the closure of the convex hull of the set of local utilities.\textsuperscript{13} As \( \bar{U}_p \) consists of linear utility functions, it is easy to verify that \( \bar{U}_p \) identically ranks \( p \) against other lotteries.

**Proposition 6** For every \( p \in \Delta(X) \) there is a closed convex set \( \bar{U}_p \) of utilities \( u_p : X \mapsto \mathbb{R} \) such that for every \( q \in \Delta(X) \),
\[
p \succ q \iff \sum_{x \in X} p(x)u_p(x) > \sum_{x \in X} q(x)u_p(x), \ \forall \nu_p \in \bar{U}_p.
\]

As each \( \bar{U}_p \) fully characterizes \( \succ \) in a neighborhood around \( p \), we can define the set of utility and weight pairs \((u, w)\) that map to a local utility in \( \bar{U}_p \) around every \( p \) as \( \hat{V} = \{(u^k, w^k) : \nu^k_p \in \hat{U}_p, \ \forall \ p \in \Delta(X)\} \). We refer to \( \hat{V} \) as the exhaustible set of normalized utility and weight pairs that everywhere agree with the ordering of lotteries prescribed by \( \succ \). The uniqueness theorem presented below asserts that two utility representations are identical if and only if they are characterized by identical exhaustive sets.

**Theorem 2** For \( j = 1, 2 \), let \( \succ^j \) be a binary relation over \( \Delta(X) \) that has a multiple weighted expected utility representation by a set \( \mathcal{V}^j \) of utility \( u : X \mapsto \mathbb{R} \) and weight \( w : X \mapsto \mathbb{R}_{++} \) function pairs \((u, w)\). The preferences are identical \( \succ^1 \equiv \succ^2 \) if and only if \( \mathcal{V}^1 \equiv \mathcal{V}^2 \).

## 4 Special Cases

Weighted utility theory with incomplete preferences admits incompleteness arising either from conflicting perceptions, represented by multiple weight functions, or from indecisive tastes, represented by multiple utility functions. Thus, the general framework we have devised admits a pair of special cases, those of multiple utilities paired with a single weight function \( \mathcal{V} = \mathcal{U} \times \{w\} \) or a single utility paired with multiple weights \( \mathcal{V} = \{u\} \times W \). Each of these may be regarded as a partial completion of an incomplete preference relation.

\textsuperscript{11}Given \((u^1, w^1), (u^2, w^2) \in \hat{V} \) and \( \kappa \in (0, 1) \), we could take a direct convex combination by setting \( u^\kappa = \kappa u^1 + (1 - \kappa) u^2 \) and \( w^\kappa = \kappa w^1 + (1 - \kappa) w^2 \), but unless \( u^1 = u^2 \), this would not preserve weighted linearity. Alternatively, we could take the weighted convex combination and set \( u^\kappa = \frac{\kappa u^1 + (1 - \kappa) w^1 \bar{u}_p^1}{w^1 + (1 - \kappa) w^2} \) and \( w^\kappa = \kappa w^1 + (1 - \kappa) w^2 \). This however would not necessarily preserve the ordering of lotteries, as \( u^k(x) > u^k(x) \) for \( k = 1, 2 \) does not imply \( u^\kappa(x) > u^\kappa(x) \).

\textsuperscript{12}Since \( u^k(x_1) = u^k(x_2) = 1 \), we have that \( u^k_p(x_1) = u^k_p(x_2) = 0 \) and \( u^k_p(x_1) = u^k_p(x_2) = 1 \). Furthermore,
\[
\sum_{x \in X} p(x)u^k_p(x) = \sum_{x \in X} p(x)u^k(x)u^k(x) + \left[1 - \sum_{x \in X} p(x)u^k(x)\right] \frac{\sum_{x \in X} p(x)u^k(x)u^k(x)}{\sum_{x \in X} p(x)u^k(x)} = \frac{\sum_{x \in X} p(x)u^k(x)u^k(x)}{\sum_{x \in X} p(x)u^k(x)}.
\]

\textsuperscript{13}The closure is with respect to the \( \mathbb{R}^n \) topology.
In non-expected utility, local risk attitudes are captured by the local utility functions and global risk attitude depends on the variations of the local risk attitudes, as in Machina (1982). In weighted utility, the utility and weight functions play distinct roles, with the shape of the utility function capturing the decision maker’s risk attitude while the weight function captures the nature and degree of the variation in local attitudes. Specifically, the weight function reflects the extent to which the indifference map exhibits the fanning in or fanning out structure described by Machina (1982).

A decision maker with multiple utility functions and a single weight function has incomplete preferences solely due to his indecisive risk attitude, and has no more difficulty evaluating a lottery than he would evaluating each of its possible outcomes. On the other hand, a decision maker with a single utility function and multiple weight functions is sure of his risk attitude, but is indecisive when comparing lotteries because he is unsure of how to perceive randomness, and thus cannot always rate lotteries properly even if he knows how would rank their components.

4.1 Multiple Utilities

A decision maker whose preference relation is represented by multiple utilities paired with a single weight function is indecisive about the valuation of each of the outcomes in $X$, but is confident of how much attention he should pay to each of the possible payoffs. To ensure that a preference relation $\succ$ has such a representation, we adopt a stronger variant of the partial substitution axiom.

\textbf{(A.5) (Parallel Substitution)} For all $p, q \in \Delta(X)$, $p \succ q$ if and only if for every $\beta \in (0, 1)$ there is a unique $\gamma \in (0, 1)$ such that $\beta p + (1 - \beta)q \succeq \gamma q + (1 - \gamma)p$ for all $r \in \Delta(X)$.

Under this assumption, for every $p \succ q$ there must be a unique substitution ratio $T(p, q) = \{\tau_{p,q}\}$. The following lemma shows that for every $p$ we can pair a unique weight $\tau_p$ with any of the utility values $\alpha_p \in A(p)$.

\textbf{Lemma 5} If $\succ$ satisfies (A.1)-(A.3),(A.5) then, for all $p \in \Delta(X)$, there is $\tau_p > 0$ such that $\Phi(p) = A(p) \times \{\tau_p\}$.

This result leads directly into the following representation theorem.

\textbf{Theorem 3} Let $\succ$ be a binary relation over $\Delta(X)$. Then $\succ$ satisfies (A.1)-(A.3),(A.5) if and only if there is a set $U$ of utility functions $u : X \mapsto \mathbb{R}$ and a weight function $w : X \mapsto \mathbb{R}_{++}$ such that for every $p, q \in \Delta(X)$,

$$p \succ q \iff \sum_{x \in X} p(x)u(x) > \sum_{x \in X} q(x)w(x), \quad \forall u \in U.$$ 

It follows that a multiple utility, single weight representation is equivalent to applying a one-time transformation to the entire probability space and constructing a multiple expected utility representation over the transformed probability space. For any $p$, define the transformed lottery $p^w(x) = \frac{p(x)u(x)}{\sum_{x \in X} p(x)w(x)}$ for every $x \in X$, and define the relation $\succ^w$ such that $p^w \succ^w q^w$ if and only if $p \succ q$. Then it immediately follows that $\succ^w$ has a multiple expected utility representation as

$$p^w \succ^w q^w \iff \sum_{x \in X} p^w(x)u(x) > \sum_{x \in X} q^w(x)u(x), \quad \forall u \in U.$$ 

While this transformation may appear to indicate that we could apply the uniqueness results from multiple expected utility to $U$, by Theorem 2 there is in fact a broader set of equivalent representations. For example, suppose we had $U = \{u^1, u^2\}$ and set $\tilde{w}^j = \frac{au^j + b}{cu^j + d}$ and $\tilde{w}^j = w[cu^j + d]$ for $j = 1, 2$. Then $\tilde{V} = \{(\tilde{u}^1, \tilde{w}^1), (\tilde{u}^2, \tilde{w}^2)\}$ would represent the same preferences as $U \times \{w\}$, even though the former has multiple weight functions while the latter has only one.

\footnote{Fanning out reflects a decision maker who underweights the median outcome relative to the extremes, corresponding to monotonically increasing local risk aversion with respect to first order stochastic dominance, whereas fanning in reflects an overweight of the median outcome and hence decreasing local risk aversion with respect to first order stochastic dominance.}
4.2 Multiple Weights

A decision maker whose preferences are represented by a single utility function paired with multiple weight functions is confident of how he would evaluate all of the outcomes in X, but is indecisive over how much importance each of these outcomes carries when evaluating a lottery \( p \in \Delta(X) \). Such a decision maker is sure of his tastes, but when trying to compare alternative lotteries is unable to determine what aspects to focus on and attach more weight to. To ensure that a preference relation has such a representation, we impose the following axiom.

\[ \text{(A.6) (Partial Completeness)} \quad \succ \text{ is negatively transitive over } \{ \delta_x : x \in X \} \cup \{ \zeta_\alpha : \alpha \in [0, 1] \}. \]

It is immediate that (A.6) implies that there for each \( x \in X \) a unique \( \alpha \in [0, 1] \) such that \( \delta_x \succ \zeta_\alpha \). Thus, every degenerate lottery has only a single utility value so that \( A(\delta_x) = \{ \alpha_i \} \) for every \( i = 1, \ldots, n \), but may take multiple weight values so that \( T(\delta_x, \zeta_\alpha) \) need not be a singleton, and hence non-degenerate lotteries \( p \in \Delta(X) \setminus X \) may still have multiple utility values. Imposing this assumption leads to a single utility, multiple weight representation.

**Theorem 4** Let \( \succ \) be a binary relation over \( \Delta(X) \). Then \( \Delta(X) \) satisfies (A.1)-(A.4),(A.6) if and only if there is a utility function \( u : X \mapsto \mathbb{R} \) and a set \( W \) of weight functions \( w : X \mapsto \mathbb{R}_{++} \) such that for every \( p, q \in \Delta(X) \),

\[
p \succ q \iff \sum_{x \in X} p(x)w(x)u(x) \geq \sum_{x \in X} q(x)w(x)u(x), \quad \forall w \in W.
\]

Unlike in the multiple utility, single weight case, there is no simple transformation here that we can apply to produce a more familiar expected multi-utility representation. In this case, the decision maker is unsure of how to perceive randomness, as each of his weight functions distort his focus differently. While he is able to rank all of the outcomes, his ability to evaluate lotteries is compromised by his inability to determine which components he should be paying attention to.

5 Behavioral Implications

5.1 Inertia

To illustrate the differences between the two sources of indecisiveness, recall that the main empirical manifestation of incompleteness is *inertia*. Given an alternative \( a \) in some choice set, there is a range of non-comparable alternatives that will not be accepted if they were offered in exchange for \( a \). In weighted utility theory with incomplete preferences, the nature of inertia depends on the source of indecisiveness. Specifically, if his indecisiveness is due to conflicting risk attitudes then the decision maker displays inertia everywhere. By contrast, if the source of his indecisiveness is incomplete perception then the decision maker displays inertia everywhere except at degenerate lotteries \( \delta_x \). These observations have testable implications. For example, the subject in an experiment may receive \( \delta_x \) by default and be offered the opportunity to trade it for some lottery \( \zeta_\alpha \). Using standard experimental methods it is possible to verify if the subject switches at a single \( \alpha \), thus indicating preferences that can be described by a multiple weight representation, or choose to hold on to \( \delta_x \) over a range \( \alpha \), indicating that a multiple utility representation is more appropriate.

