Generic Analysis of Endogenous Growth Models

The neoclassical theory of economic growth, as formulated by Solow (1956), and Cass (1965)-Koopmans (1965), attributes virtually all long-run growth to technological progress. The level of technology is taken to be an exogenously growing factor outside the model. So the baseline model of growth says that growth is caused mostly by a factor that is not in the model.

It is important to understand why Solow and others made this assumption. The answer is that models of perfect competition are the simplest existing models of firm behavior, with well-understood implications. But models with perfect competition require constant returns to scale, and say that all factors of production are paid their marginal product; and [EulersTheorem]in MathFacts says that the sum of the factor payments exhausts all output:

$$Y = F_K K + F_L L. \tag{1}$$

Thus, the perfectly competitive firm has no money left over with which to finance basic research, invent patentable technologies, or do anything other than meet the payroll of production workers and pay the cost of capital. Thus, no firm can afford to pay for technological research, and there is therefore no alternative to the assumption that technological progress occurs exogenously.

This unsatisfactory situation persisted for a long time, but a famous paper by Paul Romer (1986) led the way to the formulation of a new generation of models that allow a role for investment in knowledge to affect growth.

The Romer paper spawned a great deal of further theoretical work by a host of others, but a penetrating paper by Sergio Rebelo (1991) provided a succinct summary of the key feature of all of these models. This handout summarizes the Rebelo point as interpreted by Barro and Sala-i-Martin (1995).

1 The Key Point

Rebelo's key observation is as follows. Consider the class of models with a Cobb-Douglas aggregate production function in capital and labor:

$$Y_t = A K_t^\alpha L_t^\nu \tag{2}$$

where no restriction is made on the ν and α coefficients. (Recall that Solow, Cass, and Koopmans all assumed $\nu + \alpha = 1$; Rebelo is relaxing this restriction). Now suppose for simplicity that in this economy saving is a constant proportion of gross income. In continuous time, the growth of the capital stock is given by

$$\dot{K}_t = s A K_t^\alpha L_t^\nu - \delta K_t. \tag{3}$$

Suppose further that population growth is constant at

$$\dot{L}_t / L_t = \xi, \tag{4}$$

and as usual define per-capita variables as the aggregate normalized by population, e.g. $k_t = K_t/L_t$. Then the aggregate per-capita accumulation equation can be rewritten

$$\dot{k}_t = sAk_t^{\alpha}L_t^{\nu+\alpha-1} - (\delta + \xi)k_t$$
$$\dot{k}_t/k_t = sAk_t^{\alpha-1}L_t^{\nu+\alpha-1} - (\delta + \xi).$$

$$(5)$$

2 Characteristics of the Steady State

If this model has a steady-state growth rate, that rate must satisfy

$$\dot{k}_t/k_t = \gamma \tag{6}$$

for some constant γ. (The value of A does not affect the conclusions from here on, and so we will assume without loss of generality that $A = 1$). From (5), this implies that

$$\gamma = sk_t^{\alpha-1}L_t^{\nu+\alpha-1} - (\delta + \xi). \tag{7}$$

Take the time derivative of this equation to obtain

$$0 = s\left((\alpha - 1)k_t^{\alpha-2}\dot{k}_t L_t^{\nu+\alpha-1} + k_t^{\alpha-1}(\nu + \alpha - 1)\dot{L}_t L_t^{\nu+\alpha-2}\right)$$
$$= k_t^{\alpha-1}L_t^{\nu+\alpha-1}\left((\alpha - 1)\dot{k}_t/k_t + (\nu + \alpha - 1)(\dot{L}_t/L_t)\right)$$
$$0 = (\alpha - 1)\dot{k}_t/k_t + (\dot{L}_t/L_t)(\nu + \alpha - 1)$$
$$= (\alpha - 1)\gamma + \xi(\nu + \alpha - 1).$$

$$(8)$$

Using this equation, we can construct a complete catalog of the possible circumstances under which steady-state growth γ can be different from zero endogenously.

2.1 Possibilities for Steady State Growth

1. $\nu + \alpha = 1$ (Constant Returns Models)

   a) $\{\nu, \alpha\} < 1 \rightarrow \gamma = 0$ (Solow case)

   b) $\nu = 0, \alpha = 1$: Rebelo (1991) AK growth model

2. $\nu + \alpha > 1$ (Increasing Returns Models)

   a) $\{\nu, \alpha\} < 1$

   i. $\xi > 0 \rightarrow \gamma = \left(\overbrace{\frac{\nu + \alpha - 1}{(1 - \alpha)}}^{>0} \right) \xi$

   ii. $\xi = 0 \rightarrow \gamma = 0$

   b) $\nu > 0, \alpha = 1$

   i. $\xi > 0 \rightarrow 0 = \xi(\nu + \alpha - 1)$ which is not satisfied for any γ

   ii. $\xi = 0 \rightarrow 0 = 0$ which can be satisfied for any γ

So whatever the details of endogenous growth models may be, in the end any model that generates perpetual growth without exogenous technological progress must be mathematically reducible to a form like that of either 1.b., 2.a.i., or 2.b.ii.

It is worth delving a bit further into why the Solow case cannot generate perpetual growth. The answer can be understood using the Solow growth accounting framework, which says that there are only three sources of long-run growth: technology, labor, and capital. Thus, there are only two potential sources of growth of output per unit of labor: Technology and an increase in the capital/labor ratio. But if $\alpha < 1$, the gross marginal product of capital approaches zero as the capital/labor ratio approaches infinity; subtracting out depreciation, eventually the net marginal product of capital becomes negative. Thus, capital accumulation can sustain growth only so long.

The bottom line is that there are only two configurations of the model that are capable of generating perpetual growth in a way that makes any sense: 1.b. (the Rebelo AK model) and 2.a.ii.

What this means is that any model that aims to permit perpetual long-run growth must ultimately boil down to a structure in which there are constant returns to scale for some set of factors of production that can jointly be accumulated forever.

References

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