Lecture 2: Collateral Equilibrium, Endogenous Leverage, and the Credit Surface.

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Goal of Lecture 1

- Introduce Collateral General Equilibrium.
- Endogenous leverage. Liquidity Value. Liquidity Wedge.
- Concept of Credit Surface.
- Characterization of LTV in equilibrium. Collateral Value and Failure of Efficient Markets
- The effect of leverage on asset prices.
Road Map of Lecture 1

- Introduce Collateral General Equilibrium.

- Examples to illustrate main concepts.

- Theoretical results.
Outline

1. Introduction
2. Model
3. Examples
4. Theoretical Results
5. Summary
We present a two-period collateral general equilibrium model that will be the base for the results in lectures 2, 3 and 4.
Time and Assets

Time $t = 0, 1$.

$s \in S$ states of nature, $s = 0$ at time 0 and $s \in S_T$ terminal at time 1.

$L_s$ commodities in $s \in S$.

Let $p_s \in \mathbb{R}^{L_s}_{+}$ the vector of commodity prices for each state $s \in S$. 
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Agents

Each investor $h \in H$ is characterized by Bernoulli utilities, $u^h_s$, $s \in S$, a discount factor, $\beta_h$, and subjective probabilities, $\gamma^h_s$, $s \in S_T$.

$u^h_s : \mathbb{R}^{L_s}_+ \rightarrow \mathbb{R}$, are differentiable, concave, and weakly monotonic (more of every good in any state strictly improves utility).

Expected utility to agent $h$ is

$$U^h = u^h_0(x_0) + \beta_h \sum_{s \in S_T} \gamma^h_s u^h_s(x_s).$$

Investor $h$’s endowment of the commodities is denoted by $e^h_s \in \mathbb{R}^{L_s}_+$ in each state $s \in S$. 
Production: Intra-Period

For each $s \in S$ and $h \in H$, let $Z^h_s \subset \mathbb{R}^{L_s}$ denote the set of feasible intra-period production for agent $h$.

Commodities can enter as inputs and outputs of the intra-period production process; inputs appear as negative components $z_l < 0$ of $z \in Z^h_s$, and outputs as positive components $z_l > 0$ of $z \in Z^h_s$.

We assume that $Z^h_s$ is convex, compact and that $0 \in Z^h_s$. 
Example 3: Production: Intra-Period
For each $h \in H$, $s \in S_T$, let $F^h_s : \mathbb{R}^L_0 \rightarrow \mathbb{R}^L_s$ be a linear inter-period production function connecting a vector of commodities $x_0$ at state $s = 0$ with the vector of commodities $F^h_s(x_0)$ it becomes in each state $s \in S_T$. 
Production enables our model to include many different kinds of commodities. Commodities could either be:

- perishable consumption goods (like food),
- durable consumption goods (like houses),
- they could represent assets (like Lucas trees) that pay dividends.
Arrow Debreu Equilibrium

\[(p, (z^h, x^h)_{h \in H}) \in \mathbb{R}^S \otimes L_s \times (\mathbb{R}^S \otimes L_s \times \mathbb{R}^S \otimes L_s)^H\text{ such that}
\]

\[\sum_{h \in H} (x^h_0 - e^h_0 - z^h_0) = 0.
\]

\[\sum_{h \in H} (x^h_s - e^h_s - z^h_s - F^h_s(x^h_0)) = 0, \forall s \in S_T.
\]

\[(z^h, x^h) \in B^h(p, \pi) = \{(z, x) \in \mathbb{R}^S \otimes L_s \times \mathbb{R}^S \otimes L_s : p_0 \cdot (x_0 - e^h_0 - z_0) + \sum_{s \in S_T} p_s \cdot (x_s - e^h_s - z_s - F^h_s(x_0)) = 0, z_s \in Z^h_s, \forall s \in S\}, \forall h (z, x) \in B^h(p, \pi) \Rightarrow U^h(x) \leq U^h(x^h), \forall h. \text{ One}
\]

budget constraint, present value prices
Financial Contracts and Collateral

The heart of our analysis involves contracts and collateral.

In Arrow Debreu the question of why agents honor their promises is ignored.

We explicitly incorporate in our model repayment enforceability problems.

Collateral is the only enforcement mechanism: agents cannot be coerced into honoring their promises except by seizing collateral agreed upon by contract in advance.
Financial Contracts and Collateral

But exclude cash flow problems.

But there is no doubt what the collateral will pay, conditional on the future state of nature.
A financial contract is an ordered pair

\[ j = ((j_s)_{s \in S_T}, c_j) \]

**Promise:** \( j_s \in R^L_s \) commodities in each final state \( s \in S_T \).

This allows for non contingent promises of different sizes, as well as contingent promises.

**Collateral:** backed by collateral \( c_j \in R^L_0 \).
Price of contract $j \in J$ is $\pi_j$. Let $\theta^h_j \geq 0 (\varphi^h_j \geq 0)$ be the number of contracts $j$ bought (sold) at time 0 by agent $h$.

An investor can borrow $\varphi^h_j \pi_j$ today by selling $\varphi^h_j \geq 0$ and lend $\theta^h_j \pi_j$ by buying $\theta^h_j \geq 0$. 
We wish to exclude cash flow problems (like adverse selection or moral hazard). For that:

- we restrict the sale of each contract $j$ to a set $H(j) \subset H$ of traders with the same durability functions, $F^h(c_j) = F^{h'}(c_j)$ if $h, h' \in H(j)$.

- we suppose that if $c_j$ is the collateral for some contract $j$, then there is a “large” contract $j'$ with $c_{j'} = c_j$ and $H(j') = H(j)$ and $j'_s \geq F^H_s(c_j)$ for all $s \in S_T$. 
Financial Contract Delivery

We assume that the maximum borrowers can lose is their collateral if they do not honor their promise. Under this no-recourse hypothesis, everybody would make the same choice.

