Measuring Consumer Behavior: Semiparametric Estimation and Revealed Preference*

Richard Blundell†
University College London and Institute for Fiscal Studies

Ely Lectures
John Hopkins University
October 2003
Draft for the Overview Lecture on 7th March

Abstract
There are three key objectives of these lectures: First, to provide a powerful test of integrability conditions on individual household data without the need for parametric models of consumer behavior. Second, to provide tight bounds on welfare costs of relative price and tax changes. Third, to provide tight bounds on demand responses. The aim is to allow price response parameters to vary across points in the income distribution. The idea is to fully exploit micro data on consumer expenditures and incomes across a finite set of discrete relative price or tax regimes. This is achieved by combining the theory of revealed preference with the semiparametric estimation of consumer expansion paths (Engel curves). Recent developments on semiparametric estimation of shape invariant models with endogenous regressors are exploited. Empirical examples are given from the analysis of cost of living indices and from tax policy reform.

*These lectures draw on joint research with James Banks, Martin Browning, Xiaohong Chen, Ian Crawford, Alan Duncan, Arthur Lewbel and Krishna Pendakur. I am grateful to them for comments and inspiration. This study is part of the program of research of the ESRC Centre for the Microeconomic Analysis of Public Policy at IFS. The financial support of the ESRC is gratefully acknowledged. Material from the FES made available by the ONS through the ESRC Data Archive has been used by permission of the controller of HMSO. Neither the ONS nor the ESRC Data Archive bear responsibility for the analysis or the interpretation of the data reported here. The usual disclaimer applies.

†Department of Economics, University College London, Gower Street, London, WC1E 6BT and Institute for Fiscal Studies, r.blundell@ucl.ac.uk, http://www ifs.org.uk
1. Introduction

Measuring the responses of consumers to variation in prices and income is at the centre of applied welfare economics, it is a vital ingredient of tax policy reform analysis and it is also key to the measurement of market power in modern empirical industrial economics. Parametric models have dominated applications in this field but, I will argue in these lectures, this is both unwise and unnecessary. To quote Dan McFadden in his presidential address to the Econometric Society: “[parametric regression] interposes an untidy veil between econometric analysis and the propositions of economic theory”.

There are three key objectives of these lectures: First, to provide a powerful test of integrability conditions on individual household data without the need for parametric models of consumer behavior. Second, to provide tight bounds on welfare costs of relative price and tax changes. Third, to provide tight bounds on demand responses (and elasticities) to relative price and tax changes. The aim is to allow these response parameters to vary across points in the income distribution in a fully nonparametric way. Parametric models place strong assumptions on both income and price responses and this papers accomplishes all that is required from parametric demand models with only nonparametric regression and revealed preference theory. The idea is to fully exploit micro data on consumer expenditures and incomes across a finite set of discrete relative price or tax regimes. The objectives are achieved by combining the theory of revealed preference with the semiparametric estimation of consumer expansion paths (Engel curves).

Historically parametric specifications in the analysis of consumer behavior have been based on the Working-Leser or Piglog form of preferences in which budget shares are linear in the log of total expenditure (see Muellbauer (1976)). These underlie the popular Almost Ideal and Translog demand models of Deaton and Muellbauer (1980) and Jorgenson, Lau and Stoker (1982). However, in the last decade empirical studies on individual data have suggested further nonlinear terms, in particular quadratic logarithmic income terms, provide a much more reliable specification (see, for example, Hausman, Newey, Ichimura and Powell (1995), Lewbel (1991), Blundell, Pashardes and Weber (1993)). This was brought together in the Quadratic Almost Ideal Demand Sys-

1
tem (QUAIDS) by Banks, Blundell and Lewbel (1998) which provided a fully integrable system consistent with the quadratic logarithmic Engel curve specification and allowing second order flexibility relative price responses. However, relative price effects remain constrained in an unnatural way across individuals with different incomes. In this lecture I want to take this line of research one important step further and allows the fully nonparametric estimation of Engel curves and, by using revealed preference restrictions alone at each point in the income distribution, also allows price responses to be quite unrestricted across individuals with different incomes.

