Theory Dec. (2016) 81:467–478
DOI 10.1007/s11238-016-9545-0

An expected utility theory for state-dependent preferences

Edi Karni[1,2] · **David Schmeidler**[3]

Published online: 16 March 2016

Abstract This note is a generalization and improved interpretation of the main result of Karni and Schmeidler (An Expected utility theory for state-dependent preferences. Working paper no. 48-80 of the Foerder Institute for Economic Research, Faculty of Social Sciences, Tel Aviv University, 1980). A decision-maker is supposed to possess a preference relation on acts and another preference relation on state-prize lotteries, both of which are assumed to satisfy the von Neumann–Morgenstern axioms. In addition, the two preference relations restricted to a state of nature are assumed to agree. We show that these axioms are necessary and sufficient for the existence of subjective expected utility over acts with state-dependent utility functions and a subjective probability measure. This subjective probability measure is unique when conditioned on the set of states of nature in which not all the prizes are equally desirable.

Keywords Subjective expected utility · State-dependent preferences · State-dependent utility · Subjective probabilities

Very useful comments and suggestions by an anonymous referee and Jean Baccelli are gratefully acknowledged. D. Schmeidler gratefully acknowledges the support of the Israeli Science Foundation under ISF Grant 204/13.

✉ Edi Karni
karni@jhu.edu

1 Johns Hopkins University, Baltimore, USA

2 University of Warwick, Coventry, UK

3 Tel Aviv University, Tel Aviv, Israel

1 Introduction

1.1 Prologue

This note is a generalization and improved interpretation of the main result of Karni and Schmeidler (1980). We never attempted to publish our original paper because, subsequently, we wrote another paper jointly with Karl Vind that, we thought, superseded our result (see Karni et al. 1983). Invoking the same analytical framework, the two papers give necessary and sufficient conditions for the existence of subjective expected utility representations of state-dependent preferences. The difference is that in Karni et al. (1983) we invoked only a subset of the state-prize lotteries introduced in Karni and Schmeidler (1980). At the time, our main interest was the characterization, in terms of preference relations, of the existence of unique subjective probabilities on the state space. Because the two models yielded the sought-after result and the result with Karl Vind required weaker hypothesis, we shelved our original result.

What we failed to fully appreciate at the time is the significance of the fact that the subjective probabilities and state-dependent utility function obtained in Karni et al. (1983) depend on an arbitrary choice of the subset of state-prize lotteries. Specifically, subsets of state-prize lotteries with different marginal probabilities on the states yield distinct subjective probabilities and utilities. Consequently, there is no guarantee that the subjective probabilities thus obtained represent the decision-maker's beliefs regarding the likely realization of the states, and there is no reason to suppose that the utility function represents his evaluation of the outcomes in the different states. This failure became apparent in Karni and Mongin (2000) who observed that the subjective probabilities in Karni and Schmeidler (1980) are the unique subjective probabilities that are independent of the marginal probabilities on the states. Furthermore, even if the underlying preference relation displays state independence, if there is a discrepancy between the subjective expected utility representation of Anscombe and Aumann (1963) and that of Karni and Schmeidler (1980), the latter probabilities and utilities are the correct representations of the decision-maker's beliefs and his evaluation of the consequences in the different states.[1]

For given marginal probabilities on the states, the utility function in Karni et al. (1983) is unique up to cardinal unit-comparable transformations. Hence it does not allow a meaningful way of comparing the utility of a prize in distinct states. By contrast, in Karni and Schmeidler (1980) the subjective probability is unique and the utility function is unique up to positive linear transformation, which permits the aforementioned comparisons.

In retrospect, the main result of our 1980 paper, summarized here, turns out to be more useful for a certain strain of literature, despite its more restrictive assumptions. This result is discussed in Nau (2001), Drèze and Rustichini (2004), Karni (2009), and Lu (2015). It was invoked by and played a crucial role in recent works by Riedener (2015), Baccelli (2016), and Karni and Safra (2016).

[1] As Karni and Mongin (2000) note "...we conclude that even under state independence, the KS (Karni and Schmeidler) rather than the conventional AA (Anscombe and Aumann) probability should be taken to be the decision maker's subjective probability." (p. 234).

1.2 Motivation of the original work

In the standard formulation of subjective expected utility theory, the preferences on alternative courses of actions are assumed to be state independent. The representation of these preferences consists of a subjective probability measure on the set of states, supposedly representing the decision-maker's beliefs regarding the likely realization of the different events (that is, subsets of the set of states of nature) and a utility index representing his evaluation of the consequences, independent of the underlying events. The imposition of state-independent utility is irreconcilable with some applications including the choice of life insurance, certain aspects of health and disability insurance, and insurance of family heirlooms. In these instances, the decision-maker's evaluation of the pecuniary outcome is not independent of the underlying state of nature. Hence the interest in extending the subjective expected utility model to allow for state-dependent preferences and a utility index that is assigned to each prize–state of nature pair. The utilities of two such pairs may differ, even if the prize is the same in both. One implication of this situation is that even if a risk-averse decision-maker is offered fair insurance terms, he may not choose full insurance (see Cook and Graham 1977; Hirshleifer and Riley 1979; Karni 1985). Clearly, the extension of the subjective expected utility model to include state-dependent preferences requires more information (observations) than the state-independent theory.