5.2 Portfolio Selection

To further illustrate the applicability of multiple weighted utility models, consider a simple example of portfolio choice. Let there be two financial assets, a risk-free asset whose rate of return is zero and a risky asset that pays \( x > 0 \) with probability \( p \) and \( y < 0 \) with probability \( 1 - p \), where \( px + (1 - p)y \geq 0 \). A decision maker whose initial wealth is \( z_0 \) must select a portfolio \((A, B)\), where \( A \) is the dollar holding of the risk-free asset and \( B = z_0 - A \) is the investment in the risky asset. This problem can be usefully formulated as the choice of a portfolio composed of Arrow securities. For this purpose there are two states \( s = 1, 2 \) whose probabilities are \( p \) and \( 1 - p \) respectively and corresponding Arrow securities \( a_s \), each paying one dollar contingent on the realization of state \( s \). The decision maker’s initial position
consists of \(z_0\) shares of each security, and her decision to invest \(B\) in the risky asset corresponds to buying \(Bx\) shares of \(a_1\) and selling \(By\) shares of \(a_2\). The corresponding terminal wealth is \(z_1 = z_0 + Bx\) with probability \(p\) and \(z_2 = z_0 + By\) with probability \(1 - p\).

Consider a weighted utility maximizing decision maker whose utility and weight functions are \(u\) and \(w\) respectively. Her problem is to choose a portfolio \((z_1, z_2)\) to maximize 

\[
\frac{pw(z_1)w(z_2) + (1 - p)w(z_2)u(z_2)}{pw(z_1) + (1 - p)w(z_2)}
\]

subject to the budget constraint \(z_1 + rz_2 = z_0 + r_0\), where \(r = \frac{y}{1-y}\) is the price of \(a_1\) in terms of \(a_2\).

Denote by \(D_J\) the set of cumulative distribution functions with a compact support \(J \subseteq \mathbb{R}\). A weighted utility maximizing decision maker evaluates the lottery \(F \in D_J\) by the weighted utility functional 

\[
U(F) = \int_{J} u(t)w(t)dF(t)
\]

and attaches weight \(W(F) = \int_{J} w(t)dF(t)\). Following Machina (1982), the local utility function is the Gateaux derivative at \(F\):

\[
u_{F}^{(u,w)}(t) = \frac{w(t)[u(t) - U(F)]}{W(F)}.
\]

The decision maker displays local risk aversion at \(F\) if the \(u_{F}^{(u,w)}\) is monotonic increasing and concave, and displays global risk aversion if she displays local risk aversion at all \(F \in D_J\). The decision maker’s problem is to choose an optimal portfolio along the path

\[
F(z_1) = \left\{p\delta z_1 + (1 - p)\delta z_2 + \frac{w(t)}{\int_{J} w(t) dF(t)} : z_1 \in [0, z_0]\right\}.
\]

If \(r = \frac{1-p}{1+p}\), that is, the expected rate of return on the risky asset is the same as that of the risk free asset, \(px + (1 - p)y = 0\), then all weighted-utility maximizers whose utility function \(u\) is monotonic increasing and concave, would choose an optimal portfolio position, \((z_1, z_2) = (z_0, z_0)\). That is, they avoid investing in the risky asset, choosing \(B^* = 0\). If the expected rate of return on the risky asset is positive, that is, if \(r < \frac{1-p}{1+p}\), then every weighted-utility maximizer displaying local risk aversion at the initial \((z_0, z_0)\) would choose a portfolio position, \((z_1^*, z_2^*)\) such that \(z_1^* > z_0\) and \(z_2^* < z_0\), so the optimal level of investment in the risky asset is positive \(B^* > 0\). Moreover, if two weighted-utility maximizing decision makers characterized by utility-weight pairs \((u, w)\) and \((\hat{u}, \hat{w})\), where everywhere along the path of portfolios the local utility \(u_{F(\cdot|z_1)}^{(u,w)}\) displays greater risk aversion than \(u_{F(\cdot|z_1)}^{(\hat{u},\hat{w})}\). Then \(z_1^{(u,w)} > z_1^{(\hat{u},\hat{w})}\) and \(z_2^{(u,w)} < z_2^{(\hat{u},\hat{w})}\), or alternatively \(B^{(u,w)} > B^{(\hat{u},\hat{w})}\), so that the more risk averse decision maker invests less in the risky asset.

\[\text{Figure 5: Portfolio Selection}\]

Thus far we reviewed some results concerning the simple portfolio choice of weighted-utility maximizing decision makers whose preference relations are complete. When a weighted-utility maximizer’s preference relation is incomplete, there is a set \(V\) of utility-weight pairs that must agree in order for one risky

\[\text{Note that on the certainty line, the weight function plays no role in defining risk attitude.}\]
prospect to be strictly preferred over another. Let \( Z := \{ (z_1^{(u,w)}, z_2^{(u,w)}): (u, w) \in V \} \), or alternatively \( B := \{ B^{(u,w)}: (u, w) \in V \} \), be the set of optimal portfolio positions corresponding to each of the pairs of utility and weight functions \((u, w) \in V\). Since the various utility-weight pairs represent distinct risk attitudes, the portfolio positions in \( Z \) are non-comparable. Hence, the decision maker exhibits indecisiveness with regard to the portfolio positions in \( Z \), which may be resolved by random choice. However, once the decision maker chooses a portfolio composition it becomes the status quo or default position, and she displays inertia by avoiding making portfolio adjustments for some range of variations in the relative price \( r \). Inertia was suggested by Bewley (2002) and may be summarized by the dictum “if in doubt do nothing.”

This situation is depicted in Figure 5 where the set of optimal portfolios corresponds to the interval \( Z = \{ (z_1^M, z_2^M) \} \). On this interval, the decision maker has randomly chosen \((z_1^M, z_2^M)\), which is now the status quo position. As the upper contour set at \((z_1^M, z_2^M)\) has a kink, some variations in \( r \) will not induce portfolio adjustment, because the distinct pairs \((u, w) \in V\) disagree on the desirable adjustments. On the other hand, the decision maker would exhibit inertia with respect only to increasing \( r \) at \((z_1^M, z_2^M)\) and inertia only with respect to decreasing \( r \) at \((z_1^L, z_2^L)\). By contrast to the expected multi-utility model, where the set of local utilities is the same everywhere and hence the decision maker displays equal inertia everywhere, in the weighted multi-utility model the set of local utilities \( U_F = \{ u_F^{(u,w)}: (u, w) \in V \} \) varies along the portfolio path \( F(z_1) \). As each of the local utilities \( u_F^{(u,w)} \in U_F \) may show distinct patterns of variation in risk attitude depending on the reference point \( F \), these differences may result in variations along the portfolio path both in the range of indecisiveness and the degree of inertia with respect to changes in \( r \).

6 Concluding Remarks

In this paper, we have considered a model of decision making under risk for preferences that satisfy neither independence nor completeness, and obtain a utility representation by the agreement of a set of utilities, as in multiple utility theory, each of which is weighted linear in the probabilities, as in weighted utility theory, thus uniting these separate strands in the literature under a unified framework. This representation further admits a variety of additional cases with distinct interpretations, as incomplete preferences may be due to ambivalent risk attitudes or incognizance of the relative salience of the possible outcomes. By directly imposing additional axioms that eliminate either of these possibilities, we obtain special cases where the multiplicity in the representation is restricted to either the utility or weight functions alone. The general framework we have devised thus serves as a useful foundation for studying decision making under risk from a variety of different perspectives.

7 Proofs

7.1 Proofs of Propositions

7.1.1 Proof of Proposition 1

Fix \( p \in \Delta(X) \). Define \( \overline{\alpha} = \inf \{ \alpha : p < \zeta_\alpha \} \). Suppose that for \( \alpha' \leq \overline{\alpha} \) we have \( p < \zeta_{\alpha'} \), then by (A.2) there is \( \beta \in (0, 1) \) such that \( p < \beta \zeta_{\alpha'} + (1 - \beta)\overline{\alpha} = \zeta_{\alpha''} \). Since \( \alpha'' = \beta \alpha' < \overline{\alpha} \), this contradicts the definition of \( \overline{\alpha} \). Now suppose that some for \( \alpha' > \overline{\alpha} \) we have \( p < \zeta_{\alpha'} \), then for all \( \alpha'' < \alpha' \) we have \( p < \zeta_{\alpha''} \) or else \( p < \zeta_{\alpha''} < \zeta_{\alpha'} \), which implies that \( \alpha' \leq \inf \{ \alpha : p < \zeta_\alpha \} = \overline{\alpha} \), a contradiction. Thus \( p < \zeta_{\alpha'} \) if and only if \( \alpha' \leq \overline{\alpha} \). Now define \( \underline{\alpha} = \sup \{ \alpha : p < \zeta_\alpha \} \), by a similar argument \( p < \zeta_{\alpha'} \) if and only if \( \alpha' \geq \underline{\alpha} \). Therefore, we have that \( p \approx \zeta_{\alpha'} \) if and only if \( \alpha' \in [\underline{\alpha}, \overline{\alpha}] \). \( \square \)
7.1.2 Proof of Proposition 2

Fix $p, q \in \Delta(X)$ such that $p \succ q$. For every $r \in \Delta(X)$, define

$$T^L(p, q, r) = \left\{ \tau > 0 : \exists \beta, \beta p + (1 - \beta) r > \frac{\beta \tau q + (1 - \beta) r}{\beta \tau + (1 - \beta)} \right\},$$

$$T^R(p, q, r) = \left\{ \tau > 0 : \exists \beta, \beta p + (1 - \beta) r < \frac{\beta \tau q + (1 - \beta) r}{\beta \tau + (1 - \beta)} \right\},$$

$$T(p, q, r) = \left\{ \tau > 0 : \beta p + (1 - \beta) \tau r \geq \frac{\beta \tau q + (1 - \beta) r}{\beta \tau + (1 - \beta)} \right\}.$$

Let $R = \{ r \in \Delta(X) : \neg(r \sim p) \lor \neg(r \sim q) \}$ denote the set of all lotteries that are comparable with either $p$ or $q$. We will establish that for every $r \in R$, $T(p, q, r)$ is a closed interval $[\tau_l, \tau_r]$, and for every $r \notin R$, there is $s \in R$ such that $T(p, q, r) \supseteq T(p, q, s)$. Taken together these will allow us to conclude that $T(p, q)$ is given by the intersection of closed intervals and is hence itself a closed interval.

