# Probabilistic Sophistication and Reverse Bayesianism

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**Abstract**

This paper extends our earlier work on reverse Bayesianism by relaxing the assumption that decision makers abide by expected utility theory, assuming instead weaker axioms that merely imply that they are probabilistically sophisticated. We show that our main results, namely, (modified) representation theorems and corresponding rules for updating beliefs over expanding state spaces and null events that constitute "reverse Bayesianism," remain valid.

**Keywords:** Awareness, unawareness, reverse Bayesianism, probabilistic sophistication

**JEL classification:** D8, D81, D83

# 1 Introduction

Probabilistically sophisticated choice is characterized by a unique subjective probability measure on a state space by which acts (that is, mappings from the set of states to the set of consequences) are transformed to lotteries on the set of consequences, a utility function on the set of these lotteries, and choice behavior that maximizes the utility over the lotteries corresponding to the feasible set of acts. In two seminal papers, Machina and Schmeidler (1992, 1995) axiomatized probabilistically sophisticated choice in the analytical frameworks of Savage (1954) and Anscombe and Aumann (1963). The upshot of these contributions is that a choice-based subjective Bayesian prior exists even if decision makers do not abide by the stricture of the subjective expected utility model.

In Karni and Vierø (2012) we introduced a model describing the evolution of the beliefs of subjective expected utility maximizing decision makers' as they discover new acts, consequences and information pertaining to links between acts and consequences. In this paper we extend our earlier work and show that our results are not predicated on subjective expected utility maximizing behavior. To do so we depart from the expected utility model, assuming instead that decision makers are merely probabilistically sophisticated. Invoking the analytical framework of Anscombe and Aumann (1963), we demonstrate that our main results, namely, (modified) representations of preferences and corresponding rules for updating beliefs over expanding state spaces that constitute "reverse Bayesianism," hold when preferences are probabilistically sophisticated. Since the analytical framework and the related literature were discussed in Karni and Vierø (2012), in what follows, we review briefly those aspects of the model necessary to make the exposition self-contained, underscoring instead the adjustment necessary for the transition from expected utility to probabilistically sophisticated choice.

In the next section we revisit our analytical framework and the axiomatic structure characterizing probabilistic sophistication. In section 3 we expose the representation theorems and analyze the evolution of beliefs in the wake of discovery of new consequences, acts and links between them. Concluding remarks appear in section 4. The proofs are collected in section 5.

## 2 The Analytical Framework

Building upon Schmeidler and Wakker (1987) and Karni and Schmeidler (1991), Karni and Vierø (2012) introduced a unifying framework within which growing awareness due to the discovery of new acts and consequences as well as revising beliefs in light of new information regarding their links may be described and analyzed. We recall this framework below.

### 2.1 Conceivable states and acts

Let $F$ be a finite, nonempty set of *feasible acts*, and let $C$ be a finite, nonempty set of *feasible consequences*. Together these sets determine a *conceivable state space*, $C^F$, whose elements depict the resolutions of uncertainty. In other words, a *state* is a function from the set of feasible acts to the set of feasible consequences which, once known, resolves all uncertainty. As an illustration, let there be two feasible acts, $F = \{f_1, f_2\}$, and two consequences, $C = \{c_1, c_2\}$. The resulting conceivable state space is $C^F$, consisting of four states as depicted in the following matrix:

$$\begin{array}{ccccc} F\backslash C^F & s_1 & s_2 & s_3 & s_4 \\ f_1 & c_1 & c_2 & c_1 & c_2 \\ f_2 & c_1 & c_1 & c_2 & c_2 \end{array} \tag{1}$$

As this example makes clears, states are defined by the consequences associated with the feasible acts. In this sense states and events (that is, subsets of the set of conceivable states) are, respectively, complete and partial resolutions of uncertainty that are, in principle observable.

Once the set of conceivable states is fixed, the set of acts can be expanded to include all *conceivable acts*. As in Anscombe and Aumann (1963) the set of conceivable acts consists of all the mappings from the conceivable state space to lotteries on the set of consequences. Formally, the set of conceivable acts is given by:

$$\hat{F} := \{f : C^F \to \Delta(C)\}, \tag{2}$$

where $\Delta(C)$ is the set of all lotteries with consequences in $C$ as prizes.[1] Conceivable acts are imaginable given the decision maker's awareness of feasible acts and consequences and

[1]Formally, $p \in \Delta(C)$ is a function $p : C \to [0, 1]$ satisfying $\Sigma_{c \in C} p(c) = 1$. Notice that with this definition

the corresponding conceivable state space. In other words, it is possible to envisage placing bets on the outcomes of the feasible acts, whose payoffs are lotteries on the set of feasible consequences.

A decision maker's conceivable state space expands due to discovery of new feasible acts and/or consequences. Consider the two-act two-consequences example depicted above and imagine that a third feasible consequence was discovered, so that the new set of feasible consequences is $C' = (c_1, c_2, c_3\}$. The feasible acts need to be redefined, since choosing the act $f_i, i = 1, 2$ conceivably may result in any of the three consequences. We denote the redefined set of feasible acts by $F^*$ and the corresponding conceivable state space is given by:

| $F^* \backslash (C')^{F^*}$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ | $s_8$ | $s_9$ |
|---|---|---|---|---|---|---|---|---|---|
| $f_1$ | $c_1$ | $c_2$ | $c_1$ | $c_2$ | $c_3$ | $c_3$ | $c_1$ | $c_2$ | $c_3$ |
| $f_2$ | $c_1$ | $c_1$ | $c_2$ | $c_2$ | $c_1$ | $c_2$ | $c_3$ | $c_3$ | $c_3$ |

The event $(C')^{F^*} - C^F = \{s_5, \ldots, s_9\}$ represents the expansion of the decision maker's conceivable state space due to the discovery of the consequence $c_3$.

Discovery of new feasible acts also alters the conceivable state space, albeit in a different way. Consider again the two-act two-consequences example above and suppose that a new feasible act, $f_3$, is discovered. The new set of feasible acts is $F' = \{f_1, f_2, f_3\}$ and the corresponding conceivable state space, $C^{F'}$, consists of eight states as follows:

| $F' \backslash C^{F'}$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ | $s_8$ |
|---|---|---|---|---|---|---|---|---|
| $f_1$ | $c_1$ | $c_2$ | $c_1$ | $c_2$ | $c_1$ | $c_2$ | $c_1$ | $c_2$ |
| $f_2$ | $c_1$ | $c_1$ | $c_2$ | $c_2$ | $c_1$ | $c_1$ | $c_2$ | $c_2$ |
| $f_3$ | $c_1$ | $c_1$ | $c_1$ | $c_1$ | $c_2$ | $c_2$ | $c_2$ | $c_2$ |

The expanded state space $C^{F'}$ is a finer partition of the original state space $C^F$. Specifically, each element of $C^F$ is a non-degenerate event in the expanded state space $C^{F'}$. For example, the state $s_1 := (c_1, c_1) \in C^F$ is the event $E = \{s_1, s_5\}$ in the state space $C^{F'}$.