Actual delivery of contract \( j \) in states \( s \in S_T \) is:

\[
\delta_s(j) = \min \{ p_s \cdot j_s, p_s \cdot F_s^{H(j)}(c_j) \}
\]

The value of the collateral in each future state does not depend on the size of the promise, or on what other choices the seller \( h \in H(j) \) makes, or on who owns the asset at the very end.
Budget Set

\[ B^h(p, \pi) = \]
\[ \{(z, x, \theta, \varphi) \in R^{S \otimes L_s} \times R^{S \otimes L_s}_+ \times R^J_+ \times R^J_+: \]
\[ p_0 \cdot (x_0 - e^h_0 - z_0) + \sum_{j \in J} (\theta_j - \varphi_j) \pi_j \leq 0 \]
\[ p_s \cdot (x_s - e^h_s - z_s - F^h_s(x_0)) \leq \]
\[ \leq \sum_{j \in J} (\theta_j - \varphi_j) \min\{p_s \cdot j_s, p_s \cdot F^{H(j)}_s(c_j)\}, \forall s \in S_T \]
\[ z_s \in Z^h_s, \forall s \in S \]
\[ \varphi_j > 0 \text{ only if } h \in H(j) \]
\[ \sum_{j \in J} \varphi_j c_j \leq x_0 \} \]
Collateral Equilibrium

\(((p, \pi), (z^h, x^h, \theta^h, \varphi^h), h \in H) \in R^S \otimes L_s \times R^J \times (R^S \otimes L_s \times R^S \otimes L_s \times R^J \times R^J)^H\) such that

\[ \sum_{h \in H} (x^h_0 - e^h_0 - z^h_0) = 0. \]

\[ \sum_{h \in H} (x^h_s - e^h_s - z^h_s - F^h_s(x^h_0)) = 0, \forall s \in S_T. \]

\[ \sum_{h \in H} (\theta^h - \varphi^h) = 0, \]

\((z^h, x^h, \theta^h, \varphi^h) \in B^h(p, \pi), \forall h\)

\((z, x, \theta, \varphi) \in B^h(p, \pi) \Rightarrow U^h(x) \leq U^h(x^h), \forall h.\)
Financial Innovation

Changes in the set $J$ of financial contracts.

The most primitive situation occurs when there are no financial contracts and $J = \emptyset$. In that case nobody can borrow.
Summary

Three examples to illustrate main theoretical results we develop later:

- Example 1: No uncertainty/Utility from collateral.


Each of the example will illustrate several main lessons:

-Lesson 1: Endogenous Leverage and Credit Surface.


-Lesson 3: Characterization of equilibrium LTV.

-Lesson 4: Collateral Values and Failure of Efficient Markets Pricing.

-Lesson 5: Effect of leverage on asset prices. Comparison with Arrow-Debreu.
Example 1: This example is characterized by:

- No uncertainty.

- Utility from collateral.
Example 1: Time and Commodities

Food and Housing in periods 0 and 1. Food is completely perishable and Housing is completely durable.

\[ L_0 = \{x_{0F}, x_{0H}\}, L_1 = \{x_{1F}, x_{1H}\} \]

\[ F_1(x_{0F}, x_{0H}) = (0, x_{0H}) \]

No intra-period production.
Example 1: Utilities and Endowments

\[ U^A(x_0F, x_0H, x_1F, x_1H) = x_0F + x_0H + x_1F + x_1H \]

\[ U^B(x_0F, x_0H, x_1F, x_1H) = 9x_0F - 2x_0^2 + 15x_0H + x_1F + 15x_1H \]

\( (e^A_0F, e^A_0H, e^A_1F, e^A_1H) = (20, 1, 20, 0) \)

\( (e^B_0F, e^B_0H, e^B_1F, e^B_1H) = (4, 0, 50, 0) \)
Example 1: Arrow Debreu Equilibrium

\[(p_{0F}, p_{0H}, p_{1F}, p_{1H}) = (1, 30, 1, 15)\]

\[(x^A_{0F}, x^A_{0H}, x^A_{1F}, x^A_{1H}) \simeq (22, 0, 48, 0)\]

\[(x^B_{0F}, x^B_{0H}, x^B_{1F}, x^B_{1H}) \simeq (2, 1, 22, 1)\]
Example 1: No-Contract Equilibrium

\[(p_{0F}, p_{0H}, p_{1F}, p_{1H}) = (1, 16, 1, 15)\]

\[(x_{0F}^A, x_{0H}^A, x_{1F}^A, x_{1H}^A) \approx (22.2, \frac{6}{7}, 33, 0)\]

\[(x_{0F}^B, x_{0H}^B, x_{1F}^B, x_{1H}^B) \approx (1.8, \frac{1}{7}, 37, 1)\]
Example 1: Collateralized Financial Contracts

\[( (j_{1F}, j_{1H}), (c_{j0F}, c_{j0H}))_{j \in J} = ( (j, 0), (0, 1))_{j \in J} \]

Assume \( j = 15 \in J \)

Every contract \( j \) is non-contingent promise of \( j \) units of food.
Example 1: Collateral Equilibrium

\[(p_{0F}, p_{0H}, p_{1F}, p_{1H}), (\pi_j)_{j \in J} = (1, 18, 1, 15), (\min(j, 15))_{j \in J}\]

\[(x^A_{0F}, x^A_{0H}, x^A_{1F}, x^A_{1H}), (\theta^A_j, \phi^A_j)_{j \in J} \simeq (23, 0, 35, 0), ((1, 0)_{j=15}, (0, 0)_{j \neq 15})\]

\[(x^B_{0F}, x^B_{0H}, x^B_{1F}, x^B_{1H}), (\theta^B_j, \phi^B_j)_{j \in J} \simeq (1, 1, 35, 1), ((0, 1)_{j=15}, (0, 0)_{j \neq 15})\]
Example 1: Non-Contingent Promises

Real Gross Interest Rate

\[ 1 + r_j = \frac{j}{\pi_j} \]

For \( j \leq 15 \), \( 1 + r_j = 1 \), the riskless rate of interest.