Suppose $r \in R$. Then if $r \succ q$, define $\tau_r = \inf T^L(p, q, r)$ and $\tau_r = \sup T^R(p, q, r)$. Suppose that for $r' \leq \tau_r$ we have $r' \in T^L(p, q, r)$, then there is $r'' \in T^L(p, q, r)$ such that $r'' < r' \leq \tau_r$, contradicting the definition of $\tau_r$. Now suppose that for $r' > \tau_r$ we have $r' \notin T^L(p, q, r)$, then we must have $r'' \notin T^L(p, q, r)$ for every $r'' < r'$, and therefore $r' \leq \tau_r$, a contradiction. Thus $r' \in T^L(p, q, r)$ if and only if $r' \succ \tau_r$. This implies that $r' \in T(p, q, r)$ if and only if $r' \in [\tau_r, \tau_s]$. If $r \prec q$, then we can define $\tau_r = \inf T^R(p, q, r)$ and $\tau_r = \sup T^L(p, q, r)$ and apply a similar argument to the above. If $r \equiv q$ and either $r > p$ or $r < p$, we can again repeat the argument above by switching $p$ and $q$ and noting that $r' \in T(p, q, r)$ if and only if $\frac{1}{2} \in T(q, p, r)$. Thus for every $r \in R$, $T(p, q, r)$ is a closed interval $[\tau_r, \tau_s]$.

Now suppose $r \notin R$. Then if there is $r' \in \Delta(\{p, q, r\})$ such that $r' > q$, there are $\lambda, \alpha \in [0, 1]$ such that $r' = \lambda[p + (1 - \alpha) r] + (1 - \lambda) q > q$, so that by betweenness we have $s = \alpha p + (1 - \alpha) r > q$. This implies that there is $s \in R$ such that $T(p, q, s) \supseteq T(p, q, r)$. A similar result follows if we have $s \in \Delta(\{p, q, r\})$ such that $s < q$. Likewise, if there is $s \in \Delta(\{p, q, r\})$ such that $s \succ p$ or $s < p$, we repeat the argument again noting that $r' \in T(p, q, r)$ if and only if $r' \sim \tau_r$, $T(p, q, r) \supseteq T(p, q, s)$ if and only if $T(p, q, s) \supseteq T(p, q, r)$.

Now suppose that for $s \in \Delta(\{p, q, r\})$ we have $s \prec p$ and $s \equiv q$. If for some $\theta \in (0, 1)$ we have $s \succ \theta p + (1 - \theta) q$, then by betweenness $p \equiv \theta p + (1 - \theta) q$ and by the argument above, $T(p, \theta p + (1 - \theta) q) \supseteq T(p, \theta p + (1 - \theta) q)$, which in turn implies $T(p, q) \supseteq T(p, q)$. A similar result follows if for some $\theta \in (0, 1)$ we have $s \prec \theta p + (1 - \theta) q$. Finally, if for all $s \in \Delta(\{p, q, r\})$ and $\theta \in (0, 1)$ we have

16Let $\tau_s = \infty$ if $T^L(p, q, r) = \emptyset$ and $\tau_r = 0$ if $T^R(p, q, r) = \emptyset$.

17If $r' \in T^L(p, q, r)$, then $r' \in \{1\}$ such that $\beta p + (1 - \beta) r > \frac{\beta r' q + (1 - \beta) r}{\beta r' + (1 - \beta)}$. This implies by (A.2) that there is $\lambda \in (0, 1)$ such that

$$\beta p + (1 - \beta) r > \frac{\lambda \beta r' q + (1 - \lambda) r}{\lambda \beta r' + (1 - \lambda)} = \frac{\beta r' q + (1 - \lambda) r}{\beta r' + (1 - \lambda)}.$$

Letting $\tau'_r = \frac{\lambda \beta r' q + (1 - \lambda) r}{\beta r' + (1 - \lambda)} < r'$ completes the argument.

18Otherwise if some $r'' \in T^L(p, q, r)$, then for some $\beta \in (0, 1)$ we have $\beta p + (1 - \beta) r > \frac{\beta r' q + (1 - \beta) r}{\beta r' + (1 - \beta)}$, implying $r'' \in T^L(p, q, r)$ as well.

19As before, let $\tau_r = \infty$ if $T^R(p, q, r) = \emptyset$ and $\tau_s = 0$ if $T^L(p, q, r) = \emptyset$.

20Pick $\tau \in T(p, q, r)$ and for any $\beta \in (0, 1)$, let $\beta' = \beta + (1 - \beta) \alpha$ so that $p' = \beta p + (1 - \beta) s = \beta' p + (1 - \beta) r$ and let $q' = \frac{\beta' r + (1 - \beta) s}{\beta' r + (1 - \beta)}$. Then we have that

$$p' \sim \frac{\beta r + (1 - \beta) s}{\beta r + (1 - \beta)} = \frac{\beta r + (1 - \beta) s}{\beta r + (1 - \beta)} = \frac{(1 - \lambda) \beta p + \beta' \tau r + (1 - \beta) s}{(1 - \lambda) \beta p + \beta' \tau r + (1 - \beta) s}.$$
\[ s \leq \theta p + (1 - \theta)q, \text{ then } T(p, q, r) = \mathbb{R}_{++},^{22} \text{ so that } T(p, q, r) \supseteq T(p, q, s) \text{ for all } s \in R. \]

By definition we have that \( T(p, q, s) = \bigcap_{r \in \Delta(X)} T(p, q, r) \). Note that \( T(p, q, r) \) is bounded if any \( T(p, q, r) \) is bounded, which we can establish by setting \( r = \delta_\tau \).^{23} Since for every \( r \notin R \) there is \( s \in R \) such that \( T(p, q, r) \supseteq T(p, q, s) \), we have that \( T(p, q, s) = \bigcap_{r \in R} T(p, q, r) = \bigcap_{r \in R} [\tau_r, \tau_r] \). Letting \( \tau = \sup_{r \in R} \tau \), we have that \( T(p, q) = [\tau, \tau] \). \qed

### 7.1.3 Proof of Proposition 3

Necessity is immediate from Lemma 1 and the definition of \( \Omega \). To prove sufficiency suppose that for \( p, q \in \Delta(X) \) there is \( \tau > 0 \) such that \( o = \frac{p - \tau q}{1 - \tau} \in \Omega \). We need to show that \( p \succ q \). Since \( o \in \Omega \), there are \( p', q' \in \Delta(X) \) such that \( p' \succ q' \) and \( \tau' \in T(p', q') \) such that

\[ o = \frac{p - \tau q}{1 - \tau} = \frac{p' - \tau' q'}{1 - \tau'}. \]

Suppose that the line defined by \( p \) and \( q \) intersects the interior of the simplex so that there are \( \lambda_u, \lambda_v \), assuming without loss of generality that \( \lambda_u > \lambda_v \), such that

\[ u := \lambda_u p + (1 - \lambda_u) q \in \text{int} \Delta(X), \]
\[ v := \lambda_v p + (1 - \lambda_v) q \in \text{int} \Delta(X), \]
\[ o = \frac{u - \frac{\lambda_u + (1 - \lambda_u)}{\lambda_v + (1 - \lambda_u)} v}{1 - \frac{\lambda_u + (1 - \lambda_u)}{\lambda_v + (1 - \lambda_u)}} v := u - \hat{v} v. \]

Pick \( \lambda_u \approx \lambda_v \) so that \( \hat{\tau} \approx 1 \). Define \( s' \) to be the intersection of the lines \( p'u \) and \( q'v \). Then as \( u \) and \( v \) are close together, we have

\[ s' := \frac{1 - \tau'}{\hat{\tau} - \tau'} u - \frac{1 - \hat{\tau}}{\hat{\tau} - \tau'} \frac{\hat{\tau} q - (1 - \hat{\tau}) q'}{1 - \tau'} \in \text{int} \Delta(X), \]
\[ u = \frac{1 - \hat{\tau}}{1 - \tau'} p' + \frac{\hat{\tau} - \tau'}{1 - \tau'} v' = \hat{\beta} p' + (1 - \hat{\beta}) s', \]
\[ v = \frac{1 - \tau'}{1 - \tau'} q' + \frac{\hat{\tau} - \tau'}{1 - \tau'} v' = \hat{\gamma} q' + (1 - \hat{\gamma}) s'. \]

Since \( p' \succ q' \) and \( \tau' = \frac{\gamma(1 - \gamma)}{\beta(1 - \beta)} \in T(p', q') \), we have

\[ u = \lambda_u p + (1 - \lambda_u) q = \frac{\beta p' + (1 - \beta) s'}{\beta(1 - \beta)} = \lambda_u p + (1 - \lambda_u) q = v. \]

Therefore since \( \lambda_u > \lambda_v \), if \( p \succ q \) then by betweenness \( u \succ v \), a contradiction, so \( \neg(p \succ q) \) and by a similar argument \( \neg(p \prec q) \), hence \( p \approx q \).

That, again by betweenness,

\[ \beta p + (1 - \beta) t = \frac{\theta \tau[p + (1 - \beta) t] + (1 - \theta)[\beta t + (1 - \beta) t]}{\theta \tau + (1 - \theta)[\beta t + (1 - \beta) t]} = \frac{\beta \tau(p + (1 - \theta) q) + (1 - \beta)[\theta \tau + (1 - \theta) t]}{\beta \tau + (1 - \beta)[\theta \tau + (1 - \theta) t]} \]

Taking the odds ratio of the above, we conclude that \( \tau \in T(p, q, t) \) if and only if \( \frac{\tau}{\theta \tau + (1 - \theta) t} \in T(p, \theta p + (1 - \theta) q, t) \). Thus \( T(p, \theta p + (1 - \theta) q, r) \supseteq T(p, q, r) \) and \( T(p, q, r) \supseteq T(p, q, s) \).

\( ^{22} \) Pick \( \beta \in (0, 1) \) and let \( s = \beta \theta p + (1 - \beta) r \), and for any \( \tau > 0 \) let \( \theta = \frac{1}{1 - \tau} \). Then, by assumption, \( \beta \theta p + (1 - \beta) r \approx \frac{\tau}{1 - \tau} \) and invoking betweenness yet again we have that

\[ \beta \theta p + (1 - \beta) r \approx \frac{[\beta p + (1 - \beta) r] - \beta(1 - \tau)}{1 - \beta(1 - \tau)} = \frac{\beta p + (1 - \beta) r}{\beta \tau + (1 - \beta) r}. \]

This implies that \( \tau \in T(p, q, r) \) for every \( \tau > 0 \).