As the decision maker's conceivable state space expands, so does the set of conceivable acts. In the wake of the discovery of a new consequence, the new set of conceivable acts is

of $\Delta(C)$ we have that, for any $C \subset C'$, any $p \in \Delta(C)$ is also an element of $\Delta(C')$ with $p(c) = 0$ for all $c \in C' - C$. Likewise, $q \in \Delta(C')$ is an element of $\Delta(C)$ if $q(c) = 0$ for all $c \in C' - C$.

$\hat{F}^* := \{f : (C')^{F^*} \to \Delta(C')\}$, and in the aftermath of the discovery of a new feasible act, the new set of conceivable acts is $\hat{F}' := \{f : C^{F'} \to \Delta(C)\}$.

At this point the reader may find it disturbing that we expand the conceivable state space in the wake of discovery of new feasible acts but we do not expand the state space when we introduce new conceivable acts. The reason is that conceivable acts are bets on the conceivable states (or events). As we have seen, the discovery of new feasible acts expands the conceivable state space by assigning to every existing state the set of all the consequences, thereby "splitting" it to generate a refined state space. By contrast a new conceivable act assigns to every existing state a unique outcome. Hence, for conceivable acts the subjective uncertainty regarding the payoffs of all acts, feasible or otherwise, is completely resolved once the original state is known. Consequently, the introduction of conceivable acts do not change the conceivable state space.

Decision makers are supposed to be able to express preferences among conceivable acts. Because the set of conceivable acts is a variable in our model, we denote the preference relation on $\hat{F}$ by $\succcurlyeq_{\hat{F}}$, and denote by $\succ_{\hat{F}}$ and $\sim_{\hat{F}}$ the asymmetric and symmetric parts of $\succcurlyeq_{\hat{F}}$, respectively. These derived relations are given the usual interpretation of strict preference and indifference, respectively. With the usual abuse of notation, we denote by $p$ the constant act that assigns $p$ to each $s \in C^F$ and by $c$ the degenerate lottery $\delta^c$ that assigns the unit probability mass to the consequence $c$.

## 2.2 Feasible states

Decision makers entertain beliefs about the possible links between feasible acts and their potential consequences. These beliefs manifest themselves in, and may be inferred from, the decision makers' choice behavior.

Consider a decision maker whose choices are characterized by a preference relation $\succcurlyeq_{\hat{F}}$ on $\hat{F}$. For any $f \in \hat{F}$, $p \in \Delta(C)$, and $s \in C^F$, let $p_{\{s\}}f$ be the act in $\hat{F}$ obtained from $f$ by replacing its $s-th$ coordinate with $p$. Following Savage (1954), a state $s \in C^F$ is said to be *null* if $p_{\{s\}}f \sim_{\hat{F}} q_{\{s\}}f$, for all $p, q \in \Delta(C)$. A state is said to be *nonnull* if it is not null. Denote by $E^N$ the set of null states and let $S(F,C) = C^F - E^N$ be the set of all nonnull states. Henceforth, we refer to $S(F,C)$ as the *feasible state space*. Note that a conceivable state is null if it includes an assignment of a feasible consequence to a feasible act that the decision maker believes to be impossible.

New information may change the decision maker's beliefs concerning the links between feasible acts and consequences and, consequently, his perception of the feasible state space. Unlike the discovery of new feasible consequences and/or new feasible acts, which expands both the set of conceivable and the set of feasible states, changes of the decision maker's beliefs concerning the links between them will expand or contract the set of feasible states without affecting the conceivable state space.

Expansion of the feasible state space entails updating zero probability events, while contraction of it entails nullifying positive probability events that are no longer considered possible. When new links become possible, the decision maker includes the consequences $f(s)$, for all $f \in F$ and some $s \in C^F - S(F, C)$, in the ranges he considers possible of the feasible acts. Vice versa when old links are eliminated. We denote the newly defined feasible state space by $S'(F, C)$ and, to underscore the changing nature of the set of conceivable acts when the feasible set of states changes, we denote the corresponding set of conceivable acts by $\hat{F}_{S'}$, and the decision maker's posterior preference relation by $\succcurlyeq_{\hat{F}_{S'}}$.

## 2.3 Basic preference structure

Let $F$ and $C$ be finite sets of feasible acts and consequences, respectively. The set of conceivable states is given by $C^F$ and the corresponding set of conceivable acts by $\hat{F} := \{f : C^F \to \Delta(C)\}$ as described above. We assume throughout that each set of consequences has a most preferred and a least preferred element. Formally, there exist $c^*(C), c_*(C) \in C$ such that the constant act that assigns $c^*(C)$ to every state is strictly preferred over any other constant act in $\hat{F}$ and the constant act that assigns $c_*(C)$ to every state is strictly less preferred than any other constant act in $\hat{F}$.

As described above, when the state space expands in the wake of discoveries of new feasible consequences, the set of conceivable acts must be expanded and the preference relations must be redefined on the extended domain. For instance, if $\hat{F}^*$ is the expanded set of conceivable acts, then the corresponding preference relation is denoted by $\succcurlyeq_{\hat{F}^*}$. If the state space expands as a result of the discovery of new feasible acts, then the new set of conceivable acts is denoted by $\hat{F}'$ and the extended preference relation by $\succcurlyeq_{\hat{F}'}$.

For each set, $\hat{F}$, of conceivable acts and $\alpha \in [0, 1]$ define the convex combination $\alpha f + (1 - \alpha) g \in \hat{F}$ by: $(\alpha f + (1 - \alpha) g)(s) = \alpha f(s) + (1 - \alpha) g(s)$, for all $s \in C^F$. Then, $\hat{F}$ is a convex subset in a linear space.

