CRRA Portfolio Choice with Two Risky Assets

Merton (1969) and Samuelson (1969) study optimal portfolio allocation for a consumer with Constant Relative Risk Aversion utility $\mathrm{u}(c) = (1-\rho)^{-1}c^{1-\rho}$ who can choose among many risky investment options.

Using their framework, here we study a consumer who has wealth a_t at the end of period t, and is deciding how much to invest in two risky assets with lognormally distributed return factors $\mathbf{R}_{t+1} = (\mathbf{R}_{1,t+1}, \mathbf{R}_{2,t+1})'$, $\log \mathbf{R}_{t+1} = \mathbf{r}_{t+1} = (\mathbf{r}_{1,t+1}, \mathbf{r}_{2,t+1})' \sim (\mathcal{N}(\mathbf{r}_1, \sigma_1^2), (\mathcal{N}(\mathbf{r}_2, \sigma_2^2))'$, with covariance matrix

$$\begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}.$$

If the period-t consumer invests proportion ς_i of a_t in risky asset i, $i = 1, 2$ (so that $\varsigma_1 = (1-\varsigma_2)$ and vice-versa), spending all available resources in the last period of life[1] $t+1$ will yield:

$$c_{t+1} = \underbrace{(\varsigma \cdot \mathbf{R}_{t+1})}_{\equiv \mathbb{R}_{t+1}} a_t$$

where $\mathbb{R}_{t+1}$ is the portfolio-weighted return factor.

Campbell and Viceira (2002) point out that a good approximation to the portfolio rate of return is obtained by

$$\mathbb{r}_{t+1} = \mathbf{r}_{1,t+1} + \varsigma_2(\mathbf{r}_{2,t+1} - \mathbf{r}_{1,t+1}) + \varsigma_2(1-\varsigma_2)\eta/2 \tag{1}$$

where

$$\eta = (\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}).$$

Using this approximation, the expectation as of date t of utility at date $t+1$ is:

$$\begin{aligned} \mathbb{E}_t[\mathrm{u}(c_{t+1})] &\approx (1-\rho)^{-1}\,\mathbb{E}_t\left[\left(a_t e^{\mathbf{r}_{1,t+1}} e^{\varsigma_2(\mathbf{r}_{2,t+1}-\mathbf{r}_{1,t+1})+\varsigma_2(1-\varsigma_2)\eta/2}\right)^{1-\rho}\right] \\ &\approx \underbrace{(1-\rho)^{-1}a_t^{1-\rho}}_{\text{constant } < 0}\, \underbrace{e^{(1-\rho)\varsigma_2(1-\varsigma_2)\eta/2}\,\mathbb{E}_t\left[e^{(\mathbf{r}_{1,t+1}+\varsigma_2(\mathbf{r}_{2,t+1}-\mathbf{r}_{1,t+1}))(1-\rho)}\right]}_{\text{excess return utility factor}} \end{aligned}$$

where the first term is a negative constant under the usual assumption that relative risk aversion $\rho > 1$.

Our foregoing assumptions imply that

$$(1-\rho)(\varsigma_1\mathbf{r}_{1,t+1} + \varsigma_2\mathbf{r}_{2,t+1}) \sim \mathcal{N}((1-\rho)(\varsigma_1\mathbf{r}_1 + \varsigma_2\mathbf{r}_2), (1-\rho)^2(\varsigma_1^2\sigma_1^2 + \varsigma_2^2\sigma_2^2 + 2\varsigma_1\varsigma_2\sigma_{12}))$$

(using [LogELogNormTimes]). With a couple of extra lines of derivation we can show that the log of the expectation in (2) is

$$\log \mathbb{E}_t\left[e^{(\mathbf{r}_{1,t+1}+\varsigma_2(\mathbf{r}_{2,t+1}-\mathbf{r}_{1,t+1}))(1-\rho)}\right] = (1-\rho)(\varsigma_1\mathbf{r}_1 + \varsigma_2\mathbf{r}_2) + (1-\rho)^2(\varsigma_1^2\sigma_1^2 + \varsigma_2^2\sigma_2^2 + 2\varsigma_1\varsigma_2\sigma_{12}))/2$$

[1]The portfolio allocation solution obtained below induces back to earlier periods of life, as Samuelson (1963, 1989) famously emphasized.

Substituting from (2) for the log of the expectation in (2), the log of the 'excess return utility factor' in (2) is

$$(1-\rho)\varsigma_2(1-\varsigma_2)\eta/2+(1-\rho)(\mathbf{r}_1+\varsigma_2(\mathbf{r}_2-\mathbf{r}_1))+(\rho-1)^2(\sigma_1^2+\varsigma_2^2\eta+2\varsigma_2(\sigma_{12}-\sigma_2^2))/2.$$

The ς that minimizes this log will also minimize the level; the FOC for minimizing this expression is

$$\begin{aligned}(1-2\varsigma_2)\eta/2+\mathbf{r}_2-\mathbf{r}_1+(1-\rho)(\varsigma_2\eta+(\sigma_{12}-\sigma_1^2))&=0\\ (\mathbf{r}_2-\mathbf{r}_1+\frac{\eta}{2})+(1-\rho)(\sigma_{12}-\sigma_1^2)&=\rho\eta\varsigma_2.\end{aligned} \tag{2}$$

So

$$\varsigma_2=\left(\frac{\mathbf{r}_2-\mathbf{r}_1+\eta/2+(1-\rho)(\sigma_{12}-\sigma_1^2)}{\rho\eta}\right) \tag{3}$$

and note that if the first asset is riskfree so that $\sigma_1=\sigma_{12}=0$ then this reduces to

$$\varsigma_2=\left(\frac{\mathbf{r}_2-\mathbf{r}_1+\sigma_2^2/2}{\rho\sigma_2^2}\right) \tag{4}$$

but the log of the expected return premium (in levels) on the risky over the safe asset in this case is $\varphi\equiv\log\mathbf{R}_2/\mathbf{R}_1=\mathbf{r}_2-\mathbf{r}_1+\sigma_2^2/2$ (recalling that we have assumed $\sigma_{12}=\sigma_1^2=0$), so (4) becomes

$$\varsigma_2=\left(\frac{\varphi}{\rho\sigma_2^2}\right) \tag{5}$$

which corresponds to the solution obtained for the case of a single risky asset in `Portfolio-CRRA`.

References

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