\( ^{23} \) Since for any \( \beta \in (0, 1) \) we have \( r \succ \beta p + (1 - \beta) r \), by (A 2) there is \( \gamma \in (0, 1) \) such that \( \gamma q + (1 - \gamma) r \succ \beta p + (1 - \beta) r \), implying \( \tau_r > \frac{\gamma(1 - \gamma)}{\beta(1 - \beta)} \). Likewise for any \( \tau \in (0, 1) \) we have \( r \succ \tau q + (1 - \tau) r \), so there is \( \beta \in (0, 1) \) such that \( \beta p + (1 - \beta) r \succ \tau q + (1 - \tau) r \) so that \( \tau_r < \frac{\gamma(1 - \gamma)}{\beta(1 - \beta)} \).
Now suppose that the line defined by \( p \) and \( q \) lies entirely on the boundary of the simplex. Then pick \( t \in \text{int}\Delta(X) \) such that \( t \succ p \) and \( t \succ q \) and, for every \( n \), let
\[
P^n = \frac{1}{n} t + \left( 1 - \frac{1}{n} \right) p, \quad q^n = \frac{1}{n} t + \left( 1 - \frac{1}{n} \right) q.
\]
Thus \( \lim_{n\to\infty} p^n = p \) and \( \lim_{n\to\infty} q^n = q \). For each \( n \), we have that the line \( p^n q^n \) intersects \( \Delta(X) \) and can verify that
\[
o = \frac{p^n - \left[ \frac{1}{n} + (1 - \frac{1}{n}) \tau \right] q^n}{1 - \left[ \frac{1}{n} + (1 - \frac{1}{n}) \tau \right]}.
\]
Hence, by the argument above \( p^n \succ q^n \) for every \( n = 1, 2, \ldots \). Suppose that \( p \succ q \), then there is \( s \in \Delta(X) \) such that \( p \succ s \succ q \). By betweenness, since \( t \succ p \) every \( p^n \succ p \succ s \). By the strong Archimedean, since \( s \succ q \) there is \( \alpha \in (0, 1) \) such that \( s \succ \alpha q + (1 - \alpha) t \). Hence, for any \( n \geq \frac{1 - (1 - \alpha)(1 - \tau)}{1 - \alpha \tau} \) then \( \frac{(1 - \frac{1}{n}) \tau}{1 - (1 - \alpha)(1 - \tau) / (1 - \alpha \tau)} \geq \alpha \) we have \( s \succ \alpha q + (1 - \alpha) t \succ q^n \). By transitivity, \( p^n \succ q^n \), a contradiction. Thus \( \neg(p \succ q) \) and, by a similar argument, \( \neg(p \prec q) \) so that \( p \approx q \).

### 7.1.4 Proof of Proposition 4

Define \( v = wu \), so that for \( p \in \Delta(X) \) we may write
\[
\begin{bmatrix}
V(p) \\
W(p)
\end{bmatrix}
= \begin{bmatrix}
v(x_1) & \cdots & v(x_n) \\
w(x_1) & \cdots & w(x_n)
\end{bmatrix}
\begin{bmatrix}
p(x_1) \\
\vdots \\
p(x_n)
\end{bmatrix}
= Vp,
\]
\[
U(p) = \frac{\sum_{i=1}^n p(x_i)v(x_i)}{\sum_{i=1}^n p(x_i)w(x_i)} = \frac{V(p)}{W(p)}.
\]

For \( p, q \in \Delta(X) \), we have that \( U(p) > U(q) \) if and only if
\[
W(p)W(q)[U(p) - U(q)] = V(p)W(q) - V(q)W(p) = \begin{bmatrix} V(p) & V(q) \end{bmatrix} \begin{bmatrix} W(p) \\ W(q) \end{bmatrix} = |Vp - Vq| = |VP| > 0.
\]

Now consider a positive affine transformation
\[
\tilde{V} = \begin{bmatrix}
\tilde{v}(x_1) & \cdots & \tilde{v}(x_n) \\
\tilde{w}(x_1) & \cdots & \tilde{w}(x_n)
\end{bmatrix}
= \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix}
v(x_1) & \cdots & v(x_n) \\
w(x_1) & \cdots & w(x_n)
\end{bmatrix}
= AV.
\]

This implies that \( \tilde{U}(p) > \tilde{U}(q) \) if and only if \( |\tilde{VP}| = |A||VP| > 0 \), so that the ranking of lotteries is unchanged as long as \( |A| > 0 \), or \( ad - bc > 0 \).

### 7.1.5 Proof of Proposition 5

Let \( \mathcal{V} \) represent \( \succ \), pick any pair \((u, w) \in \mathcal{V}\), and let \( v = wu \). We begin by showing that there exists a normalized function pair \((\hat{u}, \hat{w})\) for which \((u, w)\) is a rational affine transformation, so that there are \( a, b, c, d \) such that
\[
\begin{bmatrix}
v(x_1) & v(x_n) \\
w(x_1) & w(x_n)
\end{bmatrix}
= \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix}
\hat{v}(x_1) & \cdots & \hat{v}(x_n) \\
\hat{w}(x_1) & \cdots & \hat{w}(x_n)
\end{bmatrix}
= \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}
= \begin{bmatrix} v(x_n) - v(x_1) & v(x_1) \\ w(x_n) - w(x_1) & w(x_1) \end{bmatrix}.
\]
Solving for these constants, we see that we indeed have a positive rational affine transformation as long as \( u \) ranks the best element \( x_n \) above the worst \( x_1 \). To grasp this claim, observe that
\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{-1}
= \begin{bmatrix}
v(x_n) - v(x_1) & v(x_1) \\ w(x_n) - w(x_1) & w(x_1) \end{bmatrix}.
\]
Hence,
\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix} = w(x_1)[v(x_n) - v(x_1)] - v(x_1)[w(x_n) - w(x_1)] = w(x_1)w(x_n)[u(x_n) - u(x_1)] > 0.
\]
Inverting this matrix, we transform \((u, w)\) back to the normalized \((\hat{u}, \hat{w})\).
\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}^{-1} = \frac{1}{w(x_1)[v(x_n) - v(x_1)] - v(x_1)[w(x_n) - w(x_1)]}
\begin{bmatrix}
  w(x_1) & -v(x_1) \\
  -w(x_1)u(x_n) - w(x_1)u(x_1) & w(x_1)
\end{bmatrix}.
\]
For any \(x \in X\), we have that
\[
\begin{bmatrix}
  \hat{v}(x) \\
  \hat{w}(x)
\end{bmatrix} = \begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}^{-1} \begin{bmatrix}
  v(x) \\
  w(x)
\end{bmatrix} = \frac{1}{w(x_1)[v(x_n) - v(x_1)] - v(x_1)[w(x_n) - w(x_1)]}
\begin{bmatrix}
  w(x_1) & -v(x_1) \\
  -w(x_1)u(x_n) - w(x_1)u(x_1) & w(x_1)
\end{bmatrix}
\begin{bmatrix}
  v(x) \\
  w(x)
\end{bmatrix}.
\]
This gives us the utility and weight functions
\[
\hat{u}(x) = \frac{\hat{v}(x)}{\hat{w}(x)} = \frac{w(x_1)w(x)[u(x) - u(x_1)]}{w(x_1)[v(x_n) - v(x_1)] - v(x_1)[w(x_n) - w(x_1)]},
\hat{w}(x) = \frac{w(x_1)w(x)[u(x_n) - u(x)] + w(x_1)w(x)[u(x) - u(x_1)]}{w(x_1)[v(x_n) - v(x_1)]}.
\]
It is easily verified that \((\hat{u}(x_1), \hat{w}(x_1)) = (0, 1)\) and \((\hat{u}(x_n), \hat{w}(x_n)) = (1, 1)\). Now for every \(p \in \Delta(X)\), define
\[
\hat{U}(p) = \sum_{i=1}^{n} p(x_i)\hat{u}(x_i) = \sum_{i=1}^{n} p_i \alpha_i = \alpha, \quad \hat{W}(p) = \sum_{i=1}^{n} p(x_i)\hat{w}(x_i) = \sum_{i=1}^{n} p_i \tau_i = \tau.
\]
Since the utility function is normalized by assumption, we have, for any \(\alpha\),
\[
\hat{U}(\omega) = \frac{\alpha \hat{u}(x_n)\hat{u}(x_1) + (1-\alpha)\hat{u}(x_1)\hat{u}(x_1)}{\alpha \hat{u}(x_n) + (1-\alpha)\hat{u}(x_1)} = \alpha, \quad \hat{W}(\omega) = \alpha \hat{w}(x_n) + (1-\alpha)\hat{w}(x_1) = 1.
\]
As \(\hat{U}(p) = \hat{U}(\omega_p)\) and \(\hat{W}(p) = \tau_p \hat{W}(\omega_p)\), we have for every \(\beta \in (0, 1)\) and \(r \in \Delta(X)\) that
\[
\hat{U}(\beta p + (1-\beta)r) = \frac{\beta \hat{W}(p)\hat{U}(p) + (1-\beta)\hat{W}(r)\hat{U}(r)}{\beta \hat{W}(p) + (1-\beta)\hat{W}(r)}
\]
\[
= \frac{\beta \tau_p \hat{W}(\omega_p)\hat{U}(\omega_p) + (1-\beta)\hat{W}(r)\hat{U}(r)}{\beta \tau_p \hat{W}(\omega_p) + (1-\beta)\hat{W}(r)}
\]
\[
= \hat{U}\left(\frac{\beta \tau_p \omega_p + (1-\beta)r}{\beta \tau_p + (1-\beta)}\right).
\]
This implies that every \(\beta p + (1-\beta)r \geq \frac{\beta \tau_p \omega_p + (1-\beta)r}{\beta \tau_p + (1-\beta)}\), so that \((\alpha_p, \tau_p) \in \Phi(p)\) for every \(p \in \Delta(X)\). Hence, \(\{(\hat{u}(x_i), \hat{w}(x_i))\}_{i=1}^{n} = \{\alpha_i, \tau_i\}_{i=1}^{n} \in \Psi\). \(\Box\)

7.1.6 Proof of Proposition 6

Let \(\hat{V}\) be a normalized set of utilities that represents \(\succ\), and for every \((u^k, w^k) \in \mathcal{V}\), define
\[
U^k(p) = \frac{\sum_{x \in X} p(x)u^k(x)u^k(x)}{\sum_{x \in X} p(x)u^k(x)} , \quad W^k(p) = \sum_{x \in X} p(x)w^k(x).
\]
Around any lottery \( p \in \Delta(X) \), let \( \hat{w}^k = U^k(p) \) and \( \hat{U}_p = \{ w^k = w^k u^k + (1 - w^k) \hat{w}^k : (u^k, w^k) \in \mathcal{V} \} \) be the set of normalized local utilities. For every \( q \in \Delta(X) \), let

\[
U^k(p) = \sum_{x \in X} q(x) w^k(x) = \sum_{x \in X} q(x) [w^k(x) u^k(x) + [1 - w^k(x)] U^k(p)] = W^k(q) U^k(p) + [1 - W^k(q)] U^k(p).
\]

Since \( U^k(p) = U^k(q) \), we have

\[
U^k(p) - U^k(q) = W^k(q) [U^k(p) - U^k(q)].
\]

Thus \( p > q \) if and only if \( U^k(p) > U^k(q) \) for every \( (u^k, w^k) \in \mathcal{V} \), which in turn holds if and only if \( U^k(p) > U^k(q) \) for every \( u^k \in \mathcal{U}_p \). Denote the closure of the convex hull of \( \mathcal{U}_p \) by \( \bar{\mathcal{U}}_p = \{ u^k_\pi = \sum_{u^k \in \mathcal{U}_p} \pi_k u^k_\pi : \sum_{u^k \in \mathcal{U}_p} \pi_k = 1 \} \), then

\[
U^k(p) > U^k(q), \forall u^k \in \mathcal{U}_p \iff U^\pi(p) = \sum_{u^k \in \mathcal{U}_p} \pi_k U^k(p) > \sum_{u^k \in \mathcal{U}_p} \pi_k U^k(q) = U^\pi(q), \forall u^k \in \{ \mathcal{U}_p \}.
\]

Therefore, \( p > q \) if and only if every \( U^\pi(p) > U^\pi(q) \), completing the proof.