Loan-to-Value

\[ LTV_j = \frac{j}{p_0H} \]
G (97) introduced the concept of **Credit Surface**: the equilibrium relationship between $LTV_j$ and $1 + r_j$.

Borrowers can choose any contract on the Credit Surface provided they put up the corresponding required collateral. In this sense leverage is endogenous.

In the Arrow-Debreu budget set, borrowers face in equilibrium a flat Credit surface.
Lesson 1: Endogenous Leverage and Credit Surface

But because collateral is scarce, only few contracts will be actively traded in equilibrium. So agents will choose only a few points on the credit surface.

But which contracts are traded in equilibrium?
Lesson 1: Endogenous Leverage and Credit Surface

For LTV less than 50%, the riskless interest prevails. At that point the leverage surface rises vertically. It is impossible to obtain LTV greater than 50%. Thus the leverage in the economy is endogenously chosen by the borrowers, given the menu of leverage-interest rate pairs offered by the market, as described by the credit surface.
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Lesson 2: Constrained Borrowing and Liquidity Value

For \( j \leq 15 \), the net value to agent \( B \) (measured in units of food at 0, which is the 0 numeraire) of taking out the loan, or liquidity value, is

\[
LV_j^B = \pi_j - \frac{jp_{1F}1}{9 - 4x_{0F}^B} = j - \frac{j}{5}
\]

For \( j = 15 \),

\[
LV_{15}^h = 15 - \frac{15}{5} = 12
\]

So there is a big surplus to borrowing. \( B \) feels constrained in borrowing by need to put up collateral. To put it another way, he would be willing to pay gross interest of \( 1 + r = \frac{9 - 4x_{0F}^B}{1} = 500\% \), even under penalty of death for non-delivery, yet the market rate of interest is 0\% if he puts up enough collateral to guarantee delivery.
Notice that \( LV_j^B \) is strictly increasing in \( j \) all the way up to \( j = 15 \). Thus \( j = j^* = 15 \) is uniquely the best choice of leverage for \( j \leq 15 \).

For \( j > 15 \), \( B \) will default and only deliver 15, so these contracts are equivalent to \( j^* = 15 \). Leverage is effectively uniquely chosen at 50% in equilibrium.
Lesson 4: Collateral Values

If borrowing leads to such a surplus (liquidity value) for $B$, why doesn’t he buy more collateral and borrow more?

Because the price of the collateral exceeds his marginal utility! We define this excess as the collateral value of the house to $B$.

\[
CV_{0H}^B = p_{0H} - \frac{15 + 15}{9 - 4x_{0F}^B} = 18 - \frac{30}{5} = 12
\]

Notice that the collateral value of the house equals the maximum liquidity value that can be obtained by using the house as collateral.
Lesson 4: Collateral Values and Inefficient Pricing

Collateral goods are priced above their payoff values because they can also be used to borrow money.

This general property is paradoxical at first blush, since it would seem that the limitation on borrowing to buy the collateral would hold down its price. Once we introduce production we will find that collateral restrictions can lead to bubbles and overinvestment, something that was overlooked in some of the early collateral literature.
Lesson 5: Effect of leverage on Asset Prices

This example shows that leverage price is above the No-contract level. Increasing leverage from 0% to 50% raises asset prices.

But the leverage price is below the Arrow Debreu level.
Example 2: Fostel-Geanakoplos (2014)

Example 2: This example is characterized by:

- Uncertainty.

- Financial Asset.
Example 2: States and Commodities

\[ S = \{0, U, D\}. \]

There is a single perishable consumption good \( c \) in each state. Numeraire.

There is one financial asset \( Y \) which pays dividends only in the final period: \( d_U = 1 \) and \( d_D = .2 \).

\[ L_0 = \{ c_0, Y \}, \quad L_U = \{ c_U \}, \quad L_D = \{ c_D \} \]

\[ F_s( c_0, Y ) = d_s Y \]
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Example 2: Financial Assets

Y is a Financial asset:

-it gives no direct utility to investors, and

-it pays the same dividends no matter who owns it.
Example 2: Agents

In this example we will two agents, an optimist and pessimist $H = \{O, P\}$:

\[
U^O = \log(c_0) + .9\log(c_U) + .1\log(c_D)
\]
\[
U^P = \log(c_0) + .4\log(c_U) + .6\log(c_D)
\]

\[
(e^O_{c_0}, e^O_Y, e^O_{c_U}, e^O_{c_D}) = (8.5, 1, 10, 8.5)
\]
\[
(e^P_{c_0}, e^P_Y, e^P_{c_U}, e^P_{c_D}) = (100, 1, 100, 100)
\]
Example 2: Arrow Debreu Equilibrium

\[(p_{0c}, p_{0Y}, p_{Uc}, p_{Dc}) = (1, 0.539, 0.427, 0.556)\]

\[(x^{O}_{0c}, x^{O}_{0Y}, x^{O}_{Uc}, x^{O}_{Dc}) \simeq (9.022, \?, 18.98, 1.621)\]

\[(x^{P}_{0c}, x^{P}_{0Y}, x^{P}_{Uc}, x^{P}_{Dc}) \simeq (99.47, \?, 93.01, 107.2)\]
Example 2: No-Contracts Equilibrium

\[(p_{0c}, p_{0y}, p_{uc}, p_{dc}) = (1, .609, 1, 1)\]

\[(x_{0c}^O, x_{0y}^O, x_{uc}^O, x_{dc}^O) \simeq (7.89, 2, 12, 8.9)\]

\[(x_{0c}^P, x_{0y}^P, x_{uc}^P, x_{dc}^P) \simeq (100.61, 0, 100, 100)\]

No contract price of asset is above its Arrow Debreu price. Could have changed utilities and got it below Arrow Debreu price.
Example 2: Debt Contracts

We consider only debt financial contracts of the form

\[ j = ((j, j), (0, 1)) \]

where the promise of \( j \) units of the consumption good is non-contingent, and it uses one unit of the asset as collateral.

