Contents

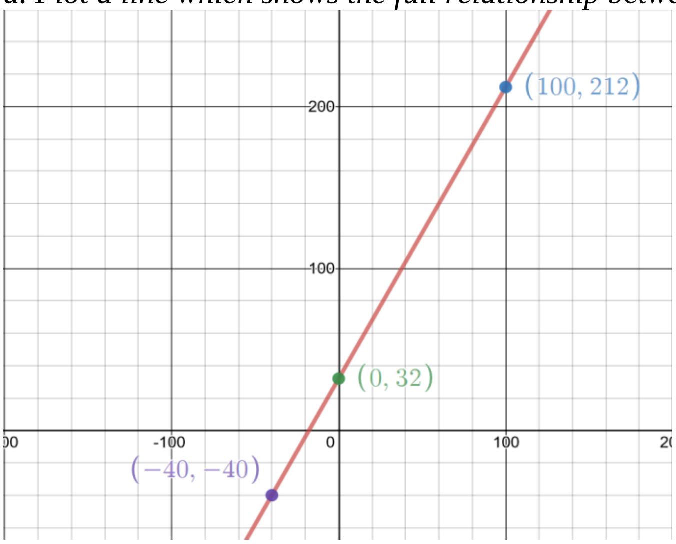
Solutions to Exercises

I. Linear equations and their graphs

a. Algebra and geometry of slopes

1. 0 degrees Celsius = 32 degrees Fahrenheit, and 100 Celsius = 212 Fahrenheit.

a. Plot a line which shows the full relationship between Celsius and Fahrenheit.



(Here we have y as F and x as C)

b. Write the equation that does the same thing.

Multiple methods:

→ Look at line between two points initially given (0, 32) and (100, 212). Slope of the line is rise over run, i.e. $\frac{212-32}{100-0} = \frac{180}{100} = 1.8$ And we can see that the y-intercept is 32. So, the equation in slope-intercept form is: $F = 1.8C + 32$

→ We can also plug each of the two points given into the point-slope form:

$32 - 212 = m(0 - 100) \rightarrow -180 = -100m \rightarrow m = 1.8$, and we know the y-int.

c. How many degrees F = 1 degree C?

Simply plug the value 1 in to our equation to find the F value → $1.8(1) + 32 = 34$

d. Is there any temperature at which C and F give the same number? If so, what is it?

HINT: You already have the equation relating C to F. Write another equation which simply says C = F. Can you find a spot where both equations are true?

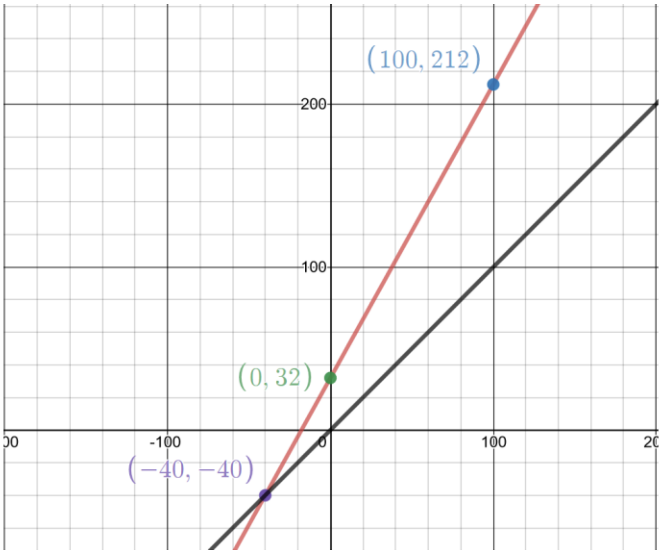
We now have an equation relating F and C, which when combined with the simple equation *C* = *F*, or *F* = *C*, can be solved as a system of two equations:

$$F = 1.8C + 32 \rightarrow 0 = 0.8C + 32 \rightarrow -32 = 0.8C \rightarrow C = -40$$
$$-(F) = -(C)$$

By multiplying each side of the equation *F* = *C* by -1, once we added the two equations together we had one equation with unknown variable, *C*, which we could easily solve for.

Keep in mind that we also could have used the substitution method by putting C in for F, or vice versa. Plugging the value of C (- 40) back into our original equation gives: $F = 1.8(-40) + 32 \rightarrow F = -40$

e. Can you find the same point by plotting C = F on the graph you already generated?



