The Brock-Mirman Stochastic Growth Model

Brock and Mirman (1972) provided the first optimizing growth model with unpredictable (stochastic) shocks.

The social planner's goal is to solve the problem:

$$\max \quad \mathbb{E}\left[\sum_{n=0}^{\infty} \beta^n \log C_{t+n}\right] \tag{1}$$

$$\text{s.t.}$$

$$K_{t+1} = Y_t - C_t$$
$$Y_{t+1} = A_{t+1}K_{t+1}^{\alpha}$$

where A_t is the level of productivity in period t, which is now allowed to be stochastic (alternative assumptions about the nature of productivity shocks are explored below). Note the key assumption that the depreciation rate on capital is 100 percent.

In this model the capital stock is not useful as a state variable: Because capital has a 100 percent depreciation rate, all that matters to the consumer when choosing how much to consume is how much income they have now, and not how that income breaks down into a part due to K and a part due to A.

The first step is to rewrite the problem in Bellman equation form

$$\mathrm{V}_t(Y_t) = \max_{C_t} \quad \log C_t + \beta\,\mathbb{E}_t[\mathrm{V}_{t+1}(Y_{t+1})] \tag{2}$$

and take the first order condition:

$$\mathrm{u}'(C_t) = \beta\,\mathbb{E}_t\left[A_{t+1}\alpha K_{t+1}^{\alpha-1}\mathrm{u}'(C_{t+1})\right]$$
$$\frac{1}{C_t} = \beta\,\mathbb{E}_t\left[\frac{A_{t+1}\alpha K_{t+1}^{\alpha-1}}{C_{t+1}}\right]$$
$$1 = \beta\,\mathbb{E}_t\left[\underbrace{\alpha A_{t+1}K_{t+1}^{\alpha-1}}_{\equiv\mathbf{R}_{t+1}}\frac{C_t}{C_{t+1}}\right]$$

where our definition of $\mathbf{R}_{t+1}$ helps clarify the relationship of this equation to the usual consumption Euler equation (and you should think about why this is the right definition of the interest factor in this model).

Now we show that this FOC is satisfied by the consumption function $C_t = \kappa Y_t$, where $\kappa = 1 - \alpha\beta$. To see this, note first that the proposed consumption rule implies that $K_{t+1} = (1-\kappa)Y_t$.

The first order condition says

$$\begin{aligned}
1 &= \beta\,\mathbb{E}_t\left[\alpha\frac{A_{t+1}K_{t+1}^{\alpha}}{K_{t+1}}\frac{\kappa Y_t}{\kappa Y_{t+1}}\right] \\
&= \beta\,\mathbb{E}_t\left[\alpha\frac{Y_{t+1}}{K_{t+1}}\frac{\kappa Y_t}{\kappa Y_{t+1}}\right] \\
&= \beta \quad \left[\alpha\frac{Y_t}{K_{t+1}}\right] \\
&= \beta \quad \left[\alpha\frac{Y_t}{Y_t - C_t}\right] \\
&= \beta \quad \left[\alpha\frac{Y_t}{Y_t(1-\kappa)}\right] \\
&= \beta \quad \left[\alpha\frac{1}{(1-\kappa)}\right] \\
(1-\kappa) &= \alpha\beta \\
\kappa &= 1-\alpha\beta.
\end{aligned}$$

An important way of judging a macroeconomic model and deciding whether it makes sense is to examine the model's implications for the dynamics of aggregate variables. Defining lower case variables as the log of the corresponding upper case variable, this model says that the dynamics of the capital stock are given by

$$\begin{aligned}
K_{t+1} &= (1-\kappa)Y_t \\
&= \alpha\beta A_t K_t^{\alpha} \\
k_{t+1} &= \log\alpha\beta + a_t + \alpha k_t
\end{aligned} \tag{3}$$

which tells us that the dynamics of the (log) capital stock have two components: One component (a_t) mirrors whatever happens to the aggregate production technology; the other is serially correlated with coefficient α equal to capital's share in output.