Attempts to construct an expected utility theory for state-dependent preferences were made by Fishburn (1973) and Drèze (1987). Fishburn assumed the existence of a preference relation over all acts conditioned on events. He required that, for every two disjoint events, not all the consequences conditioned upon one event are preferred to all the consequences conditioned upon the other. This restriction is irreconcilable with some applications (e.g., life insurance problems) that motivated our research [see Fishburn's own criticism (Fishburn 1974)]. Dreze combined state-dependent preferences with moral hazard.

1.3 Outline of the present work

Following Karni and Schmeidler (1980), we assume that there are finitely many states of nature and finitely many prizes, and it is not required that every prize be available in every state. Acts assign extraneous lotteries to each state of nature. Since the evaluation of a prize in our model depends upon the state, it is possible to think of a prize–state of nature pair as the ultimate outcome and to consider extraneous lotteries over these outcomes (state-prize lotteries for short). For a detailed discussion of the various types of state-dependent preferences and utility functions and the approach of Karni and Schmeidler (1980), see Baccelli (2016).

A decision-maker is supposed to possess a preference relation on acts and another preference relation on state-prize lotteries. The existence of a preference relation over acts is standard and requires no further elaboration. The preference relation over the state-prize lotteries requires explanation. For example, a person has to choose between two ClubMed resorts in February: ski in the Alps or Mauritius. In the event his leg is broken he prefers the latter, and otherwise he prefers ski. Here we assume that the person can rank (among others), staying the week in February in the ski resort with a broken leg, same in a good health, staying in the ClubMed Mauritius resort with a broken leg, and the same in a good health. Moreover, he can rank lotteries with these state-prize outcomes. Savage's P3, and P4 exclude the basic example's preferences. But Savage was aware of the possibility of such preferences and suggested how to deal with them in his framework: redefine outcomes to include the state-prize outcomes of our model, redefine acts accordingly, etc. Thus the same comparisons we used appear in Savage's redefined model.[2] The term 'principle' in the expression 'observable in principle' is very misleading.

The preference relation on acts and the preference relation on state-prize lotteries are assumed to satisfy the usual von Neumann–Morgenstern axioms. In addition, we impose a natural consistency axiom connecting the two preference relations. The consistency axiom requires that the preference relation on acts restricted to a state of nature agrees with the preference relation on state-prize lotteries restricted to the same state of nature. (A detailed discussion of this axiom in relation to null states appears in the last section.)

Applying the von Neumann–Morgenstern expected utility theorem to the preference relation on the state-prize lotteries, we obtain a utility index for each state-prize pair. This utility index is unique up to positive linear transformations. Applying the same theorem to the preference relation over acts yields an evaluation function on the state-prize pairs. This function is unique up to positive linear transformations, one for each state of nature, but with identical multiplicative coefficient across states. Using our consistency axiom we show that, for each state of nature, the utility function and the evaluation function are proportional. Properly normalized, the coefficients of proportionality constitute a subjective probability measure on the set of states of nature. The subjective probability measure is unique, except for the case in which all the prizes are equally desired for some states of nature.

In the process of revising our 1980 paper, we discovered that part of the consistency axiom we used is redundant. We deleted it.[3]

Section 2.1 describes the basic structure of the model and states our main result. The proof appears in the second subsection. In Sect. 3, we discuss the meaning of null states.

2 The formal model

2.1 Framework and main result

Let S be a finite, nonempty set whose elements are referred to as states of nature. For each $s \in S$, we denote by $X(s)$ a finite, nonempty, set and let $X = \cup_{s \in S} X(s)$. Elements of X are referred to as prizes and $X(s)$ is the set of feasible prizes in the state s. Define

[2] Savage's extended model contains more comparisons and it led him to suggest the baffling consequence of "being hanged without damage to his health or reputation". Drèze (1987, p. 68) . We do not require such consequences.

[3] The same redundancy appears in the statements of the consistency and strong consistency axioms in Karni et al. (1983).

$Y = \{(x, s) \in X \times S \mid x \in X(s)\}$. Let $F := \{f \in \mathbb{R}_+^Y \mid \forall s \in S,\ \Sigma_{x \in X(s)} f(x, s) = 1\}$ be the set of acts.