In what follows we will use the following notations: for any $E \subset C^F$, $p_E h \in H$ is defined by: $p_E h(s) = p$ if $s \in E$ and $p_E h(s) = h(s)$, otherwise. For all $p, q \in \Delta(C)$, $p$ dominates $q$ according to first-order stochastic dominance if $\Sigma_{\{i|c_i \preccurlyeq c\}} p(c_i) \leq \Sigma_{\{i|c_i \preccurlyeq c\}} q(c_i)$ for all $c \in C$, and $p$ strictly dominates $q$ according to first-order stochastic dominance if $p$ dominates $q$ according to first-order stochastic dominance and, in addition, $\Sigma_{\{i|c_i \preccurlyeq c\}} p(c_i) < \Sigma_{\{i|c_i \preccurlyeq c\}} q(c_i)$ for some $c \in C$. We denote these domination relations by $p \geq^1 q$ and $p >^1 q$, respectively.

Following Machina and Schmeidler (1995), we assume that, for each $\hat{F}$, $\succcurlyeq_{\hat{F}}$ adheres to the following axioms, which ensure probabilistic sophistication.

**(A.1)** (**Weak order**) For every $\hat{F}$, the preference relation $\succcurlyeq_{\hat{F}}$ is transitive and complete.

**(A.2)** (**Mixture continuity**) For each $\hat{F}$ and all $f, g, h \in \hat{F}$, if $f \succ_{\hat{F}} g$ and $g \succ_{\hat{F}} h$ then there exist $\alpha \in (0,1)$ such that $\alpha f + (1-\alpha) h \sim_{\hat{F}} g$.

**(A.3)** (**Monotonicity**) For every $\hat{F}$ and $p, q \in \Delta(C)$, if $p \geq^1 q$ then $p_{E_i} h \succcurlyeq_{\hat{F}} q_{E_i} h$, for all partitions $\{E_1, ..., E_n\}$ of $C^F$ and all $h \in \hat{F}$, with $p_{E_i} h \succ_{\hat{F}} q_{E_i} h$ if $p >^1 q$ and $E_i$ is nonnull.

**(A.4)** (**Replacement**) For every $\hat{F}$ and any partition $\{E_1, ..., E_n\}$ of $C^F$, if

$$\delta^{c^*(C)}\ {}_{E_i}\left(\delta^{c_*(C)}\ {}_{E_j}\delta^{c_*(C)}\right) \sim \left(\alpha\delta^{c^*(C)} + (1-\alpha)\,\delta^{c_*(C)}\right)_{E_i \cup E_j} \delta^{c_*(C)}$$

for some $\alpha \in [0,1]$ and pair of events $E_i, E_j$, then

$$p_{E_i}\left(q_{E_j} h\right) \sim \left(\alpha p + (1-\alpha)\,q\right)_{E_i \cup E_j} h$$

for all $p, q \in \Delta(C)$ and $h \in \hat{F}$.

**(A.5)** (**Nontriviality**) For every $\hat{F}$, $\succ_{\hat{F}} \neq \varnothing$.

To link the preference relations across expanding sets of conceivable acts, we invoke the invariant risk preferences axiom introduced in Karni and Vierø (2012), asserting the commonality of risk attitudes across levels of awareness.

**(A.6)** (**Invariant risk preferences**) For every given $\hat{F}, \hat{F}'$, if $C$ and $C'$ are the sets of consequences associated with $\hat{F}$ and $\hat{F}'$, respectively, then $p \succcurlyeq_{\hat{F}} q$ if and only if $p \succcurlyeq_{\hat{F}'} q$ for all $p, q \in \Delta(C \cap C')$.

When new consequences are discovered, $C \subset C'$, then $C \cap C' = C$. When new feasible acts are discovered, the invariant risk preferences axiom may be stated as follows: For all $F, F'$ and $p, q \in \Delta(C)$, $p \succcurlyeq_{\hat{F}} q$ if and only if $p \succcurlyeq_{\hat{F}'} q$. When new links are discovered (or old links eliminated) between the original sets of acts, $F$, and consequences, $C$, the invariant risk preferences axiom asserts that, for all $p, q \in \Delta(C)$, $p \succcurlyeq_{\hat{F}} q$ if and only if $p \succcurlyeq_{\hat{F}_{S'}} q$.

# 3 The Main Results

As in Karni and Vierø (2012), we divide the analysis of the effects of growing awareness on choice behavior and the evolution of decision makers' beliefs into three parts. First, we explore the implications of the discovery of new consequences. Second, we explore the implications of the discovery of new feasible acts. Third, we explore the implications of new information regarding acts-consequences links. The discovery of new acts or consequences increases the number of conceivable and, in general, also that of feasible states. However, unlike the discovery of new consequences, the discovery of new feasible acts increases the number of conceivable states by refining the original state space. By contrast, the discovery of new acts-consequences links changes the set of feasible states without affecting the conceivable state space.

To explore the implications of these sources of growing awareness, we introduce additional axioms, each of which modifies a corresponding axiom in Karni and Vierø (2012). These modifications are needed in order to accommodate the possibility that preference relations do not satisfy the independence axiom and, consequently, are not necessarily separable.

## 3.1 Discovery of new consequences and its representation

The following axiom requires that, when a decision maker discovers new consequences his ranking of subjective versus objective uncertainty, conditional on the original set of feasible states, remains intact. To formalize this idea, let $C' \supset C$, $F^*$, and $S(F^*, C')$ denote, respectively, the new set of consequences, the new set of feasible acts redefined to accommodate the new consequences, and the resulting new feasible state space.[2]

[2]Below, $f' = f$ on an event $E$ means that $f'(s) = f(s)$ for all $s \in E$ (i.e. it is defined pointwise for the states in $E$).

**(A.7)** (**Replacement consistency I**) For every given $F$, for all $C, C'$ with $C \subset C'$ and $S(F,C) \subseteq S(F^*,C')$, for all $s \in S(F,C)$, $\eta \in [0,1]$, $f,g \in \hat{F}$, and $f',g' \in \hat{F}^*$, if $f = \delta^{c^*(C)}{}_{\{s\}}\delta^{c_*(C)}$, $g = \eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}$ on $C^F$, $g' = \eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}$ on $(C')^{F^*}$, $f' = f$ on $S(F,C)$ and $f' = g'$ on $S(F^*,C') - S(F,C)$, then it holds that $f \succcurlyeq_{\hat{F}} g$ if and only if $f' \succcurlyeq_{\hat{F}^*} g'$.

Axiom (A.7) concerns bets that involve only the best and worst consequences in $C$. It ensures consistency between the ranking of subjective versus objective uncertainty given awareness of conceivable acts that correspond to nested sets of feasible consequences. The axiom asserts that, conditional on the prior subjective state space, the decision maker's ranking of the subjective bet that pays off in state $s$ and the objective bet that pays off with probability $\eta$, is preserved when he discovers new feasible consequences which expand the conceivable state space.

