# Consumption Models with Habit Formation

## 1 The Problem

Consider a consumer whose goal at date $t$ is to solve the problem[1]

$$\max \sum_{n=0}^{T-t} \beta^n \mathrm{u}(c_{t+n}, h_{t+n}) \tag{1}$$

where $h_{t+n}$ is the habit stock, and all other variables are as usually defined. The DBC is

$$m_{t+1} = (m_t - c_t)\mathsf{R} + y_{t+1}. \tag{2}$$

However, when habits affect utility we must also specify a process that describes how habits evolve over time. Our assumption will be:

$$h_{t+1} = c_t. \tag{3}$$

Bellman's equation for this problem is therefore

$$\mathrm{v}_t(m_t, h_t) = \max_{\{c_t\}} \quad \mathrm{u}(c_t, h_t) + \beta \mathrm{v}_{t+1}((m_t - c_t)\mathsf{R} + y_{t+1}, c_t). \tag{4}$$

To clarify the workings of the Envelope theorem in the case with two state variables, let's define a function

$$\underline{\mathrm{v}}_t(m_t, h_t, c_t) = \mathrm{u}(c_t, h_t) + \beta \mathrm{v}_{t+1}((m_t - c_t)\mathsf{R} + y_{t+1}, c_t) \tag{5}$$

and define the function $\mathbf{c}_t(m_t, h_t)$ as the choice of $c_t$ that solves the maximization (4), so that we have

$$\mathrm{v}_t(m_t, h_t) = \underline{\mathrm{v}}_t(m_t, h_t, \mathbf{c}_t(m_t, h_t)). \tag{6}$$

### 1.1 Optimality Conditions

#### *1.1.1 The First Order Condition*

The first order condition for (4) with respect to $c_t$ is (dropping arguments for brevity and denoting the derivative of $f$ with respect to $x$ at time $t$ as $f_t^x$):

$$0 = \mathrm{u}_t^c + \beta \left( \mathrm{v}_{t+1}^h - \mathsf{R}\mathrm{v}_{t+1}^m \right) \tag{7}$$

or, equivalently,

$$\mathrm{u}_t^c = \beta \left( \mathsf{R}\mathrm{v}_{t+1}^m - \mathrm{v}_{t+1}^h \right). \tag{8}$$

The intuition is as follows. Note first that if utility is not affected by habits, then $\mathrm{v}_{t+1}^h = 0$ and equation (8) reduces to the usual first order condition for consumption,

[1] This handout is a simplified version of Carroll (2000).

which tells us that increasing consumption by $\epsilon$ today and reducing it by $\mathsf{R}\epsilon$ in the next period must not change expected discounted utility. With habits, an increase in consumption today has a consequence beyond its effect on tomorrow's resources $m_{t+1}$: tomorrow's habit stock will be changed as well. An increase in consumption today of size $\epsilon$ increases the size of the habit stock which tomorrow's consumption is compared to, and therefore reduces tomorrow's utility by an amount corresponding to the marginal utility of higher habits tomorrow $\mathrm{v}^h_{t+1}$. Since $\mathrm{v}^h_{t+1}$ is negative (higher habits make utility lower), this tells us that the RHS of equation (8) will be a larger positive number than it would be without habits. This means that the level of $c_t$ that satisfies the first order condition will be a lower number (higher marginal utility) than before. Hence, habits increase the willingness to delay spending, and increase the saving rate.

Note that the first order condition also implies that

$$\frac{d\underline{\mathrm{v}}_t}{dc_t} = 0 \tag{9}$$

when evaluated at $c_t = \mathbf{c}_t(m_t, h_t)$.

*1.1.2 Envelope Conditions*

Now consider the total derivative of $\underline{\mathrm{v}}_t(m_t, h_t, \mathbf{c}_t(m_t, h_t))$ with respect to $m_t$. (To reduce clutter, I will write $d\mathbf{c}_t(m_t, h_t)/dm_t$ as $d\mathbf{c}_t/dm_t$). The chain rule tells us that