The novel aspect of our (1980) formulation is the introduction of the set of prize–state lotteries. Formally, $L := \{\ell \in \mathbb{R}_+^Y \mid \Sigma_{(x,s) \in Y} \ell(x, s) = 1\}$ is the set of prize–state lotteries. (In Karni et al. (1983) we considered subsets of L characterized by a fixed marginal distribution on the set of states.) For all $f, f' \in F$ and $\alpha \in (0, 1)$ define $\alpha f + (1-\alpha) f' \in F$ by $(\alpha f + (1-\alpha) f')(x, s) = \alpha f(x, s) + (1-\alpha) f'(x, s)$ and, for all $\ell, \ell' \in L$ and $\alpha \in (0, 1)$ define $\alpha\ell + (1-\alpha)\ell' \in L$ by $(\alpha\ell + (1-\alpha)\ell')(x, s) = \alpha\ell(x, s) + (1-\alpha)\ell'(x, s)$. Under these definitions, both F and L are convex sets.

We repeat here basic definition from the von Neumann–Morgenstern expected utility theory and a standard version of the representation theorem in this theory:

A binary relation $\trianglerighteq$ on a convex set C is said to be a weak order if it is complete (that is, for all $c, c' \in C$, either $c \trianglerighteq c'$ or $c' \trianglerighteq c$) and transitive. The corresponding strict preference relation, $\triangleright$ and the indifference relation, $\approx$ are defined as the asymmetric and symmetric parts of $\trianglerighteq$, respectively. The relation $\trianglerighteq$ is said to be Archimedean (or continuous) if for all $c, c', c'' \in C$, if $c \triangleright c'$ and $c' \triangleright c''$ then there exist $\alpha, \beta \in (0, 1)$ such that $\alpha c + (1-\alpha)c'' \triangleright c'$ and $c' \triangleright \beta c + (1-\beta)c''$. It satisfies independence if, for all $c, c', c'' \in C$, and $\alpha \in (0, 1]$, $c \triangleright c'$ implies $\alpha c + (1-\alpha)c'' \triangleright \alpha c' + (1-\alpha)c''$. The relation $\trianglerighteq$ is said to be nontrivial or it is non-degenerate if $\triangleright$ is nonempty.

Given these definitions, we can now state a standard version of the following well-known theorem.

Theorem 1 (Basic representation theorem (BRT)): *Given a binary relation, $\trianglerighteq$, on a convex, nonempty subset, C, of an Euclidean space, the following are equivalent:*

(i) The relation $\trianglerighteq$ is an Archimedean weak order satisfying the independence axiom.
(ii) There exist an affine real-valued function V on C such that for all $c, d \in C$

$$c \trianglerighteq d \Leftrightarrow V(c) \geq V(d). \tag{1}$$

Moreover, V is unique up to multiplication by a positive number and addition of any number.

Henceforth, we use of the symbols $\succcurlyeq$ and $\succcurlyeq^*$ to represent binary relations on F and L, respectively.

The BRT may be further specified when the set C is a simplex like L. In this case the affinity of V implies that for all $\ell \in L$,

$$V(\ell) = \sum_{s \in S} \sum_{x \in X(s)} u(x, s)\, \ell(x, s), \tag{2}$$

where $u(x, s) = V(\mathbf{1}_{x,s})$ for all $(x, s) \in Y$, and $\mathbf{1}_{x,s}$ denotes the degenerate lottery in L that assigns probability 1 to (x, s). Also the uniqueness properties of V are inherited by u.

If the set C is a product of simplices like F then, for all $f \in F$,

$$V(f) = \sum_{s \in S} \sum_{x \in X(s)} w(x, s)\, f(x, s), \tag{3}$$

where $w(x, s) = V(\mathbf{1}_{x,s})$ for all $(x, s) \in Y$, and for all $s \in S$, $\Sigma_{x \in X(s)} f(x, s) = 1$. Moreover, the uniqueness property of w is weaker than that of u in the sense that the additive constant may vary with $s \in S$.[4]

A prize–state lottery, $\ell \in L$ is said to be semipositive if $\Sigma_{x \in X(s)} \ell(x, s) > 0$, for every $s \in S$. Denote by L_{sp} the subset of L whose elements are semipositive. Let $H : L_{sp} \to F$ be a function defined by

$$H(\ell(x, s)) = \frac{\ell(x, s)}{\Sigma_{y \in X(s)} \ell(y, s)}, \quad \forall (x, s) \in Y.$$

Given $f, f' \in F$ and $s \in S$, f equals f' outside s if, for all $t = s$ and $x \in X(t)$, $f(x, t) = f'(x, t)$. Likewise for ℓ and ℓ' in L.

A state of nature s is said to be $\succcurlyeq$-nonnull if there are $f, f' \in F$ such that f equals f' outside s, and $f \quad f'$. Otherwise s is said to be $\succcurlyeq$-null . Similarly, a state of nature s is said to be $\succcurlyeq^*$-nonnull if there are $\ell, \ell' \in L$ such that ℓ equals ℓ' outside s, and $\ell \quad ^* \ell'$. Otherwise s is said to be $\succcurlyeq^*$-null.