Our first result describes the evolution of a decision maker's beliefs in the wake of discoveries of new consequences. Like Theorem 1 in Karni and Vierø (2012) this theorem asserts that, as he becomes aware of new consequences, the decision maker updates his beliefs in a way that preserves the likelihood ratios of events in the original state space. Unlike in Karni and Vierø (2012) the decision maker is not necessarily an expected utility maximizer, he is merely probabilistically sophisticated. Hence, reverse Bayesianism is independent of the expected utility hypothesis.

**Theorem 1** For each set, $\hat{F}$, of conceivable acts let $\succcurlyeq_{\hat{F}}$ be a binary relation on $\hat{F}$ then, for all $\hat{F}, \hat{F}^*$, the following two conditions are equivalent:

(*i*) $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}^*}$ each satisfy (A.1) - (A.5) and jointly, $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}^*}$ satisfy (A.6) and (A.7).

(*ii*) There exist real-valued, mixture continuous, strictly monotonic functions, $V$ on $\Delta(C)$ and $V^*$ on $\Delta(C')$, and probability measures, $\pi_{\hat{F}}$ on $C^F$ and $\pi_{\hat{F}^*}$ on $(C')^{F^*}$, such that for all $f, g \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow V\left(\sum_{s\in S(F,C)} \pi_{\hat{F}}(s)\, f(s)\right) \geq V\left(\sum_{s\in S(F,C)} \pi_{\hat{F}}(s)\, g(s)\right) \quad (3)$$

and, for all $f', g' \in \hat{F}^*$,

$$f' \succcurlyeq_{\hat{F}^*} g' \Leftrightarrow V^*\left(\sum_{s\in S(F^*,C')} \pi_{\hat{F}^*}(s)\, f'(s)\right) \geq V^*\left(\sum_{s\in S(F^*,C')} \pi_{\hat{F}^*}(s)\, g'(s)\right) \quad (4)$$

Moreover, the functions $V$ and $V^*$ are unique up to positive transformations and $V(p) = V^*(p)$ for all $p \in \Delta(C)$, the probability measures $\pi_{\hat{F}}$ and $\pi_{\hat{F}^*}$ are unique, $\pi_{\hat{F}}(S(F,C)) = \pi_{\hat{F}^*}(S(F^*,C')) = 1$, and, for all $s \in S(F,C)$.

$$\pi_{\hat{F}}(s) = \frac{\pi_{\hat{F}^*}(s)}{\pi_{\hat{F}^*}(S(F,C))}. \tag{5}$$

### 3.2 Discovery of new feasible acts and its representation

The discovery of new feasible acts expands the conceivable state space and increases the number of coordinates defining a state. To state the next axiom, which is analogous to Axiom (A.7), we introduce the following additional notations: If $F \subset F'$ then for each $s \in C^F$ there corresponds an event $E(s) \subset C^{F'}$ defined by $E(s) = \{s' \in C^{F'} \mid \boldsymbol{P}_{C^F}(s') = s\}$, where $\boldsymbol{P}_{C^F}(\cdot)$ is the projection of $C^{F'}$ on $C^F$.[3] For $s \in C^F$, we refer to the set $E(s)$ as the inverse image under $\boldsymbol{P}_{C^F}$ of $s$ on $C^{F'}$.

**(A.8)** (**Replacement consistency II**) For every given $C$, all pairs of feasible acts $F$ and $F'$ such that $F \subset F'$, all $s \in S(F,C)$, $\eta \in [0,1]$, $f, g \in \hat{F}$, and $f', g' \in \hat{F}'$, if $f = \delta^{c^*(C)}{}_{\{s\}}\delta^{c_*(C)}$, $g = g' = \eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}$, and $f' = \delta^{c^*(C)}{}_{E(s)}\delta^{c_*(C)}$, then it holds that $f \succcurlyeq_{\hat{F}} g$ if and only if $f' \succcurlyeq_{\hat{F}'} g'$.

Like Axiom (A.7), Axiom (A.8) concerns bets that involve only the best and worst consequences in $C$. It asserts that the ranking of subjective bets that pay off in state $s$ and the objective bets that pay off with probability $\eta$, conditional on a given set of conceivable acts, is the same as the ranking of subjective bets that pay off in the event $E(s)$ and the objective bets that pay off with probability $\eta$ conditional on the set of conceivable acts spanned by the discovery of new feasible acts. In other words, the axiom asserts that the decision maker's ranking of subjective versus objective uncertainty is independent of the detail with which the subjective uncertainty is described.

The representation theorem below describes how a decision maker's beliefs evolve as he becomes aware of new feasible acts. Specifically, the decision maker updates his beliefs so that the probability of each state in the original state space is equal to that of its inverse image under $\boldsymbol{P}_{C^F}$ on $C^{F'}$. In other words, since the event $E(s)$ in $C^{F'}$ is a refinement of the probability of $s$ in $C^F$ its probability is equal to that of $s$.

[3] Suppose that $\mid F \mid = r$ and $\mid F' \mid = k > r$. Let $s = (c_1, ..., c_k) \in C^{F'}$, then $\boldsymbol{P}_{C^F}(s) = (c_1, ..., c_r) \in C^F$.

**Theorem 2** For each set $\hat{F}$ of conceivable acts let $\succcurlyeq_{\hat{F}}$ be a binary relation on $\hat{F}$, then, for all $\hat{F}, \hat{F}'$, the following two conditions are equivalent:

$(i)$ $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}'}$ each satisfy (A.1) - (A.5) and jointly, $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}'}$ satisfy (A.6) and (A.8).