Yes, the solution of the system of equations is where the two lines cross, at (-40, -40)

2. There are b boys in the class. This is three more than four times the number of girls. How many girls are in the class?

We want to know x, the number of girls. We can rewrite the words above as an equation and solve for x: $b = 4x + 3 \rightarrow x = \frac{b-3}{4}$

a. If b is 11, how many girls are in the class?

All we have to do is plug 11 in for b in our equation: $x = \frac{11-3}{4} \rightarrow x = 2$

3. Many counties in Tennessee missed school days in the spring of 2010 due to a massive flood. One solution for how to make up those days was to add time to each school day for a portion of the year. Say that 10 minutes were added to each day. How many extended days would be needed to make up 5 school days, which last 6 hours each?

We need to make up $5 * 6 = 30$ hours. Each unit of extra time is 10 minutes, or $\frac{1}{6}$ of an hour. $\frac{30}{1/6} = 180$, the number of extended days needed.

4. Attorney A charges a fixed fee of $250 for an initial meeting and $150 per hour for all hours worked after that.

a. Write an equation for the amount she charges in slope-intercept form.

Call her final fee F and her hours worked h. $F = 250 + 150h$

b. Attorney B charges $150 for the initial meeting and $175 per hour. Find the charge for 26 hours of work for each attorney.

$F_A = 250 + 150h_A \rightarrow F_A = 250 + 150 * 26 =$ $4,150 for Attorney A

$F_B = 150 + 175h_B \rightarrow F_B = 150 + 175 * 26 =$ $4,700 for Attorney B

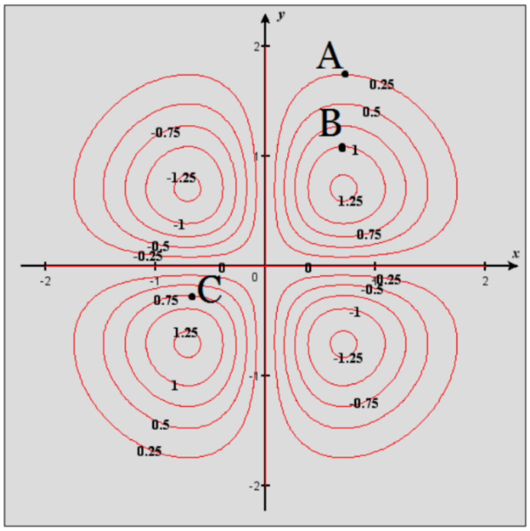
c. Which is the better deal?

For 26 hours of work, it is cheaper to hire Attorney A.

d. At how many hours does hiring Attorney B become a better deal?

Find the number of hours $h*$ where the two would charge the same fee, i.e. set $F_A = F_B$. Then we have: $250 + 150h^* = 150 + 175h^* \rightarrow 100 = 25h^* \rightarrow h^* = 4$. If the work takes fewer than 4 hours, it is cheaper to hire Attorney B. Otherwise, hiring A is cheaper.

b. Level Curves



1. a. If a person were walking straight from point A to point B, would she be walking uphill or downhill?

Uphill, because the z (elevation) values as seen on the red lines are increasing.

b. Is the slope steeper at point B or point C?

Point C, because the level curves are closer together at that point. This indicates that small changes in x and/or y result in bigger z changes than at point B.

c. Starting at C and moving so that x remains constant and y decreases, will the elevation begin to increase or decrease?

Increase. This is the same as moving straight “down” from a 2D perspective, and z increases as we move in that direction.

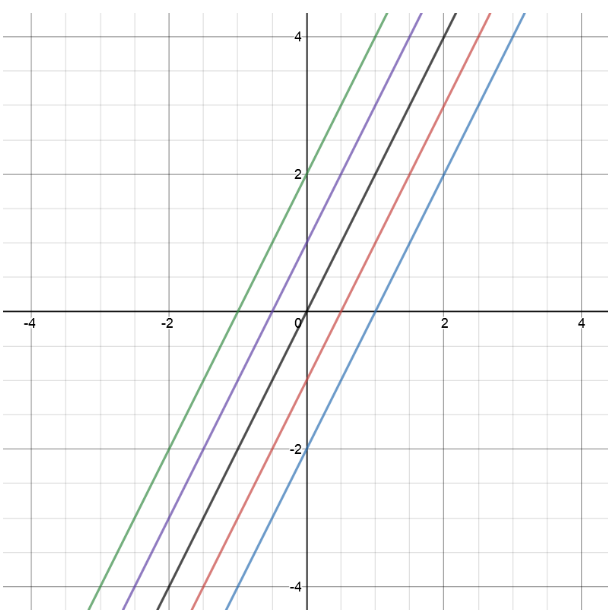
d. Starting at B and moving so that y remains constant and x increases, will the elevation begin to increase or decrease?