The next axiom requires that, for any $\succcurlyeq$-nonnull $s \in S$, $\succcurlyeq$ and $\succcurlyeq^*$ rank lotteries on $X(s)$ identically. More precisely:

Consistency: For all s in S and all semipositive ℓ,$\ell' \in L$, such that ℓ equals ℓ' outside s : if $H(\ell) \quad H(\ell')$ then $\ell \quad ^* \ell'$.

We now state the main result.[5]

Theorem 2 *Let $\succcurlyeq$ on F and $\succcurlyeq^*$ on L be binary relations, then conditions (i) and (ii) below are equivalent, and condition (ii) implies condition (iii):*

(i) The binary relations $\succcurlyeq$ on F and $\succcurlyeq^*$ on L are Archimedean weak orders satisfying independence, $\succcurlyeq$ is non-degenerate , and jointly they satisfy consistency.

(ii) There exists a real-valued function u on Y and a (subjective) probability p on S such that, for all f ,$g \in F$,

$$f \succcurlyeq g \Leftrightarrow \sum_{s \in S} \sum_{x \in X(s)} p(s) u(x, s) [f(x, s) - g(x, s)] \geqq 0, \tag{4}$$

and, for all ℓ,$\ell' \in L$,

$$\ell \succcurlyeq^* \ell' \Leftrightarrow \sum_{s \in S} \sum_{x \in X(s)} u(x, s) \left[\ell(x, s) - \ell'(x, s)\right] \geqq 0. \tag{5}$$

Moreover, there are $s \in S$, and $z, z' \in X(s)$ such that, $p(s) > 0$, and $u(z, s) = u(z', s)$.

---

[4] These representations of vNM utility already appear in Fishburn's textbook (Fishburn 1970). For a web accessible presentation where $X(s) = X$ for all s, see http://www.tau.ac.il/~schmeid/PDF/Decision_Theory_Technical_Notes.pdf.

[5] Karni's (1985) statement of the following theorem misstated the uniqueness of the utility function. The same mistake appeared in Karni and Mongin's Proposition 3 (Karni and Mongin 2000), and is implied in the restatement of the result of Karni and Schmeidler (1980) in Karni et al. (1983).

(iii) (a) *The function u in (ii) is unique up to multiplication by a positive number and addition of any number;* (b) *if a state s is $\succcurlyeq$-null and $\succcurlyeq^*$-nonnull, then* $p(s) = 0$; (c) *if a state s is $\succcurlyeq$-nonnull then* $p(s) > 0$; (d) *the probability p conditioned on the set of $\succcurlyeq^*$-nonnull states is unique.*

Comments: As we see from statement (iii) in the theorem the uniqueness of the probability p is not guaranteed. Let us denote by $\mathcal{N}$ the subset of $\succcurlyeq$-null states, and by $\mathcal{N}^*$ the subset of $\succcurlyeq^*$-null states. If $\mathcal{N}^* = \varnothing$ then by (iii) (d) the probability p in (4) is unique. From representation 5 it is obvious that a state t is $\succcurlyeq^*$-null iff $\forall y, z \in X(t)$, $u(z, t) = u(y, t)$. This in turn implies that the state t is also $\succcurlyeq$-null. Hence $\mathcal{N}^* \subset \mathcal{N}$. In this case $p(t)$ may attain any value in $[0, 1)$ without affecting the inequalities in 4.

The preference relations $\succcurlyeq$ and $\succcurlyeq^*$ are said to satisfy inverse consistency if, for all s in S and all semipositive $\ell,\ell' \in L$ such that ℓ equals ℓ' outside s, $\ell \quad {}^* \ \ell'$ implies $H(\ell) \quad H(\ell')$. The preference relations $\succcurlyeq$ and $\succcurlyeq^*$ satisfy inverse consistency if and only if $\mathcal{N} = \mathcal{N}^*$. Since $\mathcal{N}^* \subset \mathcal{N}$ to prove this assertion we need to show that $\mathcal{N} \subset \mathcal{N}^*$. But inverse consistency implies that $S \setminus \mathcal{N}^* \subset S \setminus \mathcal{N}$ or, equivalently, that $\mathcal{N} \subset \mathcal{N}^*$.

In general, we do not impose inverse consistency, thus allowing for the possibility that for some $\ell,\ell' \in L$ such that ℓ equals ℓ' outside s, $\ell \quad {}^* \ \ell'$ and $H(\ell) \sim H(\ell')$. In this case, the representation (5) implies that $\sum_{x \in X(s)} u(x, s)[\ell(x, s) - \ell'(x, s)] > 0$ and the representation (4) implies that $p(s) \sum_{x \in X(s)} u(x, s)[f(x, s) - g(x, s)] = 0$. The latter condition requires that $p(s) = 0$.