$(ii)$ There exist a real-valued, mixture continuous, strictly monotonic function $V$ on $\Delta(C)$ and probability measures, $\pi_{\hat{F}}$ on $C^F$ and $\pi_{\hat{F}'}$ on $C^{F'}$, such that for all $f, g \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow V\left(\Sigma_{s\in S(F,C)}\pi_{\hat{F}}(s)\, f(s)\right) \geq V\left(\Sigma_{s\in S(F,C)}\pi_{\hat{F}}(s)\, g(s)\right) \tag{6}$$

and, for all $f', g' \in \hat{F}'$,

$$f' \succcurlyeq_{\hat{F}'} g' \Leftrightarrow V\left(\Sigma_{s\in S(F',C)}\pi_{\hat{F}'}(s)\, f'(s)\right) \geq V\left(\Sigma_{s\in S(F',C)}\pi_{\hat{F}'}(s)\, g'(s)\right) \tag{7}$$

Moreover, the function $V$ is unique up to positive transformations, the probability measures $\pi_{\hat{F}}$ and $\pi_{\hat{F}'}$ are unique, $\pi_{\hat{F}}(S(F,C)) = \pi_{\hat{F}'}(S(F',C)) = 1$, and, for all $s \in S(F,C)$.

$$\pi_{\hat{F}}(s) = \pi_{\hat{F}'}(E(s)) \tag{8}$$

where $E(s)$ is the inverse image under $\boldsymbol{P}_{C^F}$ of $s$ on $S(F',C)$.

## 3.3 Discovery of new feasible states and the nullification of existing feasible states and their representations

When links between feasible acts and consequences that were believed to exist are discovered to be non-existent, the feasible state space contracts. Similarly, when such links that were believed not to exist are discovered to exist, the feasible state space expands. In the first instance, an event that was believed to be nonnull and was assigned positive probability becomes a null event and must be assigned zero probability. In the second instance, an event that was believed to be null and was assigned zero probability becomes a nonnull event and must be assigned positive probability.

The next axiom depicts the evolution of the preference relation in these circumstances. Clearly, the first instance described above corresponds to the usual Bayesian updating. The second instance, in which the posterior of a zero-probability event is positive, does not admit Bayesian updating. It is, however, consistent with our model of reverse Bayesianism. In fact, in our model the two instances are treated symmetrically, which is reassuring given that they depict symmetrically opposing discoveries.

**(A.9)** (**Replacement consistency III**) For all pairs of conceivable acts $\hat{F}$ and $\hat{F}_{S'}$, all $f, g \in \hat{F}$ and $f', g', \in \hat{F}_{S'}$, if $g = g' = \eta \delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}$ for some $\eta \in [0,1]$ and $f = f' = \delta^{c^*(C)}{}_{\{s\}}\left(\delta^{c_*(C)}{}_{S(F,C)\cap S'(F,C)} g\right)$, for some $s \in S(F,C) \cap S'(F,C)$, then it holds that $f \succcurlyeq_{\hat{F}} g$ if and only if $f' \succcurlyeq_{\hat{F}_{S'}} g'$.

The next representation theorem describes how a decision maker's beliefs are updated as he discovers that links between feasible acts and consequences that he believed impossible are in fact possible and when he discovers that links that he believed possible are in fact impossible.

**Theorem 3** For each set of conceivable acts $\hat{F}$, let $\succcurlyeq_{\hat{F}}$ be a binary relation on $\hat{F}$ then, for all $\hat{F}$ and $\hat{F}_{S'}$, the following two conditions are equivalent:

($i$) Each $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}_{S'}}$ satisfy (A.1) - (A.5) and jointly $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}_{S'}}$ satisfy (A.6) and (A.9).

($ii$) There exist a real-valued, mixture continuous, strictly monotonic function $V$ on $\Delta(C)$ and, for all $\hat{F}$ and $\hat{F}_{S'}$, there are probability measures $\pi_{\hat{F}}$ and $\pi_{\hat{F}_{S'}}$ on $C^F$ such that, for all $f, g \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow V\left(\sum_{s\in S(F,C)} \pi_{\hat{F}}(s)\, f(s)\right) \geq V\left(\sum_{s\in S(F,C)} \pi_{\hat{F}}(s)\, g(s)\right) \tag{9}$$

and, for all $f', g' \in \hat{F}_{S'}$,

$$f' \succcurlyeq_{\hat{F}_{S'}} g' \Leftrightarrow V\left(\sum_{s\in S'(F,C)} \pi_{\hat{F}_{S'}}(s)\, f'(s)\right) \geq V\left(\sum_{s\in S'(F,C)} \pi_{\hat{F}_{S'}}(s)\, g'(s)\right) \tag{10}$$

Moreover, the function $V$ is unique up to positive transformations, the probability measures $\pi_{\hat{F}}$ and $\pi_{\hat{F}_{S'}}$ are unique, $\pi_{\hat{F}}(S(F,C)) = \pi_{\hat{F}_{S'}}(S'(F,C)) = 1$, and

$$\frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(s')} = \frac{\pi_{\hat{F}_{S'}}(s)}{\pi_{\hat{F}_{S'}}(s')} \tag{11}$$

for all $s, s' \in S(F,C) \cap S'(F,C)$.

If $S'(F,C) \subset S(F,C)$ then Theorem 3 describes the Bayesian updating (that is, (11) may be written as $\pi_{\hat{F}_{S'}}(s) = \pi_{\hat{F}}(s) / \pi_{\hat{F}}(S'(F,C))$, for all $s \in S'(F,C)$). If $S'(F,C) \supset S(F,C)$ then $\pi_{\hat{F}}(s) = \pi_{\hat{F}_{S'}}(s) / \pi_{\hat{F}_{S'}}(S(F,C))$ for all $s \in S(F,C)$.

## 4 Concluding Remarks

Grant and Polak (2006) propose an alternative axiomatization of probabilistically sophisticated choice behavior. Departing from the model of Machina and Schmeidler (1995), they replace the axioms of monotonicity, (A.3), and replacement, (A.4). This work is insightful in that it "decomposes" the independence assumptions that are built into the replacement axiom of Machina and Schmeidler. However, since the representations in the two models are the same, and the of axioms of both Machina and Schmeidler and Grant and Polak are necessary and sufficient, they are equivalent. Hence, if we were to replace axioms (A.3) and (A.4) with the axioms of Grant and Polak, and add our axioms (A.6) through (A.9) to characterize the evolution of beliefs in the wake of discovery of new feasible consequences, new feasible acts and new facts about the links between them, we would obtain, as corollaries, the analogues of Theorems 1, 2, and 3 in the axiomatic framework of Grant and Polak (2006).

## 5 Proofs

### 5.1 Proof of Theorem 1

$(i) \Rightarrow (ii)$. Since $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}^*}$ satisfy (A.1) - (A.5), the Theorem of Machina and Schmeidler (1995) implies (3) and (4) as well as the uniqueness of $V$ and $V^*$ and of $\pi_{\hat{F}}$ and $\pi_{\hat{F}^*}$. By (3) and (4), the restriction of $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}^*}$ to the constant acts $p \in \Delta(C)$ imply that $V(p) \geq V(q)$ if and only if $p \succcurlyeq_{\hat{F}} q$ and $p \succcurlyeq_{\hat{F}^*} q$ if and only if $V^*(p) \geq V^*(q)$. By (A.6), $p \succcurlyeq_{\hat{F}} q$ if and only if $p \succcurlyeq_{\hat{F}^*} q$. Thus, by the uniqueness of the representations, $V$ and $V^*$ can be chosen so that $V = V^*$ on $\Delta(C)$.