Similarly, since log output is simply $y = a + \alpha k$, the dynamics of output can be obtained from

$$
\begin{aligned}
y_{t+1} &= a_{t+1} + \alpha k_{t+1} \\
&= \alpha(\log K_{t+1}) + a_{t+1} \\
&= \alpha(\log \alpha\beta Y_t) + a_{t+1} \\
&= \alpha(y_t + \log \alpha\beta) + a_{t+1}
\end{aligned} \tag{4}
$$

so the dynamics of aggregate output, like aggregate capital, reflect a component that mirrors a and a serially correlated component with serial correlation coefficient α.

The simplest assumption to make about the level of technology is that its log follows a random walk:

$$a_{t+1} = a_t + \epsilon_{t+1}. \tag{5}$$

Under this assumption, consider the dynamic effects on the level of output from a unit positive shock to the log of technology in period t (that is, $\epsilon_{t+1} = 1$ where $\epsilon_s = 0 \ \forall \ s \neq t+1$). Suppose that the economy had been at its original steady-state level of output $\check{y}$ in the prior period. Then the expected dynamics of output would be given by

$$
\begin{aligned}
y_t &= \check{y} + a_t \\
\mathbb{E}_t[y_{t+1}] &= \check{y} + a_t + \alpha a_t \\
\mathbb{E}_t[y_{t+2}] &= \check{y} + a_t + \alpha a_t + \alpha^2 a_t
\end{aligned} \tag{6}
$$

and so on, as depicted in figure 1.

Also interesting is the case where the level of technology follows a white noise process,

$$a_{t+1} = \check{a} + \epsilon_{t+1}. \tag{7}$$

The dynamics of income in this case are depicted in figure 2.

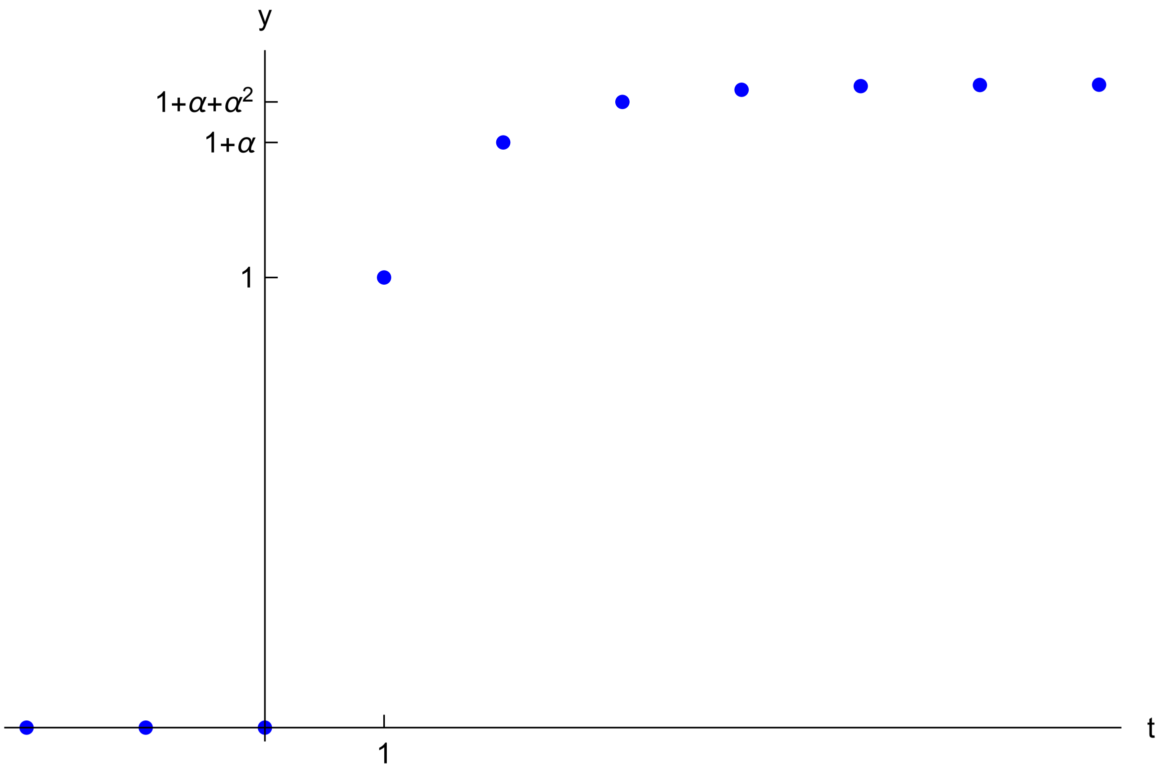
The key point of this analysis, again, is that the dynamics of the model are governed by two components: The dynamics of the technology shock, and the assumption about the saving/accumulation process.