However, the axiomatic structure does not rule out that there is another real-valued, constant, function $v(\cdot, s)$ on $X(s)$ such that representation (4) can be replaced with the following representation: $f \succcurlyeq g$ if and only if

$$\sum_{t \in S \setminus \{s\}} \sum_{x \in X(s)} p(t) u(x, t) \, [f(x, t) - g(x, t)] + p(s) \sum_{x \in X(s)} v(x, s) \, [f(x, s) - g(x, s)] \geqq 0,$$

where $p(s) \in (0, 1)$. That is, the equivalence of (i) and (ii) in the theorem holds. But $(iii)(b)$ no longer holds: the fact that s is $\succcurlyeq$-null does not imply that the probability of s is zero. This lack of determinacy of the probabilities of the null states (events) is not specific to this model. It holds in the representations of subjective expected utility in Savage (1954) and Anscombe and Aumann (1963).

2.2 Proof of the theorem

Obviously the direction that (ii) implies (i) is trivial. The last sentence of (ii) implies the nondegeneracy of $\succcurlyeq$ in (i).[6]

Invoking the implication (i) implies (ii) in the BRT, we get that (i) implies the representation (5) of $\succcurlyeq^*$ and the representation (3) of $\succcurlyeq$. The BRT also implies that

[6] This also is the trivial direction of the BRT.

(iii) (a) holds. To complete the proof that (i) implies (ii) we have to show that there exist $\alpha > 0$, $\beta : S \to \mathbb{R}$, and $p : S \to \mathbb{R}_{+}$, such that, for all $s \in S$ and $x \in X(s)$,

$$p(s)u\,(x, s) = \alpha w\,(x, s) + \beta(s). \tag{6}$$

Our only tools are the axioms of consistency and nondegeneracy of $\succcurlyeq$.

Nondegeneracy of $\succcurlyeq$ implies that there exist a state t such that, $\max_{x \in X(t)} w(x, t) > \min_{x \in X(t)} w(x, t)$. Let $\underline{x}$ and $\overline{x}$ be such that $w(\underline{x}, t) = \min_{x \in X(t)} w(x, t)$, and $w(\overline{x}, t) = \max_{x \in X(t)} w(x, t)$. Define f' and g' to coincide outside t, and $f^{'}(\overline{x}, t) = 1 = g'(\underline{x}, t)$. Then

$$\Sigma_{s \in S} \Sigma_{x \in X(s)} w\,(x, s) \left[f'\,(x, s) - g'\,(x, s)\right] = w\,(\bar{x}, t) - w\,\left(\underline{x}, t\right) > 0. \tag{7}$$

Because $w(\cdot, \cdot)$ represents $\succcurlyeq$, f' g'. Since the function H maps L_{sp} onto F, any $f \in F$ is an image of an $\ell \in L_{sp}$ defined by dividing all values of f by $\#S$. Deriving in this way $\ell^{f'}, \ell^{g'} \in L_{sp}$ from f' and g', we get that $\ell^{f'}$ and $\ell^{g'}$ are equal outside t. By consistency $\ell^{f'}$ $\ell^{g'}$ and, by the representation (5), $u(\underline{x}, t) < u(\overline{x}, t)$. Hence there are $\theta, \varphi \in \mathbb{R}$, $\theta > 0$ such that, $\theta w(\bar{x}, t) + \varphi = u(\bar{x}, t)$ and $\theta w(\underline{x}, t) + \varphi = u(\underline{x}, t)$. Define w^{1} on Y by,

$$w^{1}(x, s) = \begin{cases} \theta w(x, s) + \varphi & s = t \\ \theta w(x, s) & s = t \end{cases}$$

By the uniqueness of the representation in the BRT, (3), w^{1} represents the same preferences as w, (i.e., it represents $\succcurlyeq$).

Claim For all $y \in X(t)$, $w^{1}(y, t) = u(y, t)$.

Proof of Claim By construction $w^{1}(y, t) = u(y, t)$ holds for $y = \underline{x}$ and $y = \overline{x}$. Let $y \in X(t) \setminus \{\underline{x}, \overline{x}\}$. By way of negation, suppose that $w^{1}(y, t) < u(y, t)$. (The opposite inequality is treated analogically.) Since $w^{1}(\underline{x}, t) \leq w^{1}(y, t) \leq w^{1}(\overline{x}, t)$, at least one of these inequalities is strict.