The set of contracts is \( J = \{ \frac{1}{100}, \frac{2}{100}, \ldots, \frac{10,000}{100} \} \).
Collateral Equilibrium

Table 1: Equilibrium Static Economy.

<table>
<thead>
<tr>
<th>States</th>
<th>$s = 0$</th>
<th>$s = U$</th>
<th>$s = D$</th>
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<td>Prices and Leverage</td>
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<tr>
<td>$p_Y$</td>
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<tr>
<td>$j^*$</td>
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<tr>
<td>$r_j^*$</td>
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<tr>
<td>$LTV$</td>
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<th>Asset $Y$</th>
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<td>Pessimists</td>
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$$\pi_j = (0.4 \min(j, 1) + 0.6 \min(j, 0.2)) / (100.4/100.3), \ 1 + r_j = j / \pi_j$$
Lesson 1: Credit Surface and Endogenous Leverage

The example shows that only one contract is traded, the \( \max \ min \) contract \( j^* \) satisfying \( j^* = d_D = .2 \). The point A in the Credit Surface.

This is the maximum amount optimists can promise while guaranteeing they will not default in the future.
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Lesson 2: Constrained Borrowing

There is no default, so the Optimists are paying the riskless interest rate of 1%.

To get an extra penny of loan they would be willing to pay a rate of 37%, even if they were obliged, on penalty of death, to pay back the whole loan. But no lender will give them that deal, because there is no penalty.
Lesson 2: Constrained Borrowing

One might have thought that optimists would be so eager to borrow money that they would want to promise more than .2 per asset, happily paying a default premium in order to get more money at time 0.

According to the equilibrium, this is not the case. The threat of default is so strong, it causes the lenders to constrain the borrowers. More precisely, the offered interest rate rises too fast as a function of $j$ for the borrowers to be willing to take on more debt.
Lesson 3: LTV

When only one contract is traded in equilibrium, this uniquely pins down the leverage in the economy.

Leverage can then be characterized by

$$LTV = \frac{\pi_j^*}{p} = \frac{dD/p}{1+r_j^*} = \frac{0.199}{0.708} = 28\%.$$  

Thus $LTV$ is given by the ratio between the worst case rate of return on the asset and the riskless rate of interest.
Lesson 4: Effect of Leverage on Asset Prices.

The collateral equilibrium asset price $p = .708$ is much higher than its price in Arrow-Debreu equilibrium (.539). Thus leverage can dramatically raise asset prices above their efficient levels (contrary to Example 1 above).

What would happen if we dropped leverage, but still prohibited short selling? Get ($p = .609$), much lower than leverage price. As leverage (exogenously) increases from 0 to maxmin level, the asset price rises. One reason is that when less borrowing is allowed, period 0 consumption by the optimists would need to be lower if they continued to buy all the assets. But there is also another reason.
The payoff value of the asset for the optimist is

\[ PV^O = \sum_{s=U,D} \delta^O \gamma_s^O d_s u^O(c_s^O)/dc \]

Yet the price is \( p = .708 > PV^O = .655 \).

The reason the optimists are willing to pay more for the asset than its payoff value to them is that holding more of the asset enables them to borrow more money. This is what Fostel and Geanakoplos (2008) called Collateral Value.

This Collateral Value can create bubbles.
Example 3: This example is characterized by:

-Uncertainty.

-Financial Asset.

-This is an example of a special class of models we call C-models.
Example 3: States and Commodities in C-models

$S = \{0, U, D\}$.

There is a single perishable consumption good $c$ in states $U$ and $D$. Agents have no endowments of these.

There are two financial assets $X$, $Y$ which pay dividends only in the final period. $X$ is riskless and pays 1 in each state, $Y$ pays $d_U = 1$ and $d_D = 0.2$.

$L_0 = \{X, Y\}, L_U = \{c_U\}, L_D = \{c_D\}$

$F_s(X, Y) = 1 + d_s Y$
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\[ F_s(X, Y) = 1 + d_s Y \]
Example 3: Agents in C-Models

Continuum of investors $h \in H = [0, 1]$.

Risk neutral. No discounting. Consumption only at the end.

Expected utility to agent $h$ is

$$U^h(c_U, c_D) = \gamma^h_U c_U + \gamma^h_D c_D$$

The only source of heterogeneity is in subjective probabilities, $\gamma^h_U$ is continuous and strictly increasing in $h$. The higher is $h$, the more optimistic the investor.
Example 3: A Particular C-model

Each agent $h \in H$ has an endowment

$$(e^h_X, e^h_Y, e^h_{Uc}, e^h_{Dc}) = (1, 1, 0, 0)$$

of one unit of assets $X, Y$ at time 0.

No intra-period production.
Example 3: Arrow Debreu Equilibrium

\[(p_0X, p_0Y, p_{UC}, p_{DC}) = (1, .56, .45, .55)\]

\[(x_{0X}^h, x_{0Y}^h, x_{UC}^h, x_{DC}^h) \approx (1, 1, 3.6, 0), h \geq .45\]

\[(x_{0X}^h, x_{0Y}^h, x_{UC}^h, x_{DC}^h) \approx (1, 1, 0, 2.7), h < .45\]
Example 3: No Contracts Equilibrium

\[(p_{0X}, p_{0Y}, p_{UC}, p_{DC}) = (1, .67, .1, 1)\]

\[(x_0^h, x_0^h, x_0^h, x_0^h) \simeq (0, 2.5, 2.5, .5), h \geq .6\]

\[(x_0^h, x_0^h, x_0^h, x_0^h) \simeq (1.7, 0, 1.7, 1.7), h < .6\]

No contract price of Y higher than Arrow debreu price. With different numbers could be lower. But leverage price always higher than both in C-models.
We consider only debt financial contracts of the form

\[ j = ((j, j), (0, 1)) \]

where the promise of \( j \) units of the consumption good is non-contingent, and it uses one unit of the asset as collateral.