For further analysis, consider a nonstochastic version of this model, with $A_t = 1 \ \forall \ t$. The consumption Euler equation is

$$\frac{C_{t+1}}{C_t} = (\beta \mathsf{R}_{t+1})^{1/\rho}$$

But this is an economy with no technological progress, so the steady-state

Figure 1 Dynamics of Output With a Random Walk Shock



interest rate must take on the value such that $C_{t+1}/C_t = 1$. Thus we must have $\beta\mathsf{R} = 1$ or $\mathsf{R} = 1/\beta$.

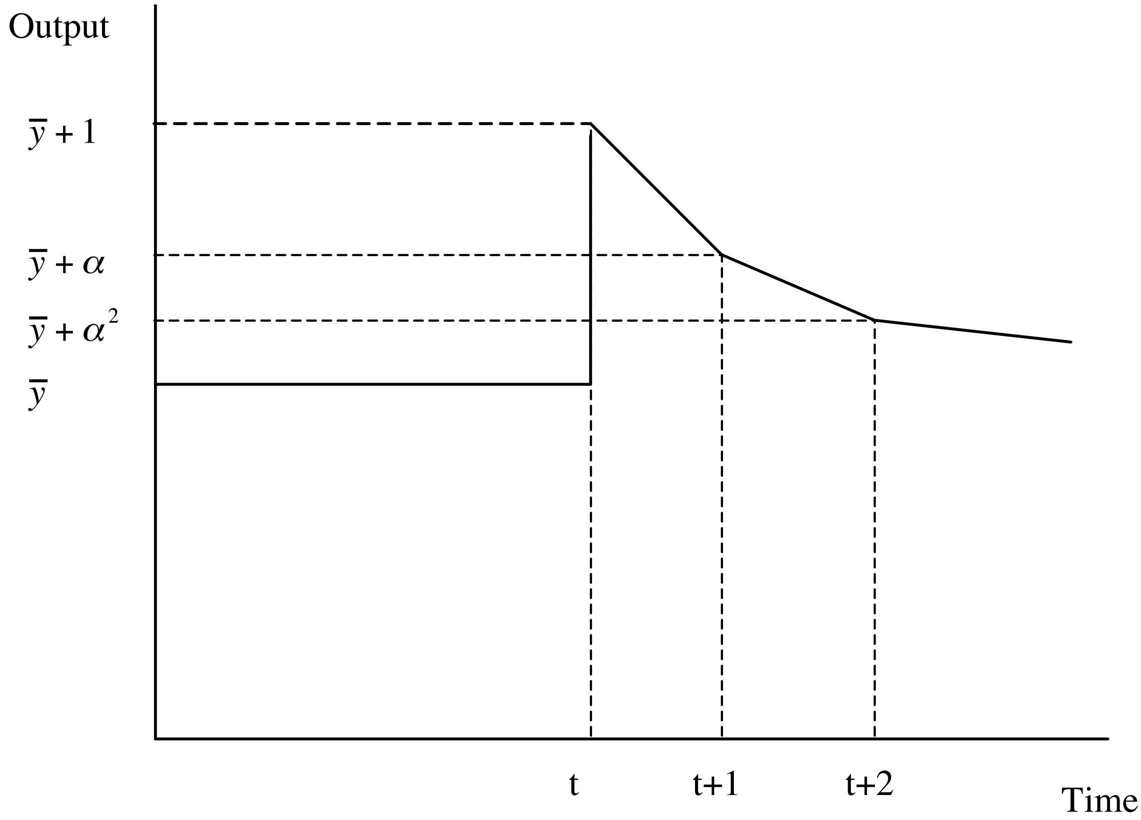
We can further derive the steady state level of capital of a nonstochastic version of the model in which $a_t = a \; \forall \; t$ from (3):