Assume that $w^{1}(y, t) < w^{1}(\overline{x}, t)$, and choose a $\mu \in (w^{1}(y, t), u(y, t))$. Define f and g in F to be equal outside t, and for $x \in X(t)$

$$f(x, t) = \begin{cases} \mu & x = \overline{x} \\ 1 - \mu & x = \underline{x} \\ 0 & x = \underline{x}, \overline{x} \end{cases}$$

$$g(x, t) = \begin{matrix} 1 & x = y \\ 0 & x = y \end{matrix}$$

Evaluating f and g with w^{1} we get that $\mu > w^{1}(y, t)$. Thus f g. On the other hand, defining $\ell^{f}, \ell^{g} \in L_{\text{sp}}$ by $\ell^{f}(x, s) = f(x, s)/\#S$, and $\ell^{g}(x, s) = g(x, s)/\#S$, and evaluating ℓ^{f} and ℓ^{g} with u we get the opposite inequality [that is, $\mu < u(y, t)$]. Thus, ℓ^{g} $^{*}\,\ell^{f}$. Since $H(\ell^{f}) = f$ and $H(\ell^{g}) = g$, this contradicts consistency.

If $w^1(y,t) > w^1(\bar{x},t)$ a similar construction leads to a contradiction. This concludes the proof of the claim. □

At this stage we have w^1 representing $\succcurlyeq$. The valuating function, w^1 differs from the one derived by the BRT from the axioms on $\succcurlyeq$ in two respects: first, $w^1(\cdot,s) = \theta w(\cdot,s)$, for all $s = t$. Second, $w^1(\cdot,t) = \theta w(\cdot,t) + \varphi = u(\cdot,t) = q(t)u(\cdot,t)$, where $q(t) = 1$. Next assume that there is another $\succcurlyeq$-nonnull state, r in S. Similarly to the above let $\underline{x}(r)$ and $\overline{x}(r)$ be a minimizer and a maximizer of the function $w^1(\cdot,r)$ on $X(r)$.

Constructing the appropriate acts and the corresponding lotteries, we conclude (via consistency) that $u(\underline{x}(r),r) < u(\overline{x}(r),r)$. So there are $\theta(r), \varphi(r) \in \mathbb{R}$, $\theta(r) > 0$ such that, $\theta(r)w^1(\bar{x},t) + \varphi(r) = u(\bar{x},t)$ and $\theta(r)w^1(\underline{x},t) + \varphi(r) = u(\underline{x},t)$. Denote $q(r) = \theta(r)$ and define w^2 on Y by

$$w^2(x,s) = \begin{matrix} q(r)w^1(x,s) + \varphi(r) & s = r \\ q(r)w^1(x,s) & s = r \end{matrix}$$

It is easy to see that the Claim holds when r replaces t. Thus, w^2 represents $\succcurlyeq$, and for $s \in \{t,r\}$, $w^2(\cdot,s) = q(s)u(\cdot,s)$. This procedure can be applied to all $k = S\backslash\mathcal{N}$, $\succcurlyeq$-nonnull states. For any $\succcurlyeq$-null state, s, define $q(s) = 0$ and let $w^k(\cdot,s) = u(\cdot,s)$. Normalizing the vector q yields the representation (4) in (ii).

Next we prove that (ii) implies (iii). That (ii) implies (iii) (a) has already been established. Considering representation (4) of $\succcurlyeq$ it is obvious that a state s is $\succcurlyeq$-null if and only if at least one of the two following conditions holds: (A) for all $x, y \in X(s)$, $u(x,s) = u(y,s)$ or (B) $p(s) = 0$. From representation (5) it is clear that a state s is $\succcurlyeq^*$-null if and only if, for all $x, y \in X(s)$, $u(x,s) = u(y,s)$ (that is, if and only if (A) holds). Hence (iii) (b) and (iii) (c) hold.

To prove (iii) (d) note first that, for all $f, g \in F$,

$$\sum_{s\in\mathcal{N}} \sum_{x\in X(s)} p(s)u(x,s)\,[f(x,s) - g(x,s)] = 0. \tag{8}$$

Hence, in (4) we can replace

$$f \succcurlyeq g \Leftrightarrow \sum_{s\in S} \sum_{x\in X(s)} p(s)u(x,s)\,[f(x,s) - g(x,s)] \geqq 0 \tag{9}$$

with

$$f \succcurlyeq g \Leftrightarrow \sum_{s\in S\diagdown\mathcal{N}} \sum_{x\in X(s)} p(s)u(x,s)\,[f(x,s) - g(x,s)] \geqq 0, \tag{10}$$

where, by (iii) (c), for all $s \in S \diagdown \mathcal{N} : p(s) > 0$. Suppose that for each $s \in S \diagdown \mathcal{N}$ there is $q(s) \in \mathbb{R}_+$ such that

$$\sum_{s \in S \diagdown \mathcal{N}} p(s) = \sum_{s \in S \diagdown \mathcal{N}} q(s) = 1 \tag{11}$$

and, for all $f, g \in F$,

$$f \succcurlyeq g \Leftrightarrow \sum_{s \in S \diagdown \mathcal{N}} \sum_{x \in X(s)} q(s) u(x, s) [f(x, s) - g(x, s)] \geqq 0. \tag{12}$$