$$\begin{aligned} \frac{d\underline{\mathrm{v}}_t}{dm_t} &= \frac{d\mathbf{c}_t}{dm_t}\mathrm{u}^c_t + \overbrace{\frac{dh_t}{dm_t}}^{=0}\mathrm{u}^h_t + \beta\left((\frac{d\mathbf{c}_t}{dm_t})(\mathrm{v}^h_{t+1} - \mathsf{R}\mathrm{v}^m_{t+1}) + \mathsf{R}\mathrm{v}^m_{t+1}\right) \\ &= \left(\frac{d\mathbf{c}_t}{dm_t}\right)\underbrace{\left(\mathrm{u}^c_t + \beta\left(\mathrm{v}^h_{t+1} - \mathsf{R}\mathrm{v}^m_{t+1}\right)\right)}_{=0 \text{ at } c_t = \mathbf{c}_t(m_t,h_t) \text{ from (7)}} + \beta\mathsf{R}\mathrm{v}^m_{t+1} \end{aligned} \tag{10}$$

so we have that

$$\begin{aligned} \mathrm{v}^m_t &= \frac{d\underline{\mathrm{v}}_t}{dm_t}\Big|_{c_t=\mathbf{c}_t(m_t,h_t)} \\ &= \beta\mathsf{R}\mathrm{v}^m_{t+1}. \end{aligned} \tag{11}$$

The Envelope theorem is the shortcut way to obtain this conclusion. The clearest way to use the theorem is by taking the partial derivatives of the $\underline{\mathrm{v}}_t$ function with respect to each of its three arguments, using the Chain Rule to take into account the possible dependency of $h_t$ and $c_t$ on $m_t$:

$$\begin{aligned} \mathrm{v}_t(m_t, h_t) &= \underline{\mathrm{v}}_t(m_t, h_t, \mathbf{c}_t(m_t, h_t)) \\ \mathrm{v}^m_t &= \frac{\partial\underline{\mathrm{v}}_t}{\partial m_t} + \frac{\partial\underline{\mathrm{v}}_t}{\partial h_t}\underbrace{\frac{\partial h_t}{\partial m_t}}_{=0} + \underbrace{\frac{\partial\underline{\mathrm{v}}_t}{\partial c_t}}_{=0}\frac{\partial\mathbf{c}_t}{\partial m_t} + \frac{\partial\underline{\mathrm{v}}_t}{\partial c_t}\frac{\partial c_t}{\partial h_t}\underbrace{\frac{\partial h_t}{\partial m_t}}_{=0} \end{aligned} \tag{12}$$

where the Envelope theorem is what tells you that the $\partial\underline{\mathrm{v}}_t/\partial c_t$ term is equal to zero because you are evaluating the function at $c_t = \mathbf{c}_t(m_t, h_t)$ (and $\partial h_t/\partial m_t$ is zero by the assumed structure of the problem in which $h_t$ is predetermined).

Now writing out $\partial \underline{\mathrm{v}}_t/\partial m_t$, (12) becomes

$$\mathrm{v}_t^m = \frac{\partial}{\partial m_t}\left[\beta \mathrm{v}_{t+1}((m_t - c_t)\mathsf{R} + y_{t+1}, h_{t+1})\right] \tag{13}$$

which the envelope theorem says is equivalent to

$$\mathrm{v}_t^m = \beta \mathsf{R} \mathrm{v}_{t+1}^m. \tag{14}$$

There is a potentially confusing thing about doing it this way, however: when you reach an expression like (13) it is tempting to think to yourself as follows: "$c_t$ is a function of $m_t$, and $h_{t+1} = c_t$ is also indirectly a function of $m_t$, so the chain rule tells me that when I take the derivative in (13) I need to keep track of these." In fact, you must treat $\partial c_t/\partial m_t$ and $\partial h_{t+1}/\partial m_t$ as zero here. The reason is that this is a *partial* derivative with respect to $m_t$. The dependence of $c_t$ (and indirectly $h_{t+1}$) on $m_t$ has already been taken care of in the two terms in (12) that were equal to zero. The confusion here is caused largely by the fact that partial differentiation is an area where standard mathematical notation is basically confusing and poorly chosen.[2]

The shortest way to obtain the end result is, as in the single variable problem, to start with Bellman's equation and take the partial derivative with respect to $m_t$ directly (treating the problem as though $c_t$ were a constant):

$$\begin{aligned}\mathrm{v}_t(m_t, h_t) &= \mathrm{u}(c_t, h_t) + \beta \mathrm{v}_{t+1}((m_t - c_t)\mathsf{R} + y_{t+1}, h_{t+1}) \\ \mathrm{v}_t^m(m_t, h_t) &= \beta \mathsf{R} \mathrm{v}_{t+1}^m(m_{t+1}, h_{t+1}).\end{aligned} \tag{15}$$

Whichever way you do it, substituting (14) into the FOC equation (8) gives

$$\mathrm{v}_t^m = \mathrm{u}_t^c + \beta \mathrm{v}_{t+1}^h. \tag{16}$$

The intuition for this is as follows. The marginal value of wealth must be equal to the marginal value associated with a tiny bit more consumption. In the presence of habits, the extra consumption yields extra utility today $\mathrm{u}_t^c$ but affects value next period by $\mathrm{v}_{t+1}^h$ (which is a negative number), the discounted consequence of which from today's perspective is the $\beta \mathrm{v}_{t+1}^h$ term.