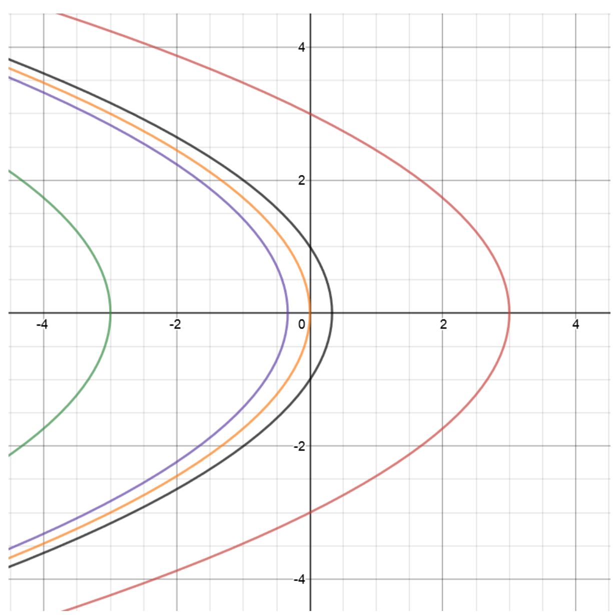
Decrease. This is the same as moving straight “right” from a 2D perspective, and z decreases as we move in that direction.

2. Plot a few points and sketch the level curves $f(x, y) = k$ *for the specified values of k.*

a. $k = 2x - y \to k = -2, -1, 0, 1, 2$

b. $k = y^2 + 3x \to k = -9, -1, 0, 1, 9$

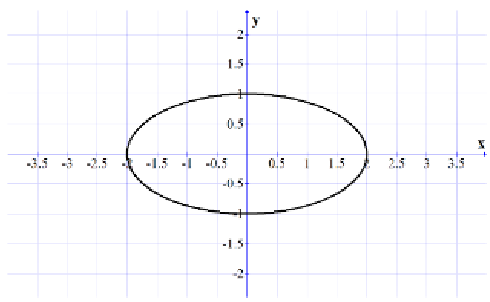




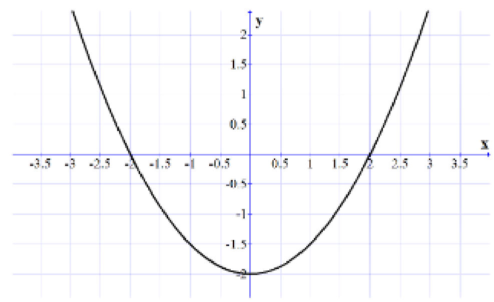
3. Multiple Choice: Which of the following graphs is the level curve for $f(x,y) = x^2 + 4y^2$ which passes through (−2, 0)?

We can eliminate (b) immediately because it does not contain the point (-2, 0). Next, we should plug (-2, 0) into the equation: $f = (-2)^2 + 4(0)^2 = 4$. So we should be expecting every point (x, y) on the graph of the curve we choose to have the value 4 when plugged into the equation. Graph (c) has the point (0, 2) which would yield $f = 16$, so that option is eliminated. Graph (d) has the point (0, -2), which also yields $f = 16$ and is not the correct answer. Graph (e) looks suspicious if we keep in mind that the graph should include all points where $f = 4$. Is there some point where x is 0 and $f = 4$? Yes, we simply solve: $4 = 0 + 4y^2 \rightarrow y = \pm 1$. Since graph (e) does not include either (-1, 0) or (1, 0), it cannot be correct. Then graph (a) is our answer, which makes sense if you notice that the original equation is for a slightly modified circle.