To prove (5) suppose that, for some $s \in S(F, C)$,

$$\pi_{\hat{F}}(s) \neq \frac{\pi_{\hat{F}^*}(s)}{\pi_{\hat{F}^*}(S(F, C))}.$$

Without loss of generality, let

$$\pi_{\hat{F}}(s) > \frac{\pi_{\hat{F}^*}(s)}{\pi_{\hat{F}^*}(S(F, C))} := \pi_{\hat{F}^*}(s \mid S(F, C)).$$

Then there is $\eta \in \left(\pi_{\hat{F}^*}(s \mid S(F, C)), \pi_{\hat{F}}(s)\right)$. By the representation in (3), $f = \delta^{c^*(C)}{}_{\{s\}}\delta^{c_*(C)} \sim_{\hat{F}} \pi_{\hat{F}}(s)\,\delta^{c^*(C)} + \left(1 - \pi_{\hat{F}}(s)\right)\delta^{c_*(C)}$.

Since $\pi_{\hat{F}}(s) > \eta > \pi_{\hat{F}^*}(s \mid S(F,C))$, by Axiom (A.3), we have the following ranking of lotteries:

$$\pi_{\hat{F}}(s)\,\delta^{c^*(C)} + \left(1 - \pi_{\hat{F}}(s)\right)\delta^{c_*(C)} \succ_{\hat{F}} \eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)} \tag{12}$$

and

$$\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)} \succ_{\hat{F}^*} \pi_{\hat{F}^*}(s \mid S(F,C))\,\delta^{c^*(C)} + \left(1 - \pi_{\hat{F}^*}(s \mid S(F,C))\right)\delta^{c_*(C)} \tag{13}$$

for all $\eta \in \left(\pi_{\hat{F}^*}(s \mid S(F,C)), \pi_{\hat{F}}(s)\right)$.

Now, by (13) and Axiom (A.3),

$$\begin{aligned} & \left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right)_{S(F,C)} \left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right) \\ \succ_{\hat{F}^*} & \left(\pi_{\hat{F}^*}(s \mid S(F,C))\,\delta^{c^*(C)} + \left(1 - \pi_{\hat{F}^*}(s \mid S(F,C))\right)\delta^{c_*(C)}\right)_{S(F,C)} \left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right), \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)} \\ \succ_{\hat{F}^*} & \left(\pi_{\hat{F}^*}(s \mid S(F,C))\,\delta^{c^*(C)} + \left(1 - \pi_{\hat{F}^*}(s \mid S(F,C))\right)\delta^{c_*(C)}\right)_{S(F,C)} \left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right), \end{aligned}$$

which, by (4), is equivalent to

$$\begin{aligned} & V^*\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right) \\ > & V^*\Big(\pi_{\hat{F}^*}(S(F,C))\left(\pi_{\hat{F}^*}(s \mid S(F,C))\delta^{c^*(C)} + \left(1 - \pi_{\hat{F}^*}(s \mid S(F,C))\right)\delta^{c_*(C)}\right) \\ & + \left(1 - \pi_{\hat{F}^*}(S(F,C))\right)\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right)\Big) \qquad (14) \\ = & V^*\Big(\pi_{\hat{F}^*}(s)\delta^{c^*(C)} + \left(\pi_{\hat{F}^*}(S(F,C)) - \pi_{\hat{F}^*}(s)\right)\delta^{c_*(C)} + \left(1 - \pi_{\hat{F}^*}(S(F,C))\right)\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right)\Big). \end{aligned}$$

By (4), inequality (14) is equivalent to

$$\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)} \succ_{\hat{F}^*} \delta^{c^*(C)}{}_{\{s\}}\left(\delta^{c_*(C)}{}_{S(F,C)}\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right)\right) \tag{15}$$

Now, by (12) and (3),

$$V\left(\pi_{\hat{F}}(s)\,\delta^{c^*(C)} + \left(1 - \pi_{\hat{F}}(s)\right)\delta^{c_*(C)}\right) > V\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right)$$

which is equivalent to

$$\delta^{c^*(C)}{}_{\{s\}}\delta^{c_*(C)} \succ_{\hat{F}} \eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}. \tag{16}$$

But the act on the left hand side of (15) is the act $g'$ in Axiom (A.7), while the act on the right hand side of (15) is the act $f'$. Also, the act on the left hand side of (16) is the act $f$ in Axiom (A.7), while the act on the right hand side of (16) is the act $g$. Expressions (15) and (16) thus imply that $f \succ_{\hat{F}} g$ and $g' \succ_{\hat{F}^*} f'$, a contradiction of Axiom (A.7).

$(ii) \to (i)$. That $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}^*}$ satisfy (A.1) - (A.5) is an implication of the Theorem of Machina and Schmeidler (1995). Invariant risk preferences, (A.6), follows from the equality of $V$ and $V^*$ on $\Delta(C)$.

To show that (A.7) holds, let $f, g \in \hat{F}$ and $f', g' \in \hat{F}^*$ be as in (A.7). By (3),

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow V\left(\pi_{\hat{F}}(s)\,\delta^{c^*(C)} + \left(1-\pi_{\hat{F}}(s)\right)\delta^{c_*(C)}\right) \geq V\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right)$$

By the equality of $V$ and $V^*$ on $\Delta(C)$ and (5), the last inequality holds if and only if

$$V^*\left(\frac{\pi_{\hat{F}^*}(s)}{\pi_{\hat{F}^*}(S(F,C))}\delta^{c^*(C)} + \left(1-\frac{\pi_{\hat{F}^*}(s)}{\pi_{\hat{F}^*}(S(F,C))}\right)\delta^{c_*(C)}\right) \geq V^*\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right). \tag{17}$$

which, by (4), is equivalent to

$$\frac{\pi_{\hat{F}^*}(s)}{\pi_{\hat{F}^*}(S(F,C))}\delta^{c^*(C)} + \left(1-\frac{\pi_{\hat{F}^*}(s)}{\pi_{\hat{F}^*}(S(F,C))}\right)\delta^{c_*(C)} \succcurlyeq_{\hat{F}^*} \eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}. \tag{18}$$