If there exist $r \in S \diagdown \mathcal{N}$ such that $p(r) > q(r)$, by (11) there exist $t \in S \diagdown \mathcal{N}$ such that $p(t) < q(t)$. As $r, t \notin \mathcal{N}$, there are $y(r), z(r) \in X(r)$ and $y(t), z(t) \in X(t)$ such that, $u(y(r), r) > u(z(r), r)$ and $u(y(t), t) < u(z(t), t)$. Once again we construct f and g in F to be equal outside $\{r, t\}$, and for some numbers ς, τ, φ, and ψ in (0, 1) (to be specified later),

$$f(x, r) = \begin{cases} \varsigma & x = y(r) \\ 1 - \varsigma & x = z(r) \\ 0 & x \in X(r) \backslash \{y, z\} \end{cases} \qquad f(x, t) = \begin{cases} \tau & x = y(t) \\ 1 - \tau & x = z(t) \\ 0 & x \in X(r) \backslash \{y, z\} \end{cases}$$

$$g(x, r) = \begin{cases} \varphi & x = y(r) \\ 1 - \varphi & x = z(r) \\ 0 & x \in X(r) \backslash \{y, z\} \end{cases} \qquad g(x, t) = \begin{cases} \psi & x = y(t) \\ 1 - \psi & x = z(t) \\ 0 & x \in X(r) \backslash \{y, z\} \end{cases}$$

Given these notations the number of summands in (10) can be further reduced and we have, $f \succcurlyeq g$ if and only if

$$p(r) [u(y(r), r) (\varsigma - \varphi) + u(z(r), r) (\varphi - \varsigma)] + p(t) [u(y(t), t)(\tau - \psi) + u(z(t), t)(\psi - \tau)] \geqq 0 \tag{13}$$

or, equivalently, $f \succcurlyeq g$ if and only if

$$p(r)(\varsigma - \varphi) [u(y(r), r) - u(z(r), r)] + p(t) (\tau - \psi) [u(y(t), t) - u(z(t), t)] \geqq 0. \tag{14}$$

When we replace the probability p with q, to obtain the required contradiction, we have to show that,

$$q(r) (\varsigma - \varphi) [u(y(r), r) - u(z(r), r)] + q(t) (\tau - \psi) [u(y(t), t) - u(z(t), t)] < 0. \tag{15}$$

The inequality in (11) can be rewritten as

$$(\varsigma - \varphi) p(r) [u(y(r), r) - u(z(r), r)] \geqq (\tau - \psi) p(t) [u(z(t), t) - u(y(t), t)], \tag{16}$$

whereas the inequality (15) can be rewritten as

$$(\varsigma - \varphi)\, q(r)\, [u(y(r), r) - u(z(r), r)] < (\tau - \psi)\, q(t)\, [u(z\,(t), t) - u(y(t), t)]\,. \tag{17}$$

All the square bracketed expressions in (16) are positive. So one can find ς, τ, φ, and ψ in (0, 1), with $\varsigma > \varphi$ and $\tau > \psi$ such that equality holds in (16). Replacing $p(r)$ with the smaller $q(r)$ and replacing $p(t)$ with the larger $q(t)$ yields the inequality as stated in (17). So we get the desired contradiction: $f \succcurlyeq g$ and $g \succ f$. Thus the proof of $(iii)(d)$, and the proof of the theorem is completed. □

3 Null states

The formal definition of a null state introduced in the preceding section does not capture the intuitive notion of a null state—that is, a state whose prior (subjective) probability is zero. To see this, it is useful to classify the various situations that may arise while restricting the preference relations $\succcurlyeq$ and $\succcurlyeq^*$ to a state s in S as in Table 1. To simplify the exposition in Tables 1 and 2 below, we denote by $\succ_s$ and $\succ_s^*$ the strict preference relations on F and $\Delta(X)$, respectively, defined by: $f \succ_s g$ if f and g are equal outside s and $f \succcurlyeq g$ and $\ell \succ_s^* \ell'$ if ℓ and ℓ' are equal outside s and $\ell \succcurlyeq^* \ell'$.

The assertion $p(s) = 0$ in the bottom left corner of Table 1 is based on the representation in the theorem, and disregards our last comment in Sect. 2.1.

The right top entry of Table 1 is empty by the consistency axiom. The left bottom entry corresponds to our definition of a null state and $p(s) = 0$ by the theorem. The inequality $p(s) > 0$ in the left top entry is also implied by the theorem. The configuration ($\succ_s = \varnothing$, $\succ_s^* = \varnothing$) which corresponds to the right bottom entry does not permit us to differentiate between null and non-null s in the intuitive sense. This is illustrated in the example in Table 2.