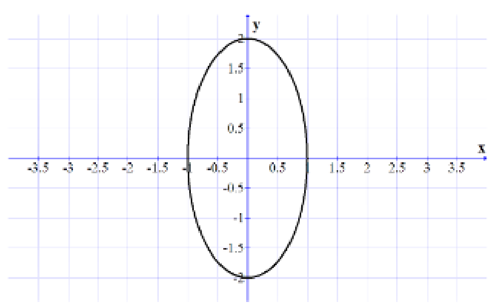
(a)



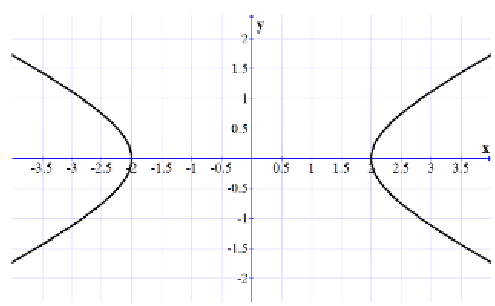
(d)



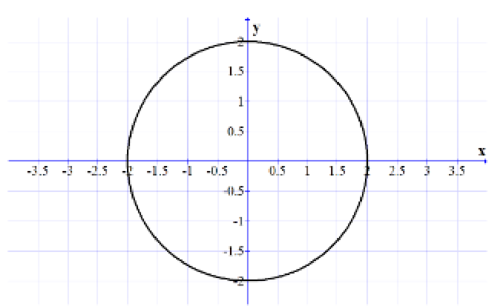
(b)



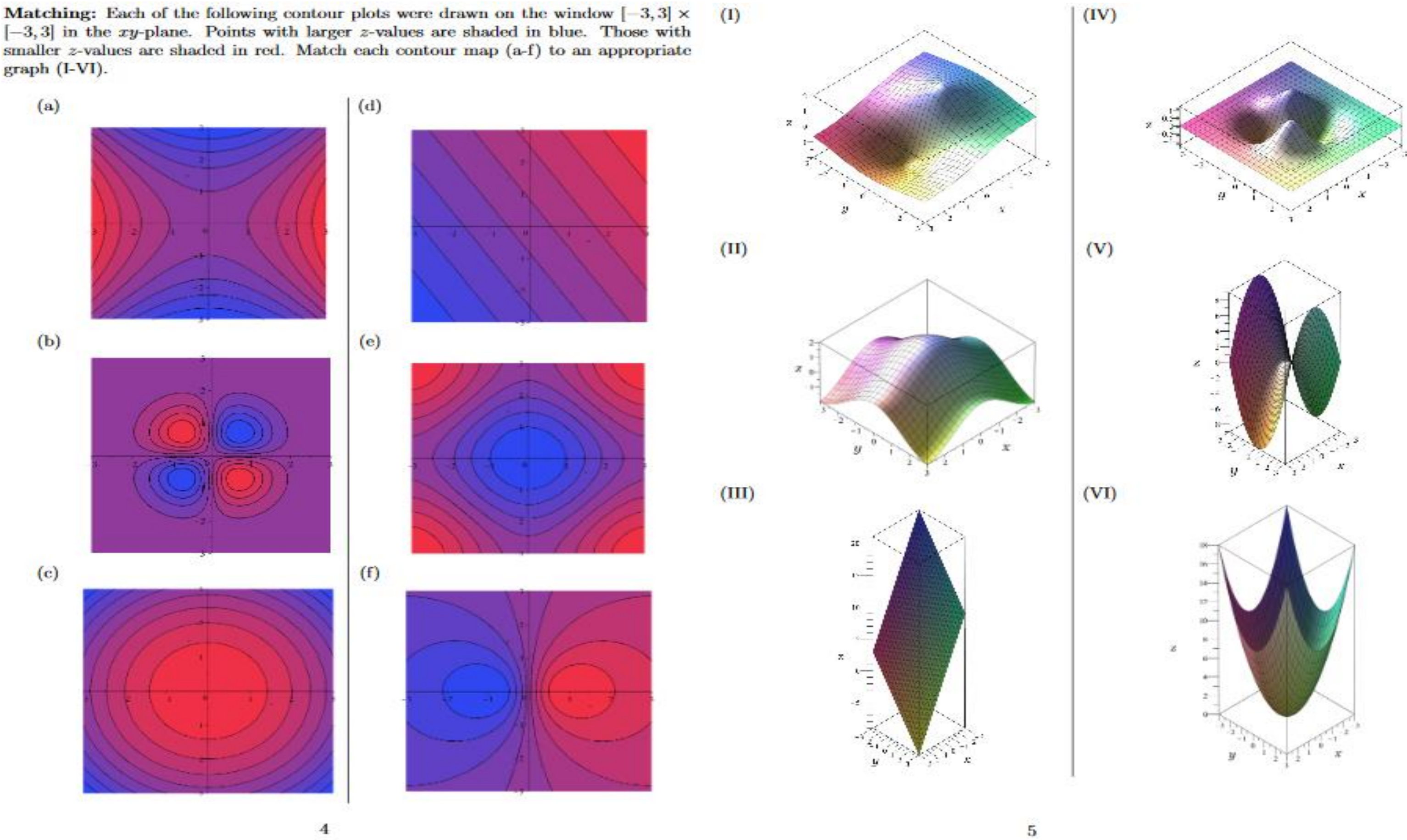
(e)