Now, since the left-hand-side lottery in (18) first-order stochastically dominates the right-hand side lottery, by Axiom (A.3)

$$\begin{aligned}&\left(\pi_{\hat{F}^*}(s \mid S(F,C))\,\delta^{c^*(C)} + \left(1-\pi_{\hat{F}^*}(s \mid S(F,C))\right)\delta^{c_*(C)}\right)_{S(F,C)}\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right)\\ \succcurlyeq_{\hat{F}^*}\;&\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right)_{S(F,C)}\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right)\end{aligned}$$

Hence, (17) holds if and only if $V^*\left(\xi\delta^{c^*(C)} + (1-\xi)\,\delta^{c_*(C)}\right) \geq V^*\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right)$, where

$$\xi := \left(\pi_{\hat{F}^*}(S(F,C))\,\frac{\pi_{\hat{F}^*}(s)}{\pi_{\hat{F}^*}(S(F,C))} + \left(1-\pi_{\hat{F}^*}(S(F,C))\right)\eta\right) = \pi_{\hat{F}^*}(s) + \left(1-\pi_{\hat{F}^*}(S(F,C))\right)\eta.$$

Since $\xi\delta^{c^*(C)} + (1-\xi)\,\delta^{c_*(C)} \in \Delta(C)$ is the constant act whose payoff is $\Sigma_{s\in S(F^*,C')}\pi_{\hat{F}^*}(s)\,f'(s)$, the representation result in (4) implies that (17) holds if and only if $f' \succcurlyeq_{\hat{F}^*} g'$.♠

## 5.2 Proof of Theorem 2

$(i) \Rightarrow (ii)$. Since $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}'}$ satisfy (A.1) - (A.5), the Theorem of Machina and Schmeidler (1995) implies a representation as in (6) as well as the uniqueness of $V$ and of $\pi_{\hat{F}}$ for each level of awareness. By (A.6), $p \succcurlyeq_{\hat{F}} q$ if and only if $p \succcurlyeq_{\hat{F}'} q$. Thus, by the uniqueness of the representations, $V$ can be chosen to be invariant to the level of awareness.

To prove (8), suppose that, for some $s \in S(F, C)$, $\pi_{\hat{F}}(s) \neq \pi_{\hat{F}'}(E(s))$. Without loss of generality, let $\pi_{\hat{F}}(s) > \pi_{\hat{F}'}(E(s))$. Then there exists $\eta \in \left(\pi_{\hat{F}'}(E(s)), \pi_{\hat{F}}(s)\right)$, and by Axiom (A.3), we have the following ranking of lotteries:

$$\pi_{\hat{F}}(s)\, \delta^{c^*(C)} + \left(1 - \pi_{\hat{F}}(s)\right) \delta^{c_*(C)} \succ_{\hat{F}} \eta \delta^{c^*(C)} + (1-\eta)\, \delta^{c_*(C)} \tag{19}$$

and

$$\eta \delta^{c^*(C)} + (1-\eta)\, \delta^{c_*(C)} \succ_{\hat{F}'} \pi_{\hat{F}'}(E(s)) \delta^{c^*(C)} + \left(1 - \pi_{\hat{F}'}(E(s))\right) \delta^{c_*(C)} \tag{20}$$

for all $\eta \in \left(\pi_{\hat{F}'}(E(s)), \pi_{\hat{F}}(s)\right)$.

However, by (6),

$$f = \delta^{c^*(C)}{}_{\{s\}} \delta^{c_*(C)} \sim_{\hat{F}} \pi_{\hat{F}}(s)\, \delta^{c^*(C)} + \left(1 - \pi_{\hat{F}}(s)\right) \delta^{c_*(C)}.$$

Also, by (7),

$$f' = \delta^{c^*(C)}{}_{E(s)} \delta^{c_*(C)} \sim_{\hat{F}'} \pi_{\hat{F}'}(E(s))\, \delta^{c^*(C)} + \left(1 - \pi_{\hat{F}'}(E(s))\right) \delta^{c_*(C)}.$$

Thus, the preference in (19) is equivalent to $f \succ_{\hat{F}} g$, while the preference (20) is equivalent to $g' \succ_{\hat{F}'} f'$. This contradicts Axiom (A.8).

$(ii) \rightarrow (i)$ That $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}'}$ satisfy (A.1) - (A.5) is an implication of the Theorem of Machina and Schmeidler (1995). Invariant risk preferences, (A.6), follows from the function $V$ being independent of $\hat{F}$.

To show that (A.8) holds, let $f, g \in \hat{F}$ and $f', g' \in \hat{F}'$ be as in (A.8). By (6),

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow V\left(\pi_{\hat{F}}(s)\, \delta^{c^*(C)} + \left(1 - \pi_{\hat{F}}(s)\right) \delta^{c_*(C)}\right) \geq V\left(\eta \delta^{c^*(C)} + (1-\eta)\, \delta^{c_*(C)}\right)$$

By (8), the last inequality holds if and only if

$$V\left(\pi_{\hat{F}'}(E(s))\, \delta^{c^*(C)} + \left(1 - \pi_{\hat{F}'}(E(s))\right) \delta^{c_*(C)}\right) \geq V\left(\eta \delta^{c^*(C)} + (1-\eta)\, \delta_{c_*(C)}\right) \tag{21}$$

By (7), the expression in (21) is equivalent to $f' \succ_{\hat{F}'} g'$. Thus, Axiom (A.8) must hold.♠

## 5.3 Proof of Theorem 3

$(i) \Rightarrow (ii)$ . Since $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}_{S'}}$ satisfy (A.1) - (A.5), the Theorem of Machina and Schmeidler (1995) implies a representation as in (9) as well as the uniqueness of $V$ and of $\pi_{\hat{F}}$ for each level of awareness. By (A.6), $p \succcurlyeq_{\hat{F}} q$ if and only if $p \succcurlyeq_{\hat{F}_{S'}} q$. Thus, by the uniqueness of the representations, $V$ can be chosen to be invariant to the level of awareness.