The four entries in Table 2 represent utilities. Based on them we construct a unique well-defined preference relation, $\succcurlyeq^*$ on L. For any probability vector p satisfying the restriction $p(s) > 0$ of Table 2, we construct a unique well-defined preference relation, $\succcurlyeq$ on F. Moreover, all relations $\succcurlyeq$ on F constructed in this manner are identical. Therefore, using the representations in the theorem to deduct the probability p from this relation, we show that $p(t)$ can be any number in the interval [0, 1). The state t corresponds to the right bottom entry of Table 1. Although formally defined

Table 1 Null states

$\succ \setminus \succ^*$	$\succ_s^* \neq \varnothing$	$\succ_s^* = \varnothing$
$\succ_s \neq \varnothing$	$p(s) > 0$	$\varnothing$
$\succ_s = \varnothing$	$p(s) = 0$	$p(s) \in [0, 1)$

Table 2 Example

$x \setminus S$	s	t
x	3	1
y	0	1
P	$p(s) > 0$	$p(t) = 1 - p(s)$

as a null state it may be assigned positive probability. This example illustrates that although a state may be null in an intuitive sense, it may not be possible to deduce this property from (the observation) $\succcurlyeq$ and $\succcurlyeq^*$. The example in Table 2 can be easily extended to any finite number of states and prizes.

To avoid the indeterminacy of p as described above, the condition for uniqueness of p stated in the theorem as: “for all s in S, the relation $^*_s$ is nonempty”. However, we prove a somewhat stronger result. Denote by T the set of states s such that $^*_s = \varnothing$. Then the probability p from the theorem conditioned on T is unique.

It is worth emphasizing that state-independent preferences do not imply state-independent utility functions. Karni and Mongin (2000) argue that, in this case, if there is a discrepancy between the subjective probability of Anscombe and Aumann (1963) and those obtained in this model, the latter probability is the one that represents the decision-maker’s beliefs.

References

Anscombe, F. J., & Aumann, R. J. (1963). A definition of subjective probability. *Annals of Mathematical Statistics*, *43*, 199–205.

Baccelli, J. (2016). Do bets reveal beliefs? A unified perspective on state-dependent utility issues. *Synthese*.

Cook, P. J., & Graham, D. A. (1977). The demand of insurance and protection: the case of irreplaceable commodities. *Quarterly Journal of Economics*, *91*, 143–156.

Drèze, J. H. (1987). *Decision theory wit moral hazard and state-dependent preferences. In Essays on economic decisions under uncertainty*. Cambridge: Cambridge University Press.

Dreze, J. H., & Rustichini, A. (2004). State dependent utility and decision theory. In S. Barbera, P. Hammond, & C. Seidl (Eds.), *Handbook of utility theory: volume 2 extensions* (pp. 839–892). New York: Springer.

Fishburn, P. C. (1970). *Utility theory for decision making*. New York: Wiley.

Fishburn, P. C. (1973). A mixture-set axiomatization of conditional subjective expected utility. *Econometrica*, *41*, 1–25.

Fishburn, P. C. (1974). On the foundations of decision making under uncertainty. In M. S. Balch, D. L. McFadden, & S. Y. Wu (Eds.), *Essays on economic behavior under uncertainty*. Amsterdam: North-Holland.

Hirshleifer, J., & Riley, J. G. (1979). The analytics of uncertainty and information—an expository survey. *Journal of Economic Literature*, *17*, 1375–1421.

Karni, E. (1985). *Decision making under uncertainty: the case of state-dependent preferences*. Cambridge: Harvard University Press.

Karni, E. (2009). State-dependent utility. In P. Anand, P. K. Pattanaik, & C. Puppe (Eds.), *Handbook of rational and social choice*. Oxford: Oxford University Press.

Karni, E., & Mongin, P. (2000). On the Determination of subjective probability by choice. *Management Science*, *46*, 233–248.

Karni, E., & Safra, Z. (2016). A theory of stochastic choice under uncertianty. *Journal of Mathematical Economics* (**forthcoming**).

Karni, E., & Schmeidler, D. (1980). An Expected utility theory for state-dependent preferences. Working paper no. 48–80 of the Foerder Institute for Economic Research, Faculty of Social Sciences, Tel Aviv University.

Karni, E., Schmeidler, D., & Vind, K. (1983). On state dependent preferences and subjective probabilities. *Econometrica*, *51*, 1021–1032.

Lu, J. (2015). A Bayesian theory of state-dependent utilities. (**unpublished**).

Nau, R. (2001). De Finetti was right: probability does not exist. *Theory and Decision*, pp. 89–124.

Riedener, S. (2015). Maximising expected value under axiological uncertainty an axiomatic approach. PhD dissertation.

Savage, L. J. (1954). *The foundations of statistics*. New York: Wiley.

von Neumann, J., & Morgenstern, O. (1947). *Theory of games and economic behavior*. Princeton: Princeton University Press.