(c)



4. (See below)



Matching: Each of the following contour plots were drawn on the window $[-3,3] \times [-3,3]$ in the xy-plane. Points with larger z-values are shaded in blue. Those with smaller z-values are shaded in red. Match each contour map (a-f) to an appropriate graph (I-VI).

First notice that (b) must match (IV) since it shows two pairs of zones, one with high z (blue) and the other with low z (red).

Another match is (f) to (I), which shows a gradually deepening zone (red) across from a gradually growing zone (blue).

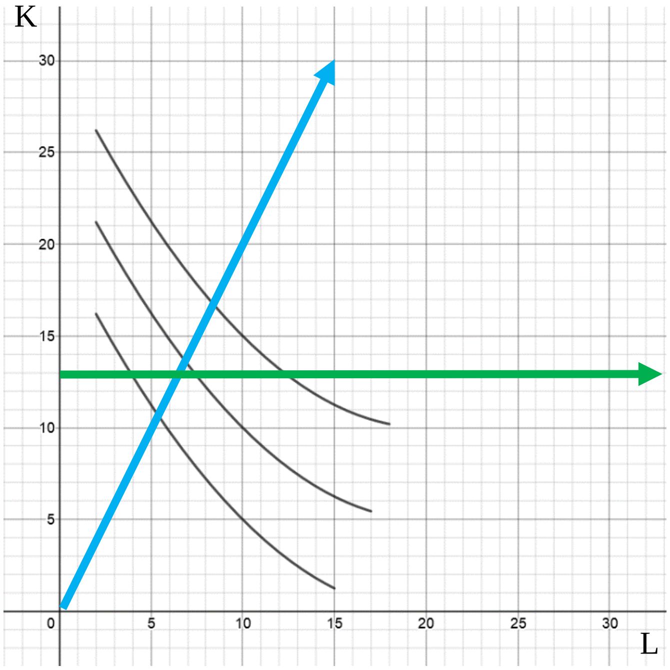
And (e) must match (II) since it depicts a central zone of high z (blue) gradually decreasing, with lowest points at the four corners of the picture (red).

We can match (c) to (VI) by noticing that the center is red and should be the lowest point, with a rapid increase in z towards the four corners (blue). Even if you were unsure about the color coding, it is apparent that the steepness of (c) is much higher than that of (e).

Then (d) matches (III), since it shows a gradual but linear progression from highest to lowest z.

And (a) matches (V), which is hard to visualize; notice that the lowest values of z are at the extremes of x (-3 and 3) where y = 0, which matches the graph.

5. On this graph of isoquants, draw a line showing all the combinations of K and L such that $K = 2L$ and a line such that K = 13 regardless of what value L takes. What does it mean to move along these lines? Explain fully.



The up-sloped teal line shows combinations of K and L such that K = 2L, and the horizontal green line shows K = 13.

The meaning of moving along the K = 2L line is that as you add units of K one unit at a time, you also add two units of L per step. In economic terms, this would suggest that producing the good is more labor-intensive than capital-intensive; each unit of capital must be matched by two units of labor.