In a problem with two state variables, the Envelope theorem can be applied to each state (and indeed in general must be applied in order to solve the model).

Again let's start the brute force way by working through the total derivative of $\underline{\mathrm{v}}_t$. For this problem, the total derivative (again denoting $d\mathbf{c}_t(m_t, h_t)/dh_t$ as $d\mathbf{c}_t/dh_t$) is:

$$\begin{aligned}\frac{d\underline{\mathrm{v}}_t}{dh_t} &= \frac{d\mathbf{c}_t}{dh_t}\mathrm{u}_t^c + \mathrm{u}_t^h + \beta\left(\frac{dh_{t+1}}{dh_t}\mathrm{v}_{t+1}^h + \frac{dm_{t+1}}{dh_t}\mathrm{v}_{t+1}^m\right) \\ &= \frac{d\mathbf{c}_t}{dh_t}\mathrm{u}_t^c + \mathrm{u}_t^h + \beta\left(\frac{dh_{t+1}}{d\mathbf{c}_t}\frac{d\mathbf{c}_t}{dh_t}\mathrm{v}_{t+1}^h + \frac{dm_{t+1}}{d\mathbf{c}_t}\frac{d\mathbf{c}_t}{dh_t}\mathrm{v}_{t+1}^m\right) \\ &= \mathrm{u}_t^h + \frac{d\mathbf{c}_t}{dh_t}\underbrace{\left(\mathrm{u}_t^c + \beta(\mathrm{v}_{t+1}^h - \mathsf{R}\mathrm{v}_{t+1}^m)\right)}_{=0 \text{ at } c_t = \mathbf{c}_t(m_t, h_t) \text{ from } (7)}\end{aligned} \tag{17}$$

[2]Google the string "partial differentiation confusing OCW" to find a fuller description of the problems of standard notation on partial differentiation.

so we have

$$\begin{aligned} \mathrm{v}_t^h &= \frac{d\underline{\mathrm{v}}_t}{dh_t}|_{c_t=\mathbf{c}_t(m_t,h_t)} \\ &= \mathrm{u}_t^h. \end{aligned} \tag{18}$$

Turning now to more direct use of the envelope theorem, the Chain Rule tells us

$$\mathrm{v}_t^h = \frac{\partial \underline{\mathrm{v}}_t}{\partial m_t} \overbrace{\frac{\partial m_t}{\partial h_t}}^{=0} + \frac{\partial \underline{\mathrm{v}}_t}{\partial h_t} + \frac{\partial \underline{\mathrm{v}}_t}{\partial c_t} \frac{\partial \mathbf{c}_t}{\partial h_t}$$

while the Envelope theorem once again says $\partial \underline{\mathrm{v}}_t / \partial c_t = 0$ at $c_t = \mathbf{c}_t(m_t, h_t)$ so we obtain

$$\begin{aligned} \mathrm{v}_t^h &= \frac{\partial \underline{\mathrm{v}}_t}{\partial h_t} \\ &= \mathrm{u}_t^h \end{aligned} \tag{19}$$

since $h_t$ appears directly only in the $\mathrm{u}(c_t, h_t)$ part of $\underline{\mathrm{v}}_t(m_t, h_t, c_t)$. And once again, the shortest way to the answer is to treat $c_t$ as though it were a constant in the value function, which yields

$$\begin{aligned} \mathrm{v}_t(m_t, h_t) &= \mathrm{u}(c_t, h_t) + \beta \mathrm{v}_{t+1}((m_t - c_t)\mathsf{R} + y_{t+1}, c_t) \\ \mathrm{v}_t^h(m_t, h_t) &= \mathrm{u}_t^h. \end{aligned} \tag{20}$$

From (16) this implies that

$$\mathrm{v}_t^m = \mathrm{u}_t^c + \beta \mathrm{u}_{t+1}^h. \tag{21}$$

Roll this equation forward one period and substitute into equation (14) to obtain:

$$\mathrm{u}_t^c + \beta \mathrm{u}_{t+1}^h = \mathsf{R}\beta \left[\mathrm{u}_{t+1}^c + \beta \mathrm{u}_{t+2}^h\right] \tag{22}$$

Note that if $\mathrm{u}_{t+1}^h = \mathrm{u}_{t+2}^h = 0$ so that habits have no effect on utility, (22) again is solved by the standard time-separable Euler equation.