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### 7.2 Proofs of Lemmas

#### 7.2.1 Proof of Lemma 1

Fix \( p, q \in \Delta(X) \) such that \( p \succ q \). Fix \( r \in \Delta(X) \) and pick \( \beta, \gamma \in (0, 1) \) that satisfy partial substitution so that \( s := \beta p + (1 - \beta) q \equiv \gamma q + (1 - \gamma) r := t \). Now pick \( \beta', \gamma' \in (0, 1) \) such that \( \tau := \gamma'/\beta' = (1 - \gamma)/\beta \), proving the proposition requires showing that \( u := \beta' p + (1 - \beta') q \equiv \gamma' q + (1 - \gamma') r := v \).

**Figure 6: Proof of Lemma 1**

As depicted in Figure 6, the extensions of the lines \( st \) and \( uv \) intersect at some source point lying outside of the simplex on the extended line \( pq \), located at \( o = \frac{s+t}{2} \). Draw parallel lines from \( s \) and \( r \) such that the line from \( s \) intersects \( uv \) at some \( s' \), and extending \( ps' \) intersects the line from \( r \) at some \( r' \), and let \( t' \) denote the intersection of \( uv \) and \( qr' \). By Desargues’ theorem, the triangles \( rst \) and \( r's't' \) are perspective from the line \( oqp \), and hence the lines \( rr', ss' \), and \( t't \) are parallel. This implies that \( s' = \beta p + (1 - \beta) r \equiv \gamma q + (1 - \gamma) r \). Since both \( s' \) and \( t' \) lie on \( uv \), we have by betweenness that \( u \equiv v \) as well, completing the proof.

#### 7.2.2 Proof of Lemma 2

We will show that \( (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i) \), and therefore the four statements are equivalent.

(\( i \Rightarrow (ii) \)) Pick \( p, q \in \Delta(X) \) and suppose that \( p \succ q \), and pick \( r \in \Delta(X) \setminus \{ p, q \} \). Letting \( \tau = \theta q + (1 - \theta) p \), we have for \( \theta \) sufficiently large that \( \tau \succ p, q \succ \tau \). For every \( \lambda \), let \( A_\lambda = \{ u \in [0, 1] : \lambda p + (1 - \lambda) q \equiv \alpha \tau + (1 - \alpha) \tau \} \). By Proposition 1, we have that each \( A_\lambda \) is a closed interval \( [\alpha \lambda_1, \alpha \lambda_2] \). If \( A^* = \bigcap_\lambda A_\lambda = \emptyset \), then there are \( \lambda_1, \lambda_2 \in [0, 1] \) and \( \alpha' \in (0, 1) \) such that \( \alpha \lambda_1 > \alpha' > \alpha \lambda_2 \), so that \( \lambda_1 p + (1 - \lambda_1) q \equiv \alpha' \tau + (1 - \alpha') \tau \equiv \lambda_2 p + (1 - \lambda_2) q \). By betweenness, if \( \lambda_1 > \lambda_2 \) then \( p \succ q \), and if \( \lambda_1 < \lambda_2 \) then \( p \prec q \). As either would contradict \( p \succ q \), we must have that \( A^* \neq \emptyset \), so letting \( r = \alpha \tau + (1 - \alpha) \tau \) for any \( \alpha \in A^* \) establishes the result.

---

\(^{24}\)That such \( r \) exists is implied by the dimensionality of \( \Delta(X) \).
(ii) ⇒ (iii) Pick $p, q \in \Delta(X)$ such that $p \succeq q$ and pick any $\tau^* \in T(p, q)$, so that there is a source point $\sigma^* = \frac{p - \tau^*}{1 + \tau^*} \in \Omega$. Pick $\tau, \rho \in \Delta(X)$ such that $\tau \succ p, q \succ \lambda$, then by the above, there is $r = \alpha \rho + (1 - \alpha)\tau \in \Delta(X)$ such that every $\lambda p + (1 - \lambda)q = r$. We need to show that there are $\tau^*_p, \tau^*_q > 0$ such that every $\lambda \tau^*_p + (1 - \lambda)\tau^*_q \in T(\lambda p + (1 - \lambda)q)$.

For every $\lambda \in [0, 1]$, let $p_\lambda = \lambda p + (1 - \lambda)q$ and $\tau^*_\lambda = \lambda \tau^* + (1 - \lambda)$, then since $\tau^* \in T(p, q)$ we have that

$$o^* = \frac{p - \tau^* q}{1 - \tau^*} = \frac{[\lambda p + (1 - \lambda)q] - [\lambda \tau^* + (1 - \lambda)]q}{1 - [\lambda \tau^* + (1 - \lambda)]} = \frac{p_\lambda - \tau^*_\lambda q}{1 - \tau^*_\lambda} \in \Omega.$$ 

This implies by Proposition 3 that $\tau^*_\lambda \in T(p_\lambda, q)$. We now claim that there is some $\tau^*_q \in T(q, r)$ such that for every $\lambda \in [0, 1]$, $\tau^*_\lambda \tau^*_q \in T(p_\lambda, q)$. Suppose not, then since by Proposition 2 the weight ranges are closed intervals $T(p_\lambda, r) = [\tau^*_\lambda, \tau^*_\lambda]$ and $T(q, r) = [\tau^*_q, \tau^*_q]$, there is $\lambda \in [0, 1]$ such that $\tau^*_\lambda > \tau^*_q$ or $\tau^*_q < \tau^*_\lambda$. Assume the former without loss of generality, then there is $\tau^*_q < \tau^*_\lambda$ and $\tau^*_q \tau^*_\lambda$ such that $\tau^*_\lambda = \frac{\tau^*_q}{1 - \tau^*_q}$. Let $s = \tau = r$, then we have that since $\tau^*_\lambda > \tau^*_\lambda, o^*_\lambda = \frac{p_\lambda - \tau^*_\lambda r}{1 - \tau^*_\lambda} \notin \Omega$, so that for some $\beta \in (0, 1), \nonumber$

$$p^*_\lambda = \beta p_\lambda + (1 - \beta) s \succ \frac{\beta \tau^*_\lambda r + (1 - \beta) s}{\beta \tau^*_\lambda + (1 - \beta)} := r''.$$ 

Now let $\gamma = \frac{\beta \tau^*_\lambda}{\beta \tau^*_\lambda + (1 - \beta)}$ so that $\tau^*_\lambda = \gamma/(1 - \gamma)$. Since $\tau^*_q \in T(q, r)$, $o^*_q = \frac{q - \tau^*_q r}{1 - \tau^*_q} \notin \Omega$ so that

$$q' = \gamma q + (1 - \gamma) s < \frac{\tau^*_q r + (1 - \gamma) s}{\gamma \tau^*_q + (1 - \gamma)} := r''.$$ 

Since by construction $\tau^*_\lambda = \tau^*_q r'$ taking the above together we have that $p^*_\lambda \succ r'' = r'''$, but as $\tau^*_\lambda \in T(p_\lambda, q)$ implies $p^*_\lambda \prec q'$, this is a contradiction. Thus we must have that $\tau^*_q \leq \tau^*_\lambda$ and a similar argument shows $\tau^*_q \geq \tau^*_\lambda$. Therefore, there is $\tau^*_q \in T(\tau^*_q, q)$ such that $\tau^*_q \tau^*_q \in T(p_\lambda, r^*)$ for every $\lambda$.

This argument is illustrated in Figure 7, showing the contradiction when the claim is violated for $\lambda = 1$, so that $\tau^* \tau^*_q \notin T(p, r)$. Letting $\tau_p = \tau^* \tau^*_q \in T(p, r)$, this implies $[\lambda \tau^* + (1 - \lambda)] \tau_q \in T(\lambda p + (1 - \lambda)q, r)$ for every $\lambda$, completing the proof.

(iii) ⇒ (iv) Pick $p, q \in \Delta(X)$ such that $p \succeq q$ and suppose there is $r \in \Delta(X)$ and $\tau_p, \tau_q > 0$ such that every $\lambda \tau_p + (1 - \lambda) \tau_q \in T(\lambda p + (1 - \lambda)q, r)$. This defines a line of source points

$$O(p, q) = \left\{ o_\lambda = \frac{\lambda p + (1 - \lambda)q - [\lambda \tau_p + (1 - \lambda) \tau_q] r}{1 - [\lambda \tau_p + (1 - \lambda) \tau_q]} : \lambda \in [0, 1] \right\} \subseteq \Omega.$$ 

Now pick $p', q' \in \Delta([p, q])$, then there are $\tau_p', \tau_q' > 0$ such that we can define source points $o^*_p = \frac{p' - \tau_p'}{1 - \tau_p'}$ and $o^*_q = \frac{q' - \tau_q'}{1 - \tau_q'}$. Now let $\tau^*_q = \frac{\tau_q'}{1 - \tau_q'}$ and $\lambda^*_q = \frac{1}{1 - \tau^*_q}$, then we have that

$$o^*_q = \lambda^*_q \left(1 - \tau^*_q\right) o'^*_p + \left(1 - \lambda^*_q\right)(1 - \tau^*_q) o'^*_q = \left(\frac{p' - \tau^*_p r'}{1 - \tau^*_p} - \frac{\tau^*_q}{1 - \tau^*_q}(q' - \tau^*_q r')\right) - \frac{1 - \tau^*_q}{1 - \tau^*_p} \in \Omega.$$ 

---

25For every $p, q \in \Delta(X)$ let $S(p, q) = \{o = \frac{p - \tau}{1 + \tau} : \tau \in T(p, q)\}$ the range of source points on the line defined by $p$ and $q$. For any $p \in \Delta(X)$, let $I(p) = \{\tau' = \alpha \tau + (1 - \alpha) \tau.p : \alpha \in [0, 1]\}$ the range of mixtures of $\tau$ and $p$ to which $p$ is incomparable, then $r \in I(p)$ and $\tau \in I(p, r)$ implies that $\tau = \frac{\beta \tau + (1 - \beta) \tau'}{\beta + (1 - \beta)} \in I(p)$ where $\beta \in [0, 1]$. As shown in Figure 6, if $o^* \in S(p, q)$, then we must be able to draw a line from it that intersects both $S(p, r)$ and $S(r, q)$, or else there are $o'_p, o'_q \notin \Omega$ that indicate that $(p')'$ and $I(p')'$ are disjoint, so that $p'$ could be $r'$ which in turn would imply $o^* \notin \Omega$.