The set of contracts is \( J = \{ \frac{1}{100}, \frac{2}{100}, \ldots, \frac{10,000}{100} \} \).
Example 3: Collateral Equilibrium Regime

- Optimists leverage Y using max min bond. (They buy Arrow U)
- Pessimists lenders buy max min bond

$h=1$

$h=0$

Marginal buyer
Example 3: Collateral Equilibrium

Table 1: Equilibrium Static Economy.

<table>
<thead>
<tr>
<th>States</th>
<th>$s = 0$</th>
<th>$s = U$</th>
<th>$s = D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prices and Leverage</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p$</td>
<td>0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h$</td>
<td>0.69</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$j^*$</td>
<td>0.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi j^*$</td>
<td>0.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r_j^*$</td>
<td>0%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$LTV$</td>
<td>0.26</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\pi_j = 0.69 \min(j, 1) + 0.31 \min(j, 0.2)
\]

\[
1 + r_j = j / \pi_j
\]
Lesson 1: Credit Surface and Absence of Default

The example shows that only one contract is traded, the \( \max \min \) contract \( j^* \) satisfying \( j^* = d_D = .2 \). The point A in the Credit Surface.

This is the maximum amount optimists can promise while guaranteeing they will not default in the future.
Lesson 1: Credit Surface and Absense of Default

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This is the maximum amount optimists can promise while guaranteeing they will not default in the future.
Lesson 2: Constrained Borrowing

One might have thought that optimists would be so eager to borrow money that they would want to promise more than .2 per asset, happily paying a default premium in order to get more money at time 0.

According to the equilibrium, this is not the case. The threat of default is so strong, it causes the lenders to constrain the borrowers. More precisely, the offered interest rate rises too fast as a function of $j$ for the borrowers to be willing to take on more debt.
Lesson 3: LTV

When only one contract is traded in equilibrium, this uniquely pins down the leverage in the economy.

Leverage can then be characterized by

$$LTV = \frac{\pi_j^*}{p} = \frac{d_D/p}{1+r_j^*} = \frac{0.2}{0.75} = 26\%.$$  

Thus $LTV$ is given by the ratio between the worst case rate of return on the asset and the riskless rate of interest.
Lesson 4: The effect of Leverage on Asset Prices

The collateral equilibrium asset price $p = 0.75$ is much higher than its price in Arrow-Debreu equilibrium (0.56). Thus leverage can dramatically raise asset prices above their efficient levels.

Now we see a second benefit using an asset as collateral can bring. By leveraging $Y$, agents can synthetically buy an Up Arrow security, even though no Arrow securities are directly marketed. If they are interested in gambling on the Up state, then they can do so via $Y$. 
Leverage raises asset prices due to the presence of Collateral Values as we saw before.

For the marginal buyer the asset price is exactly equal its payoff value, and hence the collateral value is zero. But for all the optimists, the collateral value is positive.
Over-valuation due to leverage

Proposition: Over-Valuation and Investment compared to Arrow Debreu in C-Models.

In C-Models \( p_Y^L \geq p_Y^A \).

Over-valuation due to leverage

Proposition: Over-Valuation and Investment compared to Arrow Debreu in C*-Models.

In C*-Models, where we allow for arbitrary preferences and a finite number of agents, under constant return to scale in which Y is produced from X,, \( z^L_Y \geq z^A_Y \).

Proof: Fostel-Geanakoplos (2014) will be given in lecture 4.
Outline

1. Introduction
2. Model
3. Examples
4. Theoretical Results
5. Summary
Main theoretical results:

- Binomial No-Default Theorem
- Binomial Leverage Theorem
- Collateral Values and Asset Pricing.
The No-Default Theorem

Consider the situation in which there are only two terminal states, $S = \{0, U, D\}$.

Suppose there is one consumption good in each state and a financial asset. $L_0 = \{c_0, Y\}, L_U = \{c_U\}, L_D = \{c_D\}$

Financial Asset $Y$ pays $d_U$ units of the consumption good in state $s = U$ and $0 < d_D < d_U$ in state $s = D$.

We consider only debt contracts
Binomial No-Default Theorem:

Suppose that $S = \{0, U, D\}$, that $Y$ is a financial asset, and that the max min debt contract $j^* = d_D \in J$. Then given any equilibrium $((p, \pi), (c^h, y^h, \theta^h, \phi^h)_{h \in H})$, we can construct another equilibrium $((p, \pi), (c^h, \bar{y}^h, \bar{\theta}^h, \bar{\phi}^h)_{h \in H})$ with the same asset and contract prices and the same consumptions, in which $j^*$ is the only debt contract traded, $\bar{\theta}^h = \bar{\phi}_j^h = 0$ if $j \neq j^*$. Hence equilibrium default can be taken to be zero.

Proof: Fostel-Geanakoplos (forth).
General Case

Arbitrary contracts.

Multiple assets.

Production and degrees of durability.

Multiple consumption goods.

Multiple periods

Multiple state of nature (as long as each contract take at most two values).
Absence of Default

The theorem provides a hard limit on borrowing.

It shows that there must be a robust class of economies in which agents would like to borrow more at going riskless interest rates but cannot, even when their future endowments are more than enough to cover their debts.
Absence of Default

The hard limit on borrowing is caused by the specter of default, despite the absence of default in equilibrium.

The hard limit is endogenous.