For some $s \in S(F,C) \cap S'(F,C)$, let $g = g' = \eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}$, and $f, f'$ be as in Axiom (A.9). Suppose that $f \sim_{\hat{F}} g$. But $f \sim_{\hat{F}} g$ if and only if

$$\delta^{c^*(C)}{}_{\{s\}}\left(\delta^{c_*(C)}{}_{S(F,C)\cap S'(F,C)}\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right)\right) \sim_{\hat{F}} \eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}. \tag{22}$$

By the representation in (9) the last indifference holds if and only if

$$\begin{aligned} &V\Big(\pi_{\hat{F}}(s)\delta^{c^*(C)} + \left(\pi_{\hat{F}}(S(F,C)\cap S'(F,C)) - \pi_{\hat{F}}(s)\right)\delta^{c_*(C)} + \\ &(1-\pi_{\hat{F}}(S(F,C)\cap S'(F,C)))\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right)\Big) \\ &= V\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right) \end{aligned} \tag{23}$$

But (23) holds if and only if $\pi_{\hat{F}}(s) + (1-\pi_{\hat{F}}(S(F,C)\cap S'(F,C)))\eta = \eta$. Hence,

$$\eta = \frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(S(F,C)\cap S'(F,C))}. \tag{24}$$

By Axiom (A.9), $f \sim_{\hat{F}} g$ if and only if $f' \sim_{\hat{F}_{S'}} g'$, which is equivalent to

$$\delta^{c^*(C)}{}_{\{s\}}\left(\delta^{c_*(C)}{}_{S(F,C)\cap S'(F,C)}\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right)\right) \sim_{\hat{F}_{S'}} \eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}. \tag{25}$$

By the representation in (10), (25) holds if and only if

$$\begin{aligned} &V\Big(\pi_{\hat{F}_{S'}}(s)\delta^{c^*(C)} + \left(\pi_{\hat{F}_{S'}}(S(F,C)\cap S'(F,C)) - \pi_{\hat{F}_{S'}}(s)\right)\delta^{c_*(C)} \\ &+(1-\pi_{\hat{F}_{S'}}(S(F,C)\cap S'(F,C)))\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right)\Big) \\ &= V\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right) \end{aligned} \tag{26}$$

But (26) holds if and only if $\pi_{\hat{F}_{S'}}(s) + (1-\pi_{\hat{F}_{S'}}(S(F,C)\cap S'(F,C)))\eta = \eta$. Thus, $f' \sim_{\hat{F}_{S'}} g'$ if and only if

$$\eta = \frac{\pi_{\hat{F}_{S'}}(s)}{\pi_{\hat{F}_{S'}}(S(F,C)\cap S'(F,C))}. \tag{27}$$

By (24) and (27 we have that

$$\frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(S(F,C)\cap S'(F,C))} = \frac{\pi_{\hat{F}_{S'}}(s)}{\pi_{\hat{F}_{S'}}(S(F,C)\cap S'(F,C))}. \tag{28}$$

An analogous argument applies for any $s' \in S(F,C) \cap S'(F,C)$. We therefore also have that, for any $s' \in S(F,C) \cap S'(F,C)$,

$$\frac{\pi_{\hat{F}}(s')}{\pi_{\hat{F}}(S(F,C)\cap S'(F,C))} = \frac{\pi_{\hat{F}_{S'}}(s')}{\pi_{\hat{F}_{S'}}(S(F,C)\cap S'(F,C))}. \tag{29}$$

Together (28) and (29) imply that

$$\frac{\pi_{\hat{F}_{S'}}(s)}{\pi_{\hat{F}_{S'}}(s')} = \frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(s')}. \tag{30}$$

$(ii) \to (i)$. That $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}_{S'}}$ satisfy (A.1) - (A.5) is an implication of the Theorem of Machina and Schmeidler (1995). Invariant risk preferences, (A.6), follows from the function $V$ being independent of $\hat{F}$.

To show that (A.9) holds, let $f, g \in \hat{F}$ and $f', g' \in \hat{F}_{S'}$ be as in (A.9). By (9), $f \succcurlyeq_{\hat{F}} g$ if and only if

$$V\Big(\pi_{\hat{F}}(s)\delta^{c^*(C)} + \left(\pi_{\hat{F}}(S(F,C)\cap S'(F,C)) - \pi_{\hat{F}}(s)\right)\delta^{c_*(C)}$$
$$+(1-\pi_{\hat{F}}(S(F,C)\cap S'(F,C)))\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right)\Big)$$
$$\geq V\left(\eta\delta^{c^*(C)} + (1-\eta)\,\delta^{c_*(C)}\right)$$

By first order stochastic dominance, the last inequality holds if and only if

$$\frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(S(F,C)\cap S'(F,C))} \geq \eta. \tag{31}$$

Suppose that $g' \succ_{\hat{F}_{S'}} f'$. By (10), $g' \succ_{\hat{F}_{S'}} f'$ if and only if

$$V\Big(\pi_{\hat{F}_{S'}}(s)\delta^{c^*(C)} + \left(\pi_{\hat{F}_{S'}}(S(F,C)\cap S'(F,C)) - \pi_{\hat{F}_{S'}}(s)\right)\delta^{c_*(C)}$$

$$+(1-\pi_{\hat{F}_{S'}}(S(F,C)\cap S'(F,C)))\left(\eta\delta^{c^*(C)}+(1-\eta)\,\delta^{c_*(C)}\right)\Big)$$

$$< V\left(\eta\delta^{c^*(C)}+(1-\eta)\,\delta^{c_*(C)}\right)$$

By first order stochastic dominance, this holds if and only if $\pi_{\hat{F}_{S'}}(s)+(1-\pi_{\hat{F}_{S'}}(S(F,C)\cap S'(F,C)))\eta<\eta$. Hence,

$$\eta > \frac{\pi_{\hat{F}_{S'}}(s)}{\pi_{\hat{F}_{S'}}(S(F,C)\cap S'(F,C))}. \tag{32}$$

Now, expressions (31) and (32) imply that

$$\frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(S(F,C)\cap S'(F,C))} > \frac{\pi_{\hat{F}_{S'}}(s)}{\pi_{\hat{F}_{S'}}(S(F,C)\cap S'(F,C))}. \tag{33}$$

However, by (11),

$$\frac{\pi_{\hat{F}}(s')}{\pi_{\hat{F}}(s)} = \frac{\pi_{\hat{F}_{S'}}(s')}{\pi_{\hat{F}_{S'}}(s)} \tag{34}$$

for all $s,s'\in S(F,C)\cap S'(F,C)$. Summing over $s'\in S(F,C)\cap S'(F,C)$ and rearranging, (34) implies that

$$\frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(S(F,C)\cap S'(F,C))} = \frac{\pi_{\hat{F}_{S'}}(s)}{\pi_{\hat{F}_{S'}}(S(F,C)\cap S'(F,C))}$$

which contradicts (33).♠

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