Moving along the K = 13 line means that the amount of capital is fixed, no matter what value L takes. In order to move to higher-output isoquants, only more Labor input is required.

c. Solving two linear equations

1. One number is 10 more than another. The sum of twice the smaller plus three times the larger is 55. What are the two numbers?

Call the smaller number S and the bigger one B. We can turn the words above into two equations with two unknowns each, S and B: $B = S + 10$ and $2S + 3B = 55$

Substitution works easily here: $2S + 3(S + 10) = 55 \rightarrow 2S + 3S + 30 = 55 \rightarrow S = 5,\ B = 15$

2. Austin has more money than Saul. If Austin gave Saul $20, they would have the same amount. But if Saul gave Austin $22, Austin would then have twice as much as Saul. How much does each one actually have?

Call the amount Austin has A and the amount Saul has S. The equations suggest above are: $A - 20 = S + 20$ and $2(S - 22) = A + 22$

Substitution works once we rearrange the first equation: $A - 20 = S + 20 \rightarrow A = S + 40$

$2(S - 22) = (S + 40) + 22 \rightarrow 2S - 44 = S + 62 \rightarrow S = 106, A = 146$

3. Claudia invested $30,000; part at 5%, and part at 8%. The total interest on the investment was $2,100. How much did she invest at each rate?

We have two unknowns, the 5% investment (A) and the 8% investment (B). We know their sum, $A + B = 30{,}000$, and the total interest collected on both investments: $A(0.05) + B(0.08) = 2{,}100$ If we rearrange the first equation, we can use substitution: $B = 30{,}000 - A \rightarrow A(0.05) + (30{,}000 - A)(0.08) = 2{,}100 \rightarrow$
$0.05A + 2{,}400 - 0.08A = 2{,}100 \rightarrow 300 = 0.03A \rightarrow A = 10{,}000$ and $B = 20{,}000$

4. The perimeter of Susan's rectangular garden is 60 feet. If the length of the garden is twice the width, what are the dimensions of the garden?

We have two unknowns, length (L) and width (W). We know that $2L + 2W = 60$ and that $L = 2W$. Using substitution, we can solve: $2(2W) + 2W = 60 \rightarrow W = 10, L = 20$

5. The largest of five consecutive even integers is 2 less than twice the smallest. Which of the following is the largest integer?

If we call the smallest integer X, the rest must be X+2, X+4, X+6, and X+8. Then the first sentence can be written as: $(X + 8) + 2 = 2X \rightarrow 10 = X$ so the largest integer is 18.

6. The amount of oil used by a ship traveling at a uniform speed varies jointly with the distance and the square of the speed. If the ship uses 200 barrels of oil in traveling 200 miles at 8 miles per hour, determine how many barrels of oil are used when the ship travels 360 miles at 12 miles per hour.

Call the oil used L, the speed S, and the distance D. Joint variance means that if one of these three variables increases, the other two also will. The first sentence: $L = k * S^2 * D$ with *k* being a constant. The second sentence tells us: $200 = k * 8^2 * 200 \rightarrow k = 0.0156$ Then to solve the last scenario: $L = 0.0156 * 12^2 * 360 \rightarrow L = 808.704$

7. Lidia inherited a sum of money. She split it into five equal chunks. She invested three parts of the money in a high interest bank account which added 10% to the value. She placed the rest of her inheritance plus $500 in the stock market but lost 20% on that money. If the two accounts end up with exactly the same amount of money in them, how much did she inherit?

Call X the amount of money inherited. We know that $\left(\frac{3}{5}X\right)$ was invested at 10%, while $\left(\frac{2}{5}X + 500\right)$ was invested at a loss of 20%. We also know the outcomes were equal:

$$\frac{3}{5}X * (1 + 0.1) = \left(\frac{2}{5}X + 500\right) * (1 - 0.2) \rightarrow 0.66X = 0.32X + 400 \rightarrow X = 1{,}176.471$$

d. Linear Equation with side constraint (rationing)

1. *Susan the gardener wants to optimize her flower mix in order to maximize honeybees. She can buy seeds for two different types of flowers, zinnias and nasturtiums. She estimates that each zinnia will attract 25 bees, while each nasturtium will attract 20 bees. Each plant requires a different amount of fertilizer; one zinnia requires 20 ounces of fertilizer to produce, while one nasturtium requires 12 ounces. Her supply of fertilizer is limited to at most 1800 ounces. She also needs water to grow the flowers, and about 15 flowers of either type can be grown per gallon. She may only use 8 gallons of water on the project. How many seeds of each type should she buy?*

