The Ramsey/Cass-Koopmans (RCK) Model with Government

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This handout solves the Ramsey/Cass-Koopmans (RCK) model with government.[1] For simplicity we assume no technological progress, population growth, or depreciation, and a continuum of consumers with mass 1 distributed on the unit interval as per Aggregation.

Consider first the case where government is financed by constant lump-sum taxes of amount τ per period, and spending is at rate $\mathbf{x}$ per period, and suppose the government has a balanced budget requirement so that $\mathbf{x} = \tau$ in every period and there is no government debt. We suppose further that government spending yields no utility. The individual's optimization problem (leaving out the i subscripts that we used in the previous handout, but understanding that they are implicitly present) is now

$$\int_0^\infty u(c_t)e^{-\vartheta t} \tag{1}$$

subject to the household DBC

$$\dot{a}_t = r_t a_t + w_t - c_t - \tau, \tag{2}$$

which has Hamiltonian representation

$$\mathcal{H}(a_t, c_t, \lambda_t) = u(c_t) + (r_t a_t + w_t - c_t - \tau)\lambda_t$$

The first Hamiltonian optimization condition requires $\partial \mathcal{H}_t/\partial c_t = 0$:

$$\begin{aligned} c_t^{-\rho} &= \lambda_t \\ -\rho c_t^{-\rho-1}\dot{c}_t &= \dot{\lambda}_t \end{aligned} \tag{3}$$

The second Hamiltonian optimization condition requires:

$$\begin{aligned} \dot{\lambda}_t &= \vartheta\lambda_t - (\partial \mathcal{H}_t/\partial a_t)\lambda_t \\ &= \vartheta\lambda_t - \lambda_t r_t \\ \dot{\lambda}_t/\lambda_t &= (\vartheta - r_t) \\ \dot{c}_t/c_t &= \rho^{-1}(r_t - \vartheta). \end{aligned} \tag{4}$$

Finally, the household's behavior must satisfy a transversality constraint, which is equivalent to the intertemporal budget constraint:

$$\begin{aligned} \int_0^\infty c_t \mathfrak{R}_t^{-1} &= a_0 + \int_0^\infty \mathbf{y}_t \mathfrak{R}_t^{-1} - \int_0^\infty \tau \mathfrak{R}_t^{-1} \\ \mathbf{C}_0 &= a_0 + \mathbf{Y}_0 - \mathbf{T}_0 \end{aligned} \tag{5}$$

which says that the present discounted value of consumption must equal the current net physical wealth plus human wealth minus the PDV of taxes.

[1] The treatment is similar to that in Blanchard and Fischer (1989); see that source for more details.

Now consider the problem from the standpoint of a social planner who has the same utility function as the individual consumers. If the social planner wants to spend a constant amount $\mathbf{x}$ per period, the social planner's budget constraint is

$$\dot{k}_t = \mathrm{f}(k_t) - \delta k_t - c_t - \mathbf{x} \tag{6}$$

which reflects the fact that the social planner divides total net output between consumption and government spending. This leads to Hamiltonian

$$\mathcal{H}(k_t, c_t, \lambda_t) = u(c_t) + \lambda_t(\mathrm{f}(k_t) - \delta k_t - c_t - \mathbf{x}), \tag{7}$$

yielding the first order condition

$$\dot{c}_t/c_t = \rho^{-1}(\mathrm{f}(k_t) - \delta - \vartheta). \tag{8}$$

Now recall from the handout on decentralizing the RCK model that

$$\mathrm{f}(k_t) = \hat{r}_t k_t + w_t \tag{9}$$

where the gross return on capital $\hat{r}_t$ is equal to the net return r_t plus the depreciation rate.

Thus, the social planner's DBC is:

$$\begin{aligned}
\dot{k}_t &= \mathrm{f}(k_t) - \delta k_t - c_t - \mathbf{x} \\
&= \hat{r}_t k_t - \delta k_t + w_t - c_t - \mathbf{x} \\
&= r_t k_t + w_t - c_t - \mathbf{x},
\end{aligned} \tag{10}$$

which is equivalent to the household's budget constraint (2) when $a_t = k_t$ and $\tau = \mathbf{x}$. As discussed in DecentralizingRCK, $a_t = k_t$ must hold in equilibrium for identical households, and $\tau = \mathbf{x}$ was the balanced budget assumption that we started off with.

Note that the $\dot{c}_t = 0$ locus in the phase diagram is unchanged by changing τ and $\mathbf{x}$. However, the $\dot{k}_t = 0$ locus is shifted down by amount $\tau = \mathbf{x}$.