Now assume that the utility function takes the specific form

$$\mathrm{u}(c, h) = \mathbf{f}(c - \alpha h) \tag{23}$$

which implies derivatives of

$$\begin{aligned} \mathrm{u}^c &= \mathbf{f}' \\ \mathrm{u}^h &= -\alpha \mathbf{f}'. \end{aligned} \tag{24}$$

Substituting these into equation (22) we obtain,

$$\mathbf{f}_t' - \alpha\beta \mathbf{f}_{t+1}' = \mathsf{R}\beta[\mathbf{f}_{t+1}' - \alpha\beta \mathbf{f}_{t+2}'] \tag{25}$$

Now assume that there is a solution in which marginal utility of consumption grows at a constant rate over time, $\mathbf{f}_t' = k\mathbf{f}_{t+1}'$ and substitute into (25)

$$\begin{aligned} \mathbf{f}_{t+1}'(k - \alpha\beta) &= \mathsf{R}\beta[\mathbf{f}_{t+2}'(k - \alpha\beta)] \\ k\mathbf{f}_{t+2}'(k - \alpha\beta) &= \mathsf{R}\beta[\mathbf{f}_{t+2}'(k - \alpha\beta)] \\ k &= \mathsf{R}\beta \end{aligned} \tag{26}$$

so marginal utility grows at rate $1/\mathsf{R}\beta$. Note that if we assume $\alpha = 0$ so that habits do not matter, we again obtain the standard result that $\mathrm{u}'(c_t) = \mathsf{R}\beta \mathrm{u}'(c_{t+1})$.

Now make the final assumption that $\mathbf{f}(z) = z^{1-\rho}/(1-\rho)$, implying of course that $\mathbf{f}'(z) = z^{-\rho}$. Equation (26) can be rewritten

$$1 = \mathsf{R}\beta(z_{t+1}/z_t)^{-\rho} \tag{27}$$

Now expand $z_{t+1}/z_t$

$$\begin{aligned}
\frac{c_{t+1} - \alpha c_t}{c_t - \alpha c_{t-1}} &= \frac{c_{t+1}/c_t - \alpha}{1 - \alpha c_{t-1}/c_t} && (28)\\
&\approx \frac{1 + \Delta \log c_{t+1} - \alpha}{1 - \alpha + \alpha\Delta \log c_t} && (29)\\
&= \frac{1 - \alpha + \Delta \log c_{t+1}}{1 - \alpha + \alpha\Delta \log c_t} && (30)\\
&= \frac{1 + (\Delta \log c_{t+1})/(1-\alpha)}{1 + (\alpha/(1-\alpha))\Delta \log c_t} && (31)\\
&\approx 1 + \left(\frac{1}{1-\alpha}\right)(\Delta \log c_{t+1} - \alpha\Delta \log c_t) && (32)
\end{aligned}$$

where (29) follows from (28) because $c_{t+1}/c_t = 1 + (c_{t+1} - c_t)/c_t \approx 1 + \Delta \log c_{t+1}$ and $c_{t-1}/c_t = (c_t - (c_t - c_{t-1}))/c_t \approx 1 - \Delta \log c_t$, and (32) follows from (31) because for small $\eta$ and $\epsilon$, $(1+\eta)/(1+\epsilon) \approx 1 + \eta - \epsilon$.

Substituting (32) into (27) gives

$$\begin{aligned}
1 &\approx \mathsf{R}\beta(1 + \left(\frac{1}{1-\alpha}\right)(\Delta \log c_{t+1} - \alpha\Delta \log c_t])^{-\rho}\\
0 &\approx \log[\mathsf{R}\beta] - \rho\left(\frac{1}{1-\alpha}\right)(\Delta \log c_{t+1} - \alpha\Delta \log c_t)\\
\Delta \log c_{t+1} &\approx (1-\alpha)\rho^{-1}(\mathsf{r} - \vartheta) + \alpha\Delta \log c_t.
\end{aligned}$$

Thus, this formulation of habit formation implies that the growth rate of consumption is serially correlated.

# References


Carroll, Christopher D. (2000): "Solving Consumption Models with Multiplicative Habits," *Economics Letters*, 68(1), 67–77, http://econ.jhu.edu/people/ccarroll/HabitsEconLett.pdf.