Binomial economies and their Brownian motion limit are special cases. But they are extensively used in finance. They are the simplest economies in which one can begin to see the effect of uncertainty on credit markets.
Under the assumptions of the Binomial No-Default theorem, equilibrium leverage can always be taken to be

\[ LTV_s^k = \frac{d_{sk}^k}{p_{sk}} = \frac{\frac{\text{worst case rate of return}}{\text{riskless rate of interest}}}{1 + r_s}. \]

Proof: See Fostel-Geanakoplos (forth).
Our second theorem provides a simple formula for leverage.

Equilibrium $LTV$ is given by

$$LTV = \frac{\text{worst case rate of return}}{\text{riskless rate of interest}}$$
Endogenous Leverage

Though simple and easy to calculate, our formula provides interesting insights:

-it explains which assets are easier to leverage (the ones with low tail risk).

-it explains why changes in the bad tail can have such a big effect on equilibrium even if they hardly change expected payoffs: they change leverage.
Endogenous Leverage

Collateralized loans always fall into two categories:

When an agent uses all his assets as collateral.

- In this case, debt is explained by traditional models of demand for loans without collateral.

When an agent does not use all his assets as collateral.

- Debt is determined by the maximum debt capacity of the assets, independent of agent’s preferences.

- Theorem says we can always use the same LTV. (If maxmin contract gives 80% LTV and you only want to borrow $40 on $100 asset, can instead borrow $40 using only half the asset as collateral, Still safe loan and with 80% LTV.)
The distinction between plentiful and scarce capital all supporting loans at the same $LTV$ suggests that it is useful to keep track of a second kind of leverage that we call *diluted* leverage:

$$DLTV^k_s = \frac{\sum_h \sum_{j \in J_s^k} \max(0, \varphi_{j}^h) \pi_j}{\sum_h y_{sk}^h p_{sk}} \leq LTV^k_s.$$  

Similarly one can define *diluted investor* leverage

$$DLTV^h_s = \frac{\sum_k \sum_{j \in J_s^k} \max(0, \varphi_{j}^h) \pi_j}{\sum_k y_{sk}^h p_{sk}} \leq LTV^h_s.$$
The *Payoff Value* of contract \( j \) to agent \( h \) at state \( s \) is
\[
PV_{sj}^h = \sum_{\sigma \in \{U, D\}} \delta_h \gamma_{s\sigma} \min\{b(j), p_{s\sigma k(j)} + d_{s\sigma}^k(j)\} \frac{du^h(c_{s\sigma}^h)}{dc} \frac{du^h(c_s^h)}{dc}
\]

The *Liquidity Value* \( LV_{sj}^h \) associated to contract \( j \) to agent \( h \) at \( s \) is
\[
LV_{sj}^h = \pi_j - PV_{sj}^h.
\]

The liquidity value represents the surplus a borrower can gain by borrowing money today selling a contract \( j \) backed by collateral \( k \).
The Liquidity Wedge $\omega_{sj}^h$ associated to contract $j$ for agent $h$ at state $s$ is

$$1 + \omega_{sj}^h = \frac{\pi_j}{PV_{sj}^h}$$

In the case that contract $j$ fully delivers, $\omega_{sj}^h$ defines the extra interest a potential borrower would be willing to pay above the going riskless interest rate if he could borrow an additional penny and was committed (under penalty of death) to fully deliver.

This extra interest is called the liquidity wedge; it gives a measure of how tight the contract $j$ credit market is.
Leverage and Asset Pricing II: Liquidity Wedge and Discount Factor

Discount Theorem:

Define the risk adjusted probabilities for agent $h$ in state $s$ by

$$
\mu^h_{sU} = \frac{\gamma^h_{sU} du^h(c^h_{sU}) / dc}{\gamma^h_{sU} du^h(c^h_{sU}) / dc + \gamma^h_{sD} du^h(c^h_{sD}) / dc},
$$

$$
\mu^h_{sD} = \frac{\gamma^h_{sD} du^h(c^h_{sD}) / dc}{\gamma^h_{sU} du^h(c^h_{sU}) / dc + \gamma^h_{sD} du^h(c^h_{sD}) / dc} = 1 - \mu^h_{sU}.
$$

If agent $h$ is taking out a riskless loan in state $s$, then his payoff value in state $s$ for a tiny share of arbitrary cash flows consisting of consumption goods $x = (x_{sU}, x_{sD})$ is given by

$$
PV^h(x) = \frac{\mu^h_{sU} x_{sU} + \mu^h_{sD} x_{sD}}{(1 + r_s)(1 + \omega^h_s)}.
$$

The Payoff Value of asset $k$ to agent $h$ at state $s$ is

$$PV_{sk}^h \equiv \sum_{\sigma \in \{U, D\}} \delta_h \gamma_{s\sigma}^h (p_{s\sigma k} + d_{s\sigma}^k) \frac{du^h(c_{s\sigma}^h)}{dc} = \frac{du^h(c_s^h)}{dc}$$

The Collateral Value of asset $k$ in state $s$ to agent $i$ is

$$CV_{sk}^h \equiv p_{sk} - PV_{sk}^h$$

The collateral value stems from the added benefit of enabling borrowing that some durable assets provide.

Collateral values distort pricing and typically destroy the efficient markets hypothesis.
Collateral Value = Liquidity Value Theorem:

Suppose that $y_{sk}^h > 0$ and $\varphi_j^h > 0$ for some agent $h$ and some $j \in J^k_s$. Then, in equilibrium the following holds,

$$LV_{sj}^h = CV_{sk}^h$$

The liquidity value associated to any contract $j$ that is actually issued using asset $k$ as collateral equals the collateral value of the asset.

Liquidity and Endogenous Contracts

Since one collateral cannot back many competing loans, the borrower will always select the loan that gives the highest liquidity value among all loans with the same collateral.

The following holds

\[ LV_{sj}^{h} = PV_{sj}^{h} \omega_{sj}^{h} \]

All loans that deliver for sure will have the same liquidity wedge. If this wedge is positive, the borrower will naturally choose the biggest loan, since that has the highest payoff value and therefore the highest liquidity value.

