Portfolio Choice With CARA Utility

Consider a consumer with Constant Absolute Risk Aversion utility $\mathrm{u}(c) = -\alpha^{-1}e^{-\alpha c}$, with assets a_{T-1} who is deciding how much to invest in a risky security that will earn a normally distributed stochastic return $\mathbf{R}_T \sim \mathcal{N}(\mathbf{R}, \sigma)$ versus a safe asset that will earn return $\mathsf{R} < \mathbf{R}$ (the risky asset gets a bold font because you must be a bold person to invest in a risky asset!).[1,2]

Consumption in the last period of life will be the entire amount of resources. If the consumer invests an absolute amount of money \$$S$ in the risky asset, then

$$\begin{aligned} c_T &= S\mathbf{R}_T + (a_{T-1} - S)\mathsf{R} \\ &= a_{T-1}\mathsf{R} + \underbrace{(\mathbf{R}_T - \mathsf{R})}_{\equiv \Phi_T} S \end{aligned}$$

where Φ_T is the 'excess return' realized in period T. Given S and defining the expected risk premium as the expected value of the expected return $\Phi = \mathbb{E}_{T-1}[\mathbf{R}_T - \mathsf{R}]$, the expectation as of time $T-1$ is:

$$\begin{aligned} \mathbb{E}_{T-1}[\mathrm{u}(c_T)] &= \mathbb{E}_{T-1}[-\alpha^{-1}e^{-\alpha(a_{T-1}\mathsf{R}+(\mathbf{R}_T-\mathsf{R})S)}] \\ &= -\alpha^{-1}e^{-\alpha(a_{T-1}\mathsf{R})}\,\mathbb{E}_{T-1}[e^{-\alpha(\mathbf{R}_T-\mathsf{R})S}] \\ &= -\alpha^{-1}e^{-\alpha(a_{T-1}\mathsf{R})}e^{-\alpha S\Phi+(\alpha S)^2\sigma^2/2} \\ &= -\alpha^{-1}e^{-\alpha(a_{T-1}\mathsf{R})}e^{-\alpha(S\Phi-\alpha S^2\sigma^2/2)} \end{aligned}$$

and the third line follows from the second because if $z \sim \mathcal{N}(\Phi_z, \sigma_z^2)$ then $\mathbb{E}[e^z] = e^{\Phi_z+\sigma_z^2/2}$. (See `MathFacts` [`ELogNorm`]).

Because (1) is negative, the optimal S will be the one that yields the largest negative exponent on e, which occurs at the value of S given by

$$\max_S \left\{ S\Phi - \frac{\alpha S^2\sigma^2}{2} \right\} \tag{1}$$

with FOC

$$\begin{aligned} \Phi &= \alpha S\sigma^2 \\ S &= \frac{\Phi}{\alpha\sigma^2}. \end{aligned} \tag{2}$$

This yields the intuitive result that the greater is risk aversion or the greater is the risk, the less the consumer wants to invest in the risky asset, while the greater is the

[1] The seminal paper examining this problem (in continuous time) was by Merton (1969); that paper also examines the case with CRRA utility and lognormal returns.

[2] The assumption that returns are normally distributed is highly implausible. This means that with some positive probability, $\mathbf{R}_T < 0$. So, owning a \$1 of the risky asset in period $T-1$ could result in *negative* wealth in period T. You can lose *more than everything*, which is a violation of the legal principle of limited liability. (For a detailed history of limited liability, see Micklethwait and Wooldridge (2002).) Lognormally distributed returns are therefore much more plausible.

expected excess return, the more the consumer wants to invest. Note, however, that the model implausibly says that the dollar amount invested in the risky asset does not depend on the total dollar amount of resources a_{T-1}. So, Warren Buffett and Homer Simpson should have the exact same dollar holdings of the risky asset! If Buffett is richer than Simpson, Buffett's excess wealth is held in the safe form. Not very plausible. (That is why models with CARA utility are increasingly unfashionable in the economics and finance literatures).

References

Merton, Robert C. (1969): "Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case," *Review of Economics and Statistics*, 51, 247–257.

Micklethwait, John, and Adrian Wooldridge (2002): *The Company: A Short History of a Revolutionary Idea*. Modern Library.