26Let $p' = \mu p + \mu q + (1 - \mu p - \mu q)$ and $p' = \nu p + \nu q + (1 - \nu p - \nu q)$ and $\tau^*_p = \mu \tau_p + \mu \tau_q + (1 - \mu p - \mu q)$ and $\tau^*_q = \nu \tau_p + \nu \tau_q + (1 - \nu p - \nu q)$ and $\tau^*_q = \nu \tau_p + \nu \tau_q + (1 - \nu p - \nu q)$ and $\tau^*_p = \mu \tau_q + \mu \tau_q + (1 - \mu p - \mu q)$ and $\tau^*_q = \nu \tau_q + \nu \tau_q + (1 - \nu p - \nu q)$. Then we have that

$$o^*_p = \frac{p' - \tau^*_p r}{1 - \tau^*_p} = \frac{[\mu p + \mu q + (1 - \mu p - \mu q)] r - [\mu \tau_p + \mu \tau_q + (1 - \mu p - \mu q)]}{1 - [\mu \tau_p + \mu \tau_q + (1 - \mu p - \mu q)]} = \frac{\mu \tau_p + \mu \tau_q + (1 - \mu p - \mu q)}{1 - \mu \tau_p + \mu \tau_q + (1 - \mu p - \mu q)}.$$

Letting $\lambda_p = \frac{\mu p}{\mu p + \mu q}$ shows that $o'_p \in O(p, q) \subseteq \Omega$, and by a similar argument $o'_q \in \Omega$.
As Figure 8 shows, for any \( p', q' \in \Delta(\{p, q, r\}) \) we can draw a line connecting these two lotteries that intersects the source line \( O(p, q) \) at some \( o'_t \). By Proposition 3 this implies that \( p' \succ q' \) and \( \tau'_t \in T(p', q') \).

\((iv) \Rightarrow (i)\) This is immediate.

\(7.2.3\) Proof of Lemma 3

The steps in this proof follow much of the same logic as the proof of Lemma 2. Again, we will show that 
\((i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)\), and therefore the four statements are equivalent.

\((i) \Rightarrow (ii)\) Suppose that \( P \) is an incomparability set, then if \( \dim \Delta(P) < n - 2 \) we can pick \( \hat{r} \in \Delta(X) \setminus \Delta(P) \)\(^{27}\) and let \( \tau = \theta \delta_{x_0} + (1 - \theta)\hat{r} \) and \( \xi = \theta \delta_{x_1} + (1 - \theta)\hat{r} \), such that for \( \theta \) sufficiently close to 1, \( \tau \succ p \succ \xi \) for every \( p \in \Delta(P) \). For every \( p \in \Delta(P) \) let \( A_p = \{ \alpha : p \succ \alpha \hat{r} + (1 - \alpha)\hat{r} \} = [\alpha_p, \pi_p] \). If \( A^* = \bigcap_{p \in \Delta(P)} A_p = \emptyset \), then there are \( p_1, p_2 \in \Delta(P) \) and \( \alpha' \) such that \( \alpha_{p_1} > \alpha' > \alpha_{p_2} \). But this implies that \( p_1 \succ p_2 \), a contradiction. Picking any \( \alpha \in A^* \neq \emptyset \) and letting \( r = \alpha \hat{r} + (1 - \alpha)\hat{r} \) establishes the result.

\((ii) \Rightarrow (iii)\) Fix \( r \in \Delta(X) \setminus \Delta(P) \) satisfying (ii). Pick \( P_k = \{p_1, \ldots, p_k\} \subseteq P \) such that \( \Delta(P_k) = \Delta(P) \), then it will be sufficient to show that there are \( \{\tau_1, \ldots, \tau_k\} \subseteq \mathbb{R}^k_+ \) such that for every \( q = \sum_{j=1}^k \pi_j p_j \in \Delta(P_k) \), we have that \( \tau_q = \sum_{j=1}^k \pi_j \tau_j \in T(q, r) \).

**Claim** For any \( \ell \leq k \), let \( P_\ell = \{p_1, \ldots, p_\ell\} \), then there is \( \{\tau_1, \ldots, \tau_\ell\} \subseteq \mathbb{R}^\ell_+ \) such that for every \( q = \sum_{j=1}^\ell \pi_j p_j \in \Delta(P_\ell) \), we have that \( \tau_q = \sum_{j=1}^\ell \pi_j \tau_j \in T(q, r) \).

**Proof** We establish this property by induction. If \( \ell = 1 \) then the simplex is a singleton \( \Delta(\{p_1\}) = \{p_1\} \) so the property is trivially satisfied by picking any \( \tau_1 \in T(p_1, r) \). For the inductive step,

\(^{27}\)Again, \( \hat{r} \) exists by the dimensionality of \( \Delta(X) \).
suppose that we have such \( \{\tau_1, \ldots, \tau_k\} \in \mathbb{R}^{k+}_+ \) and let \( \tau_q = \sum_{j=1}^k \pi_j \tau_j \) for every \( q \in \Delta(P_i) \). By Lemma 2, for every \( q_1, q_2 \in \Delta(P_i) \) we have that \( q_1 \preceq q_2 \) and \( \frac{\tau_{q_1}}{\tau_{q_2}} \in T(q_1, q_2) \).  

Pick any \( p \in P \setminus \Delta(P_i) \) then, since \( P \) is an incomparability set, \( p \preceq q \) for any \( q \in \Delta(P_i) \). By Lemma 2 we have that

\[
Z_p^q := \{ \tau_p \in T(p, r) : \lambda \tau_p + (1 - \lambda) \tau_q \in T(\lambda p + (1 - \lambda) q, r), \forall \lambda \in [0, 1] \} = [\frac{\tau_p^q}{\tau_p}] \neq \emptyset.
\]

By Lemma 2, \( \tau_p \in Z_p^q \) implies that \( \frac{\tau_p}{\tau_q} \in T(p, q) \). Suppose that \( Z_p^q := \bigcap_{q \in \Delta(P_i)} Z_p^q = \emptyset \), then there are \( q_1, q_2 \in \Delta(P_i) \) and \( \tau_p^q \) such that \( \tau_p^q > \tau_p^r > \tau_q^r \). This in turn implies that \( \frac{\tau_p^q}{\tau_p^r} < \inf T(p, q_1) \) and \( \frac{\tau_p^r}{\tau_q^r} > \sup T(p, q_2) \). Letting \( s = \tau > p \), for \( \beta \in (0, 1) \) we have that

\[
\beta q_1 + (1 - \beta) s > \frac{\beta \left( \frac{\tau_p^q}{\tau_p^r} \right) p + (1 - \beta) s}{\beta \left( \frac{\tau_p^r}{\tau_q^r} \right) + (1 - \beta)} > \frac{\beta \left( \frac{\tau_p^q}{\tau_p^r} \right) q_2 + (1 - \beta) s}{\beta \left( \frac{\tau_p^r}{\tau_q^r} \right) + (1 - \beta)}.
\]

This contradicts \( \frac{\tau_p^q}{\tau_q^r} \in T(q_1, q_2) \). Hence \( Z_p^r \neq \emptyset \). Let \( p = p_{\ell+1} \) and pick any \( \tau_{\ell+1} \in Z_p^r \), then we have that the set \( \{\tau_1, \ldots, \tau_{\ell+1}\} \in \mathbb{R}^{\ell+1}_+ \) has the desired property.  

This completes the proof of the claim.

Returning to the proof of the lemma, let \( \ell = k \) and choose \( \{\tau_1, \ldots, \tau_k\} \in \mathbb{R}^k_+ \) that satisfies the claim. Then for every \( p := \sum_{j=1}^k \pi_j p_j \in P \), letting \( \tau_p = \sum_{j=1}^k \pi_j \tau_j \) establishes the result.

**(iii) \implies (iv)** Fix \( r \in \Delta(X) \setminus \Delta(P) \) and \( \{\ell p \}_{p \in P} \subseteq \mathbb{R}^+_+ \) satisfying (iii). Pick \( q_1, q_2 \in \Delta(P \cup \{r\}) \), then for \( i = 1, 2 \), \( q_i = \rho_i r + (1 - \theta_i) q_i' \) for some \( q_i' := \sum_{p \in \Delta(P)} \pi_p p \in \Delta(P) \). Let \( \tau_i = \sum_{p \in P} \pi_p \tau_p \in T(q_i', r) \) and \( \tau_i = \theta_i + (1 - \theta_i) \tau_i' \), then there is a source point at \( o_i = \frac{\tau_i - \tau_i'}{1 - \tau_i} = \frac{q_i - q_i'}{1 - \tau_i} \in \Omega \). By (iii), we have that \( \lambda \tau_1 + (1 - \lambda) \tau_2 \in T(\lambda q'_1 + (1 - \lambda) q'_2, r) \). Thus, letting \( \tau^* = \tau_1^* \) and \( \lambda^* = \frac{1}{1 - \tau^*} \) we have that

\[
o^* = \lambda^*(1 - \tau_1) q_1 + (1 - \lambda^*)(1 - \tau_2) q_2 = \frac{(q_1 - \tau_1 r) - \frac{\lambda^*}{\lambda}(q_2 - \tau_2 r)}{(1 - \tau_1) - \frac{\lambda^*}{1 - \tau_1} (1 - \tau_2)} = \frac{q_1 - \tau^* q_2}{1 - \tau^*} \in \Omega.
\]

---

28Following the proof of Lemma 2, this is seen by invoking (iii) for \( \lambda = \frac{1}{1 - \tau^*} \).

29For any \( q = \sum_{j=1}^k \pi_j p_j \), let \( \lambda = \pi_{\ell+1} \) and \( q' = \frac{\sum_{j=1}^k \pi_j p_j}{\pi_{\ell+1}} \). Then since \( \tau_{\ell+1} \in Z_p^r \subseteq Z_p^{q'} \), we have that \( \sum_{j=1}^k \pi_j \tau_j = \lambda \tau_{\ell+1} \in (1 - \lambda) \tau_{q'} \in T(\lambda p_{\ell+1} + (1 - \lambda) q', r) = T(q, r) \).
By Proposition 3, this implies that \( q_1 \asymp q_2 \) and \( \tau^* \in T(q_1, q_2). \)

\((iv) \Rightarrow (i)\) This is immediate.

\[ \square \]

### 7.2.4 Proof of Lemma 4

Fix \( p, q \in \Delta(X) \). By repeated application of Lemma 3, we have that \( p \asymp q \) if and only if they both belong to the same maximal incomparability set \( P \subseteq \Delta(X) \) with \( \dim \Delta(P) = n - 2 \). As boundedness of \( \succ \) implies \( \delta_x \succ p \succ \delta_{x_1} \) for every \( p \in \Delta(X) \), by betweenness there is a unique \( \alpha \in (0, 1) \) such that \( \zeta_{\alpha} \in \Delta(P) \).