Formula also shows why, holding liquidity wedge constant, the collateral value and thus the price rises when an asset can be leveraged more.
Summary

Introduced Collateral General Equilibrium.

We show how leverage is endogenous and we introduce the concept of Credit Surface.

We provided a complete characterization of LTV in binomial economies.

We showed the presence of non-negative collateral values.

Though collateral values are never negative, the leverage price can be above or below the efficient Arrow Debreu levels.
Proof of Binomial No-Default Theorem

The proof is organized in 3 steps:

1) Payoff Cone Lemma.

2) State Pricing Lemma.

3) Construction of new equilibrium.
Payoff Cone Lemma:

The portfolio of assets and contracts that any agent $h$ holds in equilibrium delivers a payoff vector $(w^h_U, w^h_D)$ which lies in the cone positively spanned by $(d_U - j^*, 0)$, the Arrow $U$ security, and $(j^*, j^*)$, the max min bond.
1. Payoff Cone Lemma

Space of Feasible Portfolio Payoffs

Space of debt contracts

Max min debt contract $j^*$

Arrow U
Security
($d_U - j^*, 0$)
1. Payoff Cone Lemma

Family of debt contracts

Asset Y Payoff

\[ d_U \]

\[ d_D \]

\[ 45^\circ \]
1. Payoff Cone Lemma

- Family of debt contracts
- Asset Y Payoff
- Max min contract $j^*$

Diagram:
- $U$ axis
- $D$ axis
- $d_U$ and $d_D$ points
- 45° angle
1. Payoff Cone Lemma

- Family of debt contracts
- Asset Y Payoff
- Delivery of Max min contract $j^*$

Diagram:
- Axes: $U$ (vertical) and $D$ (horizontal)
- Line: $d_U$ and $d_D$
1. Payoff Cone Lemma

- **Family of debt contracts**
- **Asset Y Payoff**
- **Net payoff of buying asset on margin through contract j**: Arrow U security ($d_U - j, 0$)
- **Max min contract $j^*$**

![Graph](image-url)
1. Payoff Cone Lemma

- Family of debt contracts
- Asset Y: Payoff

Debt contract promise $j < j^*$
1. Payoff Cone Lemma

- **Asset Y Payoff**: The value of asset Y.
- **Net payoff of buying asset on margin through debt contract $j < j^*$**: The difference in value between buying the asset and the cost of the debt contract.
- **Debt contract $j < j^*$ delivery**: The delivery of the debt contract at a specific time.

The diagram illustrates the relationship between the asset value and the net payoff, showing how the payoff cone is formed by the family of debt contracts. The 45° line indicates the equal value of the asset and the net payoff.
1. Payoff Cone Lemma

- **Family of debt contracts**
- **Asset Y Payoff**
- **Net payoff of buying asset on margin through debt contract \( j < j^* \)**
- **Debt contract \( j < j^* \) delivery**
1. Payoff Cone Lemma

- **Asset Y Payoff**
- **Family of debt contracts**
- **Debt contract promise \( j > j^* \)**

Dotted line: \( d_{U} \)

Graph showing the relationship between asset Y payoff and debt contract promise.
1. Payoff Cone Lemma

- **Asset Y Payoff**
- **Family of debt contracts**
- **Debt contract promise \( j > j^* \)**
- **Debt contract \( j > j^* \) delivery**

\[
\begin{align*}
U & \quad d_U \\
D & \quad d_D
\end{align*}
\]
1. Payoff Cone Lemma

Net payoff of buying asset on margin through contract $j > j^*$: Arrow $U$ security $(d_U - j, 0)$

Debt contract promise $j > j^*$

Debt contract $j$ delivery

45°
The Arrow $U$ security and the max min debt contract $j^*$ positively span all the feasible portfolio payoff space.
1. Payoff Cone Lemma

Any portfolio payoff \((w_U, w_D)\) is the sum of payoffs from individual holdings.

The possible holdings include:

- debt contracts \(j > j^*, j = j^*, j < j^*\),
- the asset,
- the asset bought on margin by selling some debt contract \(j\).

Buying the asset on margin using any debt contract with \(d_U > j \geq j^*\) is effectively a way of buying the \(U\) Arrow payoff \((d_U - j, 0)\).
The Arrow \(U\) security and the max min debt contract positively span all the feasible payoff space.
Notice that the assumption of two states is crucial to prove the Payoff Cone Lemma.

If there were three states, it might be impossible for a portfolio holder to reproduce his original net payoffs from a portfolio in which he can only hold the asset and buy or issue the max min debt.
State Pricing Lemma:

There exist $a > 0$ and $b > 0$ such that if any agent $h$ holds a portfolio delivering $(w^h_U, w^h_D)$, the portfolio costs $aw^h_U + bw^h_D$. 
2. State Pricing Lemma

The State Pricing lemma shows the existence of state prices \( a \) and \( b \), even if short-selling is not allowed.

This implies that there are no arbitrage possibilities in trading the asset and the contracts (even if trader had infinite wealth or didn’t need to put up collateral).

Although the state prices \( a \) and \( b \) are uniquely defined, the equilibrium may not be an Arrow-Debreu equilibrium.
2. State Pricing Lemma

**Step 1:**

We price the asset and the max min bond.

\[
p = ad_U + bd_D,
\]
\[\pi_{j^*} = aj^* + bj^* \]

Two equations in two unknowns so
\[
a = \frac{p - \pi_{j^*}}{d_U - j^*} \quad \text{and} \quad b = \frac{\pi_{j^*}}{j^*} - a.\]
2. State Pricing Lemma

State prices $a$ and $b$ are positive
2. State Pricing Lemma

State prices $a$ and $b$ are positive

- Equilibrium portfolio payoff for some agent must be above riskless debt line.
- Max min debt contract $j^*$
2. State Pricing Lemma

State prices $a$ and $b$ are positive

Equilibrium portfolio payoff for some agent must be above riskless debt line.