Now what happens if the government does not face a balanced budget requirement? Specifically, suppose we continue to have the same constant amount of spending per period but now want to consider the effect of allowing taxes to vary over time, which we denote by a subscript on τ_t. Suppose d is the level of government bonds (debt); the government's Dynamic Budget Constraint is

$$\dot{d}_t = \mathbf{x} + r_t d_t - \tau_t, \tag{11}$$

which says that debt must rise by the amount by which spending exceeds taxes.

The government's IBC will be the integral of its DBC:

$$\begin{gathered}
d_0 + \int_0^\infty \mathbf{x}\mathfrak{R}_t^{-1} = \int_0^\infty \tau\mathfrak{R}_t^{-1} \\
d_0 + \mathbf{X}_0 = \mathbf{T}_0
\end{gathered} \tag{12}$$

and we assume $d_0 = 0$ so that the government starts out with no debt (to maintain comparability with the previous example).

The DBC of the idiosyncratic family also changes. They can now own either capital k_t or government debt d_t. If the family is to be indifferent between the two forms of assets, the interest rate must be the same.

$$\begin{aligned} c_t + \dot{a}_t &= w_t + r_t a_t - \tau_t \\ a_t &= k_t + d_t. \end{aligned} \tag{13}$$

Now the family's IBC becomes

$$\begin{aligned} \int_0^\infty c_t \mathfrak{R}_t^{-1} &= k_0 + d_0 + \mathbf{Y}_0 - \int_0^\infty \tau_t \mathfrak{R}_t^{-1} \\ &= k_0 + d_0 + \mathbf{Y}_0 - \mathbf{T}_0. \end{aligned} \tag{14}$$

Note: Nowhere in this equation does the time path of taxes matter; all that matters is the PDV of taxes. And the time path of taxes also does not enter the $\dot{c}/c$ equation. Thus, the path of consumption over time is unaffected by the path of taxes over time!

This is not so surprising when you realize that it is simply the Ricardian equivalence proposition in this perfect foresight framework.

However, now consider the case where there is a tax on capital income at rate τ. Furthermore, for simplicity suppose that the government rebates all of the tax revenue in a lump sum per capita, and suppose depreciation $\delta = 0$. Thus the household budget constraint becomes

$$\dot{a}_t = r_t(1-\tau)a_t + w_t - c_t + \mathbf{z}_t \tag{15}$$

where $\mathbf{z}_t$ is the per-capita size of the lump-sum rebates,

$$\mathbf{z}_t = \tau r_t k_t. \tag{16}$$

The household's Hamiltonian becomes

$$\mathcal{H}_t(c_t, a_t, \lambda_t) = u(c_t) + \lambda_t(r_t(1-\tau)a_t + w_t - c_t + \mathbf{z}_t). \tag{17}$$

The crucial difference between this situation and the previous one is that now the effective rate of return on saving has been decreased, so that $\partial \mathcal{H}_t / \partial a_t$ is now $r_t(1-\tau)$ rather than r_t. Ultimately this produces a consumption Euler equation of

$$\dot{c}_t / c_t = \rho^{-1}((1-\tau)r_t - \vartheta) \tag{18}$$

which implies that the economy will be in equilibrium at

$$\begin{aligned} \mathrm{f}(\bar{k})(1-\tau) &= \vartheta \\ \mathrm{f}(\bar{k}) &= \vartheta \end{aligned} \tag{19}$$

so that the equilibrium level of the marginal product of capital is higher, and the capital stock must therefore be lower, than before the capital taxation was instituted.

Notice, however, that because the taxes are being rebated, the *aggregate* budget constraint does not change when the tax is imposed:

$$\begin{aligned} \dot{k}_t &= r_t(1-\tau)k_t + w_t - c_t + \tau k_t r_t \\ &= r_t k_t + w_t - c_t. \end{aligned} \tag{20}$$

Thus the social planner will choose exactly the same amount of consumption as before the tax was instituted.

The crucial point is that if an individual household saves more and thus causes next year's capital stock to be a bit higher, the *personal* benefit to that household is essentially zero. The higher taxes that the household will pay next year will be distributed to the entire population in a lump sum, so the saver will get nothing. The higher saving of this individual household is basically a positive externality from the point of view of the other consumers in the economy. However, if the social planner forces the economy as a whole to save more, the social planner receives all of the extra tax revenue.

References

Blanchard, Olivier, and Stanley Fischer (1989): *Lectures on Macroeconomics.* MIT Press.