Hence, for every \( i = 2, \ldots, n - 1 \) there exists some \( p_i = \lambda_i \delta_x + (1 - \lambda_i)\zeta_{\alpha} \in \Delta(P) \cap \Delta(\{x_1, x_i, x_n\}) \).

By Lemma 3, this implies that there exist \( \{\tau_2', \ldots, \tau'_{n-1}\} \) such that every \( \tau'_i \in T(p_i, \zeta_{\alpha}) \) and hence \( (\alpha, \tau'_i) \in \Phi(p_i) \).

Let \( \tau_i = \frac{1}{\lambda_i} \tau'_i + (1 - \frac{1}{\lambda_i}) \) and \( \alpha_i = \frac{\lambda_i \delta_x + (1 - \lambda_i)\zeta_{\alpha} - [\lambda_i \tau_i + (1 - \lambda_i)]\zeta_{\alpha}}{1 - [\lambda_i \tau_i + (1 - \lambda_i)]} = \frac{p_i - \tau'_i \zeta_{\alpha}}{1 - \tau'_i} \in \Omega \).

This implies that every \( (\alpha_i, \tau_i) \in \Phi(\delta_x). \)

This argument which follows is shown in Figure 9.\(^{30}\) For every \( p' := \sum_{i=1}^n \pi_i \delta_{x_i} \in \Delta(X) \), there are linearly independent \( \{p_2, \ldots, p_{n-1}\} \subseteq P \) \( q' = \sum_{i=2}^{n-1} \pi'_i p_i \in \Delta(P) \) such that \( p' = \lambda' q' + (1 - \lambda') \zeta_{\alpha'} \).

This implies \( (\alpha'_i, \tau'_i) \in \Phi(p') \).

Furthermore, letting \( (\alpha_1, \tau_1) = (0, 1) \) and \( (\alpha_n, \tau_n) = (1, 1) \), we have that

\[ (\alpha'_p, \tau'_p) = \left( \frac{\lambda' \sum_{i=2}^{n-1} \pi'_i [\lambda_i \tau_i \alpha_i + (1 - \lambda_i)\zeta_{\alpha} + (1 - \lambda') \zeta_{\alpha'}] + \lambda' \sum_{i=2}^{n-1} \pi'_i [\lambda_i \tau_i + (1 - \lambda_i)] + (1 - \lambda') \right) \]

\[ = \left( \frac{\lambda' \sum_{i=2}^{n-1} \pi'_i [\lambda_i \tau_i \alpha_i + (1 - \lambda_i)\zeta_{\alpha} + (1 - \lambda') \zeta_{\alpha'}] + \lambda' \sum_{i=2}^{n-1} \pi'_i [\lambda_i \tau_i + (1 - \lambda_i)] + (1 - \lambda') \right) \]

Since the above holds for any \( p \in \Delta(X) \), the collection \( \{\alpha_i, \tau_i\}_{i=1}^n \in \Psi \).

Returning to the proof, we have that for every \( p, q \in \Delta(X) \) that \( p \asymp q \) if and only if they lie on some maximal incomparability set \( P \), which defines \( \{(\alpha_i, \tau_i)\}_{i=1}^n \in \Psi \). Since there is a unique \( \alpha \in [0, 1] \) for which \( p, q, \zeta_{\alpha} \in P \), we must have \( \alpha_p = \alpha_q = \alpha \), which completes the proof.

\[ \square \]

\(^{30}\)The existence of a maximal incomparability set \( P \) implies that \( \Delta(P) \) crosses every triangle \( \Delta(\{x_1, x_i, x_n\}) \), formed by the best and worst outcomes along with some third outcome \( x_i \in X \), at some \( p'_i \). Furthermore, as betweenness implies that we may have at most one \( \zeta_{\alpha} \in \Delta(P) \), every \( p \asymp \zeta_{\alpha} \), so that we may find a source point \( \alpha_i \) in the usual manner. Drawing a line from \( \alpha_i \) through \( \delta_{x_i} \) allows us to find the utility weight pair \( (\alpha_i, \tau_i) \) for \( x_i \).

By the result of Lemma 3, every point on the line connecting two source points is itself a source point \( \alpha'_p \), and drawing this through any lottery \( p' \in \Delta(X) \) produces the pair \( (\alpha'_p, \tau'_p) \) which is in turn a linear combination of the \( (\alpha_i, \tau_i) \).
7.2.5 Proof of Lemma 5
Fix \( p \in \Delta(X) \) and \((\alpha^1, \tau^1), (\alpha^2, \tau^2) \in \Phi(p)\). If \( \tau^1 \neq \tau^2 \), let
\[
\begin{align*}
\beta^* &= \frac{\tau^2 \alpha^2 - \tau^1 \alpha^1}{(1 - \tau^1)^2 \alpha^2 - (1 - \tau^2)^2 \alpha^1}, \\
\alpha^* &= \frac{\beta^* \tau^1 \alpha^1}{\beta^* \tau^1 + (1 - \beta^*)} = \frac{\beta^* \tau^2 \alpha^2}{\beta^* \tau^2 + (1 - \beta^*)} = \frac{\tau^2 \alpha^2 - \tau^1 \alpha^1}{\tau^2 - \tau^1}, \\
\tau^{1*} &= \beta^* \tau^1 + (1 - \beta^*) = \frac{(\tau^2 - \tau^1) \tau^1 \alpha^1}{(1 - \tau^1) \tau^2 \alpha^2 - (1 - \tau^2)(\tau^1 \alpha^1)}, \\
\tau^{2*} &= \beta^* \tau^2 + (1 - \beta^*) = \frac{(\tau^2 - \tau^1) \tau^2 \alpha^2}{(1 - \tau^1) \tau^2 \alpha^2 - (1 - \tau^2)(\tau^1 \alpha^1)}.
\end{align*}
\]
Then for \( j = 1, 2 \), we have
\[
\alpha^j = \frac{p - \tau^j \zeta_{\alpha^j}}{1 - \tau^j} = \frac{[\beta^* p + (1 - \beta^*) \delta_{x_1}] - [\beta^* \tau^j \zeta_{\alpha^j} + (1 - \beta^*) \delta_{x_1}]}{1 - [\beta^* \tau^j + (1 - \beta^*)]} = \frac{[\beta^* p + (1 - \beta^*) \delta_{x_1}] - \tau^j \zeta_{\alpha^*}}{1 - \tau^j} \in \Omega.
\]
This implies that \((\alpha^*, \tau^{j*}) \in \Phi(\beta^* p + (1 - \beta^*) \delta_{x_1})\) for \( j = 1, 2 \), but as \( \tau^1 \neq \tau^2 \) implies \( \tau^{1*} \neq \tau^{2*} \), this would violate (A.4), so we must have \( \tau^1 = \tau^2 \). Hence, there is a unique \( \tau_p \) such that \( \alpha_p \in A(p) \) implies \((\alpha_p, \tau_p) = \Phi(p) = A(p) \times \{\tau_p\}\). \( \square \)

7.3 Proof of Theorem 1
(Necessity) Suppose that there is such a \( \mathcal{V} \) that represents \( \succ \). Then for any \((w^k, w^k) \in \mathcal{V} \), define
\[
U^k(p) = \frac{\sum_{x \in X} p(x)w^k(x)w^k(x)}{\sum_{x \in X} p(x)w^k(x)}, \quad W^k(p) = \sum_{x \in X} p(x)w^k(x).
\]
Hence $p \succ q$ if and only if $U^k(p) > U^k(q)$ for every $(u^k, w^k) \in \mathcal{V}$. It is easily verified that $U^k$ is weighted linear. To show that $\succ$ satisfies (A.1), note that for every $p \in \Delta(X)$, $\neg(U^k(p) > U^k(p))$, so $\succ$ is irreleflexive, and that for every $p, q, r \in \Delta(X)$, $U^k(p) > U^k(q)$ implies $U^k(p) > U^k(r)$, so $\succ$ is transitive.

To show that $\succ$ satisfies (A.2), pick $p, q, r \in \Delta(X)$ such that $p \succ q$, then $U^k(p) > U^k(q)$ for every $(u^k, w^k) \in \mathcal{V}$. If $U^k(q) > U^k(r)$, define $\alpha_k \in (0, 1)$ such that

$$U^k(\alpha_k p + (1 - \alpha_k)r) = \frac{\alpha_k W^k(p)U^k(p) + (1 - \alpha_k)W^k(r)U^k(r)}{\alpha_k W^k(p) + (1 - \alpha_k)W^k(r)} = U^k(q),$$

$$\alpha_k = \frac{W^k(r)[U^k(q) - U^k(r)]}{W^k(p)[U^k(p) - U^k(q)] + W^k(r)[U^k(q) - U^k(r)].$$

If $U^k(q) \leq U^k(r)$, then pick any $\alpha_k \in (0, 1)$. Pick any $\alpha > \inf_{(w^k, w^k)\in \mathcal{V}} \alpha_k$, then $U^k(\alpha p + (1 - \alpha)r) > U^k(q)$ for every $(u^k, w^k) \in \mathcal{V}$ and hence $\alpha p + (1 - \alpha)r \succ q$. By a similar argument, there is $\alpha' \in (0, 1)$ such that $q \succ \alpha'p + (1 - \alpha')r$.

To show that $\succ$ satisfies (A.3), pick $p, q \in \Delta(X)$ such that $p \succ q$, then $U^k(p) > U^k(q)$ for every $(u^k, w^k) \in \mathcal{V}$, so that for any $\alpha \in (0, 1)$,

$$U^k(p) > \frac{\alpha W^k(p)U^k(p) + (1 - \alpha)W^k(q)U^k(q)}{\alpha W^k(p) + (1 - \alpha)W^k(q)} = U^k(\alpha p + (1 - \alpha)q) > U^k(q).$$

This implies that $p \succ \alpha p + (1 - \alpha)q \succ q$.