Max min debt contract $j^*$

Asset Y Payoff

$U \ dU$

$D \ d_D$
2. State Pricing Lemma

State prices $a$ and $b$ are positive.
2. State Pricing Lemma

Step 2: Upper-bound for all prices.

Any security that is traded with payoff \((w_U, w_D)\) in the feasible payoff space can be purchased for \(aw_U + bw_D\) by going long on the asset \(Y\) and long or short on the max min contract \(j^*\).
2. State Pricing Lemma

**Step 2:** Upper-bound for all prices.

Any security that is traded with payoff \((w_U, w_D)\) in the feasible payoff space can be purchased for \(aw_U + bw_D\) by going long on the asset \(Y\) and long or short on the max min contract \(j^*\).
2. State Pricing Lemma

**Step 2:** Upper-bound for all prices.

Any security that is traded with payoff \((w_U, w_D)\) in the feasible payoff space can be purchased for \(aw_U + bw_D\) by going long on the asset \(Y\) and long or short on the max min contract \(j^*\).
This gives an upper bound to what any buyer would pay for any traded security with payoff \((w_U, w_D)\): \(aw_U + bw_D\).

It only remains to show that \(aw_U + bw_D\) is also a lower bound. That is, no seller would sell for a lower price.
Step 3: Lower-bound for all prices.

We show now that we can find a lower bound for all prices as well.

Suppose a debt contract \( j \) with \( j \neq j^* = d_D \) is positively traded in equilibrium. Then

\[
\pi_j \geq a \cdot \min\{d_U, j\} + b \cdot \min\{j^*, j\}.
\]
2. State Pricing Lemma

Case \( j \leq j^* = d_D \).
2. State Pricing Lemma

Case $j \leq j^* = d_D$. 

![Diagram showing the relationship between $U$, $d_U$, $d_D$, and $D$ with arrows indicating the delivery of contract $j$, debt contract $j < j^*$, and max min debt contract $j^*$]
In this case the argument is very simple since contract deliveries are colinear to promises.

_Sellers:_ They could sell \( \frac{j}{j^*} \) units of \( j^* \) instead. This is feasible for them as it requires less collateral. So it must be the case that

\[
\pi_j \geq \pi_j^*(\frac{j}{j^*}) = aj + bj.
\]
2. State Pricing Lemma

Case $j > j^* = d_D$.
Case $j > j^* = d_D$. 

Diagram: 
- Arrow $U$ security 
- Delivery of contract $j$ 
- Debt Contract $j > j^*$ 
- Max min debt contract $j^*$
2. State Pricing Lemma

Case $j > j^* = d_D$.

The key idea is that the sellers are buyers of the Arrow U security $(d_U - j^*, 0)$. It costs $p - \pi_j$ which, by Step 2, is at most $a(d_U - j)$. Hence, $\pi_j \geq aj + bj^*$. 
The key idea is that the sellers are buyers of the Arrow U security \((d_U - j^*, 0)\).

It costs \(p - \pi_j\) which, by Step 2, is at most \(a(d_U - j)\).

Hence, \(\pi_j \geq aj + bj^*\).
2. State Pricing Lemma

Case $j > j^* = d_D$.

The key idea is that the sellers are buyers of the Arrow-U security $(d_U - j^*, 0)$.

It costs $p - \pi_j$ which, by Step 2, is at most $a(d_U - j)$.

Hence, $\pi_j \geq aj + bj^*$. 
The key idea is that the sellers are buyers of the Arrow U security $(d_U - j^*, 0)$.

The cost is $p - \pi_j$ which is at most $a(d_U - j)$ given Step 2.

Hence

$$p - \pi_j \leq a(d_U - j)$$

$$\pi_j \geq p - a(d_U - j)$$

$$\pi_j \geq ad_U + bd_D - a(d_U - j)$$

$$\pi_j \geq aj + bj^*$$
The only step left is to construct the new equilibrium with no default.

For that we need to construct a new portfolio for each agent with just the asset and the max min contract $j^*$

$$(\bar{y}^h, \check{\phi}^h_{j^*}) \text{ for } h \in H.$$
3. Construction of New Equilibrium

By Payoff Cone lemma, space of feasible portfolio payoffs.
3. Construction of New Equilibrium

By Payoff Cone lemma, space of feasible portfolio payoffs.

Space of feasible portfolio payoffs

long asset, long bond deliveries
3. Construction of New Equilibrium

By Payoff Cone lemma, space of feasible portfolio payoffs.
By Payoff Cone lemma, space of feasible portfolio payoffs.

Space of feasible portfolio payoffs

long asset, short bond deliveries

long asset, long bond deliveries

long asset, short bond deliveries but not feasible due to collateral constraint.
Consider the following equilibrium portfolio payoffs.
3. Construction of New Equilibrium

Consider the following equilibrium portfolios payoffs

Space of feasible portfolio payoffs

Sum of old equilibrium portfolio payoffs

U

D
3. Construction of New Equilibrium

**Step 1: Agent Maximization.**

Given the Payoff Cone Lemma we can find unique portfolios (feasibles) for each agent consisting of the asset and the max min bond that give the same payoffs.
3. Construction of New Equilibrium

New portfolio for pessimist agent $h = L$.
New portfolio for optimist agent $h = B$. 
3. Construction of New Equilibrium

From the State Pricing lemma, the newly constructed portfolios must have the same cost as well.

Since we are assuming the asset is financial, then every agent is optimizing.
3. Construction of New Equilibrium

Step 2: Market Clearing. Sum of four red vectors equals sum of two green vectors.
3. Construction of New Equilibrium

Sum of new debt contract holdings is zero.
3. Construction of New Equilibrium

Sum of new asset holdings is the same as in the previous equilibrium.