We can translate Susan's situation into a system of equations: the number of honeybees her garden will attract, *B* is: $B = 25Z + 20N$. She must choose Z and N with the goal of maximizing B, but there are two things constraining her choice – fertilizer and water. We can express these constraints with two equations: $20Z + 12N = 1800$ & $\frac{1}{15}Z + \frac{1}{15}N = 8$. Since we need both these constraints to be followed at the same time, we can set these constraining equations equal to each other to find the combination(s) of Z and N where they are both satisfied:

$$20Z + 12N = 1800 \rightarrow -8N = -600$$

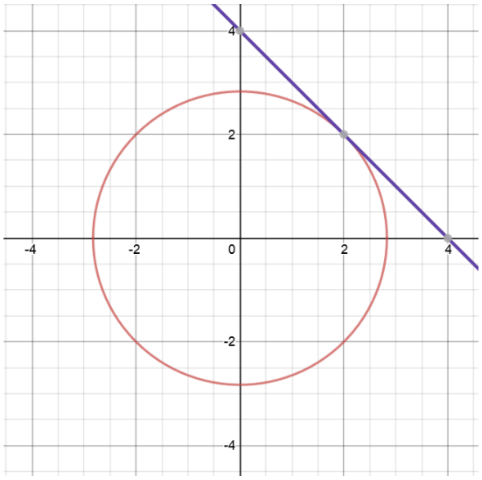
$$-20 * 15 * \left(\frac{1}{15}Z + \frac{1}{15}N\right) = (8) * 15 * -20 \rightarrow -20Z - 20N = -2400$$

By solving the system of equations, we get $N = 75$. After plugging 75 in for *N* in either of the constraint equations, we get $Z = 45$. So how many bees will Susan's garden attract?

$B = 25 * 45 + 20 * 75 = 2{,}625$. That's a lot of bees! But did she successfully maximize the number of bees her garden will attract? *Given the constraints present*, yes. There was only one combination of *Z* and *N* that satisfied the constraints, so that was the only option available to her.

e. Tangency

1. Below is the graph of the circle $x^2 + y^2 = 9$ *and the line* $y = -x + [A]$. *What is* A *equal to?*



We can rewrite the equation of the circle as: $y^2 = 9 - x^2 \to y = \sqrt{9 - x^2}$ and take its derivative: $\sqrt{9 - x^2} \to \frac{1}{2}(9 - x^2)^{-\frac{1}{2}} * -2x \to -\frac{2x}{2\sqrt{9-x^2}}$. Next, we set this phrase equal to the derivative of the line, -1, and solve:

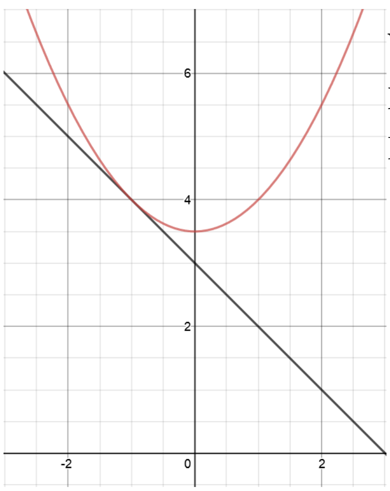
$$-\frac{x}{\sqrt{9 - x^2}} = -1 \to \sqrt{9 - x^2} = x \to 9 - x^2 = x^2 \to$$

$$9 = 2x^2 \to x = \sqrt{4.5} \approx 2.1213$$

Now plug this x value in to the original equations and solve for A: $4.5 + y^2 = 9 \to$

$$y = \sqrt{4.5} \to\to \sqrt{4.5} = -\sqrt{4.5} + A \to A = 2 * \sqrt{4.5}$$

2. Below is the graph of the parabola $\frac{1}{2}x^2 + [B]$ *and the line* $y = -x + 3$. *What is* B *equal to?*



We can use the same process as above, taking the derivative of the parabola: $\frac{1}{2}x^2 + B \to x$ and setting it equal to the derivative of the line: $x = -1$. Then plug -1 back in: $\frac{1}{2} + B = 1 + 3 \to B = 3.5$.