$$\begin{aligned} k &= \log \alpha\beta + a + \alpha k \\ (1-\alpha)k &= \log \alpha\beta + a \\ k &= \left(\frac{\log \alpha\beta}{1-\alpha}\right) + a \end{aligned} \tag{8}$$

The nonstochastic version of the model is of course not very interesting, except as a point of comparison to the stochastic version of the model. But what could be meant by the 'steady state' of a stochastic mdoel that never settles down? We can define a 'stochastic steady state' for such models in a number of (potentially) different ways:

- The location (if one exists) to which the model will converge after an

Figure 2 Dynamics of Output With A White Noise Shock



arbitrarily long period in which no shocks occurred $a_{t+1} = a_t \; \forall \; t$ (even if in every period agents *expect* that shocks will occur)

- The mean value of some variable in the model (say, K)
- The value of some state variable, say $\check{K}$, such that $\mathbb{E}_t[K_{t+1}] = K_t$ if $K_t = \check{K}$.

We consider here the last of these, which we will show reduces (in this special case) to the same equation as for the nonstochastic version of the model, $K_{t+1} = K_t$. To see this, rewrite the Euler equation as:

$$
\begin{aligned}
1 &= \beta \, \mathbb{E}_t \left[\underbrace{\alpha A_{t+1} K_{t+1}^{\alpha-1}}_{\equiv \mathbf{R}_{t+1}} \frac{C_t}{C_{t+1}} \right] \\
1 &= \beta \, \mathbb{E}_t \left[\alpha A_{t+1} K_{t+1}^{\alpha-1} \frac{Y_t}{Y_{t+1}} \right] \\
1 &= \beta \, \mathbb{E}_t \left[\alpha A_{t+1} K_{t+1}^{\alpha-1} \frac{A_t K_t^{\alpha}}{A_{t+1} K_{t+1}^{\alpha}} \right] \\
1 &= \beta \, \mathbb{E}_t \left[\alpha K_{t+1}^{\alpha-1} \frac{A_t K_t^{\alpha}}{K_{t+1}^{\alpha}} \right] \\
1 &= \beta \left[\alpha K_{t+1}^{\alpha-1} \frac{A_t K_t^{\alpha}}{K_{t+1}^{\alpha}} \right]
\end{aligned}
$$

where the expectations operator disappears because no variables are stochastic (the $A'_{t+1}s$ in the numerator and denominator cancel, and K_{t+1} is directly chosen in t so is known. For any given A_t, the steady state where $K_{t+1} = K_t = \check{K}$ is then where

$$
\begin{aligned}
1 &= \beta \alpha A_t \check{K}_t^{\alpha-1} \\
\check{k} &= \left(\frac{\log \beta \alpha}{1-\alpha} \right) + a_t
\end{aligned}
$$

which is the generalization of the nonstochastic solution derived in (8).

The result that the nonstochastic and stochastic steady states are the same is special to the Brock-Mirman model; it is NOT true of many other models of growth; it occurs here because the linearity of the consumption function, among other special assumptions. Furthermore, the third of our possible definitions of a steady state will generally differ at least a little bit from either of the first two.

References (click to download .bib file)

Brock, William, and Leonard Mirman (1972): "Optimal Economic Growth and Uncertainty: The Discounted Case," *Journal of Economic Theory*, 4(3), 479–513.