To show that $\succ$ satisfies (A.4), pick $p, q \in \Delta(X)$ such that $p \preceq q$. This implies that there are $(u^1, w^1), (u^2, w^2) \in \mathcal{V}$ such that $U^1(p) \geq U^1(q)$ and $U^2(p) \leq U^2(q)$. Define

$$U^\kappa(p) = \frac{\kappa W^1(p)U^1(p) + (1 - \kappa)W^2(p)U^2(p)}{\kappa W^1(p) + (1 - \kappa)W^2(p)}, \quad W^\kappa(p) = \kappa W^1(p) + (1 - \kappa)W^2(p).$$

Then there is some $\kappa \in [0, 1]$ such that $U^\kappa(p) = U^\kappa(q)$. For every $\beta \in (0, 1)$, fix $\gamma \in (0, 1)$ such that the odds ratio $\frac{\beta(1 - \gamma)}{(1 - \beta)} = \frac{W^\kappa(p)}{W^\kappa(q)}$. Then, for every $r \in \Delta(X)$, we have

$$U^\kappa(\beta p + (1 - \beta)r) = \frac{\beta W^\kappa(p)U^\kappa(p) + (1 - \beta)W^\kappa(r)U^\kappa(r)}{\beta W^\kappa(p) + (1 - \beta)W^\kappa(r)} = \frac{\beta W^\kappa(p)W^\kappa(q)U^\kappa(q) + (1 - \beta)W^\kappa(r)U^\kappa(r)}{\beta W^\kappa(p)W^\kappa(q) + (1 - \beta)W^\kappa(r)}.$$

Thus we can have neither that $\beta p + (1 - \beta)r \succ q + (1 - \beta)r$ nor that $\beta p + (1 - \beta)r \prec q + (1 - \beta)r$.

Hence, $\beta p + (1 - \beta)r \succeq q + (1 - \beta)r$.

**Sufficiency** Suppose $\succ$ satisfies (A.1)-(A.4). Then, for every $\psi \in \Psi$ we can construct utility and weight functions by letting $u^\psi(x_i) = \alpha_i$ and $w^\psi(x_i) = \tau_i$ for $i = 1, \ldots, n$. For every $p \in \Delta(X)$, set

$$U^\psi(p) = \alpha_p = \sum_{i=1}^{n} p_i \alpha_i = \frac{\sum_{i=1}^{n} p_i \tau_i}{\sum_{i=1}^{n} \tau_i}, \quad W^\psi(p) = \tau_p = \sum_{i=1}^{n} \tau_i = \sum_{i=1}^{n} p_i \tau_i.$$

For every $p, q \in \Delta(X)$, we have that

$$U^k(\lambda p + (1 - \lambda)q) = \frac{\sum_{x \in X} \lambda p(x)w^k(x)u^k(x)}{\sum_{x \in X} \lambda p(x)u^k(x)} = \frac{\lambda \sum_{x \in X} p(x)w^k(x)u^k(x) + (1 - \lambda) \sum_{x \in X} q(x)w^k(x)u^k(x)}{\lambda \sum_{x \in X} p(x)u^k(x) + (1 - \lambda) \sum_{x \in X} q(x)u^k(x)}.$$
Suppose that \( p > q \) then, by Lemma 4, we have that for every \( \psi \in \Psi \), \( U^\psi(p) \neq U^\psi(q) \). Suppose that, for some \( \psi \in \Psi \), \( U^\psi(p) < U^\psi(q) \), then we can pick some \( r > p > q \) and \( \beta \in (0,1) \) such that \( U^\psi(\beta p + (1-\beta)r) = U^\psi(q) \). By Lemma 4, this would imply that \( \beta p + (1-\beta)r > q \). But betweenness implies \( r > \beta p + (1-\beta)r > p > q \). Thus we must have that \( U^\psi(p) > U^\psi(q) \), for every \( \psi \in \Psi \).

Now suppose \( U^\psi(p) > U^\psi(q) \) for every \( \psi \in \Psi \). Then, by Lemma 4 \( \neg(p \leq q) \) and by the argument above \( \neg(p < q) \), so we conclude that \( p > q \). Hence \( p > q \) if and only if \( U^\psi(p) > U^\psi(q) \) for every \( \psi \in \Psi \). Letting \( V = \{(u^\psi, w^\psi) : \psi \in \Psi \} \) establishes the representation.

\[
7.4 \text{ Proof of Theorem 2}
\]

**Necessity** Suppose that \( \langle \hat{V}_1 \rangle = \langle \hat{V}_2 \rangle := V^* \), then for every \( p \in \Delta(X) \) we have that \( \langle \hat{U}_{\hat{p}}^1 \rangle = \langle \hat{U}_{\hat{p}}^2 \rangle = \{u_p = wu + (1-w)\hat{u}_p : (u,w) \in V^* \} := U_p^\delta \). By Proposition 6, this implies that for any \( q \in \Delta(X) \),

\[
p > q \iff \sum_{x \in X} p(x)u_p(x) > \sum_{x \in X} q(x)u_p(x), \forall u_p \in U_p^\delta \iff p > \gamma q.
\]

This implies that \( \gamma^{-1}=\gamma^2 \).

**Sufficiency** Suppose, without loss of generality, that there is \( (u^*, w^*) \in \langle \hat{V}_1 \rangle \setminus \langle \hat{V}_2 \rangle \). Then for some \( p \in \Delta(X) \) we have that \( u_p^* = w^*u^* + (1-w^*)\hat{u}_p \in \langle \hat{U}_{\hat{p}}^1 \rangle \setminus \langle \hat{U}_{\hat{p}}^2 \rangle \). Then by the separating hyperplane theorem, there is \( q \in \Delta(X) \) such that

\[
\sum_{x \in X} [p(x) - q(x)]u_p(x) > 0 \iff \sum_{x \in X} [p(x) - q(x)]u_p(x), \forall u_p \in \langle \hat{U}_{\hat{p}}^2 \rangle
\]

This implies on one hand that \( \sum_{x \in X} p(x)u_p(x) > \sum_{x \in X} q(x)u_p(x) \) for every \( u_p \in U_p^\delta \), so that \( p > \gamma q \), but on the other hand that \( \sum_{x \in X} p(x)u_p^*(x) \leq \sum_{x \in X} q(x)u_p^*(x) \), so that as \( u_p^* \in \langle \hat{U}_{\hat{p}}^1 \rangle \), we have \( \neg(p > \gamma q) \). Hence, \( \gamma^{-1} \neq \gamma^2 \).

Therefore, we conclude that \( \gamma^{-1}=\gamma^2 \) if and only if \( \langle \hat{V}_1 \rangle = \langle \hat{V}_2 \rangle \).

\[
7.5 \text{ Proof of Theorem 3}
\]

**Necessity** Suppose that \( \mathcal{U} \) and \( \mathcal{W} \) that represent \( \succ \). Then for every \( u^k \in \mathcal{U} \), let

\[
U^k(p) = \frac{\sum_{x \in X} p(x)w(x)u^k(x)}{\sum_{x \in X} p(x)w(x)}, \quad \text{and} \quad W(p) = \sum_{x \in X} p(x)w(x).
\]

Letting \( \mathcal{V} = \mathcal{U} \times \{w\} \), by Theorem 1 we have that \( (A.1)-(A.3) \) are satisfied. To show that \( (A.5) \) is satisfied, pick \( p, q \in \Delta(X) \) such that \( p > q \), then there are \( u^1, u^2 \in \mathcal{U} \) such that \( U^1(p) \geq U^1(q) \) and \( U^2(p) \leq U^2(q) \). For every \( \beta \in (0,1) \), fix \( \tau = \frac{\gamma(1-\tau)}{\beta} = \frac{W(p)}{W(q)} \), then for every \( r \in \Delta(X) \),

\[
U^1(\beta p + (1-\beta)r) = \frac{\beta W(p)U^1(p) + (1-\beta)W(r)U^1(r)}{\beta W(p) + (1-\beta)W(r)} \geq \frac{\beta W(p)W(q)U^1(q) + (1-\beta)W(r)U^1(r)}{\beta W(p)W(q) + (1-\beta)W(r)} = \frac{\gamma W(q)U^1(q) + (1-\gamma)W(r)U^1(r)}{\gamma W(q) + (1-\gamma)W(r)} = U^1(\gamma q + (1-\gamma)r).
\]

Likewise, \( U^2(\beta p + (1-\beta)r) \leq U^2(\gamma q + (1-\gamma)r) \), which implies that \( \beta p + (1-\beta)r \geq \gamma q + (1-\gamma)r \). To show that the substitution ratio \( \tau = \frac{W(p)}{W(q)} \) is unique let \( r = \delta_{x_n} \), then for \( \tau' < \frac{W(p)}{W(q)} \) there is \( \beta \in (0,1) \) such that \( U^k(\beta p + (1-\beta)r) < U^k(\beta x_n^* + (1-\beta)\delta_{x_n}) \) for all \( u^k \in \mathcal{U} \), and likewise for \( \tau' > \frac{W(p)}{W(q)} \).
there is $\beta \in (0,1)$ such that $U^k(\beta p + (1-\beta)r) > U^k\left(\frac{\beta r'q + (1-\beta)r}{\beta r' + (1-\beta)}\right)$ for all $u^k \in U$. This implies that $\tau$, and therefore $\gamma$, is unique, so that (A.5) is satisfied.

(Sufficiency) Suppose that $\succ$ satisfies (A.1)-(A.3),(A.5). Then by Theorem 1 we have a representation by $\mathcal{V} = \{(u^\psi, w^\psi) : \psi \in \Psi\}$. By Lemma 5, for every $x_i \in X$ there is $\tau_i > 0$ such that $\Phi(\delta x_i) = A(\delta x_i) \times \{\tau_i\}$, and thus $w^\psi(x_i) = \tau_i := w(x_i)$ for every $\psi \in \Psi$. Thus letting $\mathcal{U} = \{u^\psi : \psi \in \Psi\}$, we have that $\mathcal{V} = \mathcal{U} \times \{w\}$, so that $\succ$ has the desired representation.

This completes the proof.

7.6 Proof of Theorem 4

(Necessity) Suppose we have $u$ and $W$ that represent $\succ$. Then for every $w^k \in W$, let

$$U^k(p) = \frac{\sum_{x \in X} b(x) w^k(x) u(x)}{\sum_{x \in X} p(x) w^k(x)}, \quad W^k(p) = \sum_{x \in X} p(x) w^k(x).$$

Letting $\mathcal{V} = \{u\} \times W$, by Theorem 1 we have that (A.1)-(A.4) are satisfied. To show that (A.6) is satisfied, for every $x_i \in X$ set $\alpha_i = u(x_i) - u(x_1) - u(x_n) - u(x_1)$, so that for every $w^k \in W$ we have $U^k(\delta x_i) = u(x_i) = v u(x_n) + (1-v) u(x_1) = U(\zeta_\alpha)$ if and only if $\alpha = \alpha_i$.

(Sufficiency) Suppose that $\succ$ satisfies (A.1)-(A.4),(A.6). Then by Theorem 1 we have a representation by $\mathcal{V} = \{(u^\psi, w^\psi) : \psi \in \Phi\}$. By (A.6), for every $x_i \in X$ there is a $\alpha_i$ such that $\Phi(\delta x_i) = \{\alpha_i\} \times T(\delta x_i, \zeta_\alpha)$, and thus $w^\psi(x_i) = \alpha_i := u(x_i)$ for every $\psi \in \Psi$. Thus letting $\mathcal{W} = \{w^\psi : \psi \in \Psi\}$, we have that $\mathcal{V} = \{u\} \times \mathcal{W}$, so that $\succ$ has the desired representation.

This completes the proof. □
References