With a relatively small number of price regimes: across points in time or locations or both, nonparametric revealed preference theory provides a natural setting within which to study observed behaviour. The attraction of revealed preference theory is that it allows an assessment of the empirical validity of the usual integrability conditions without the need to impose particular functional forms on preferences. Although developed to describe individual demands by Afriat (1973) and Diewert (1973) following the seminal work of Samuelson (1938) and Houthakker (1950), it has usually been applied to aggregate data but this presents a number of problems. First, on aggregate data, ‘outward’ movements of the budget line are often large enough, and relative price changes are typically small enough, that budget lines rarely cross (see Varian (1982), Bronars (1987) and Russell (1992)). This means that aggregate data may lack power to reject revealed preference conditions. Second, if we do reject revealed preference conditions on aggregate data we have no way of assessing whether this is due to a failure at the micro level or to inappropriate aggregation across households that do satisfy the integrability conditions but who have different non-homothetic preferences. By combining nonparametric statistical methods with a revealed preference analysis of micro data we can overcome these problems.

Following from the work reported in Blundell, Browning and Crawford (2003a), there are a number of other motivations for this study. First, parametric demand studies on micro data often reject Slutsky symmetry which is one of the implications of utility maximisation subject to a linear budget constraint. Amongst the many possible explanations for this rejection are that either we have the ‘wrong’ functional form or

\footnote{See Manser and McDonald (1988), and references therein.}
that there exists no well-behaved form of preferences which can rationalise the data. Nonparametric analysis allows us to check this. Second, it has proven difficult to test for (global) negative semi-definiteness of the Slutsky matrix in parametric demand models. Using nonparametric revealed preference analysis we can simultaneously test for both symmetry and negative semi-definiteness. Third, if the integrability conditions are not rejected, we often wish to go on and use demand estimates for policy analysis. Using parametric analysis there is always some uncertainty as to how much the conclusions are driven by functional form. If we employ nonparametric techniques then we can obtain bounds on responses and use these bounds to judge the importance of the choice of functional form on the conclusions. Finally, the nonparametric analysis can aid in the development of new and parsimonious parametric demand systems.

The central contribution of the Blundell, Browning and Crawford (2003a) study was to develop a method for choosing a sequence of total expenditures that maximise the power of tests of GARP with respect to a given preference ordering. They term this the sequential maximum power (SMP) path and present some simulation evidence that shows that our GARP tests have considerable power against some alternatives, but not others. From this idea it is possible to develop a method of bounding true cost of living indices. In particular the ‘tightest’ upper and lower bounds for indifference curves passing through any chosen point in the commodity space. In turn these methods can be used to calculate tight bounds on demand responses, on annual inflation rates and on responses to tax reforms without making parametric assumptions.

In this analysis, households in the same time period and location are assumed face the same relative prices. Under this assumption, Engel curves for each location and time correspond to expansion paths for each price regime. Blundell and Duncan (1998) have shown the attraction of nonparametric Engel curves when trying to capture the shape of income effects on consumer behaviour across a wide range of the income distribution. Blundell, Duncan and Pendakur (1998) show the importance of allowing demographic composition to enter in a simple but nonlinear way when examining income responses across households of different composition.

The use of nonparametric expansion paths or Engel curves in the revealed preference analysis introduces two new econometric requirements. The first concerns semi-
parametric regression with endogenous regressors. The second, semiparametric estimation of shape invariant regression models. When there are endogenous regressors in a non/semi-parametric model two alternative, and non-nested, approaches to nonparametric regression with endogenous regressors are available: the Control Function approach and the Instrumental Variable approach. The overall relationship between these two approaches is reviewed in Blundell and Powell (2003). The control function approach has the attraction of directly conditioning out the endogeneity problem through a measurable variable. In this case it is (functions of) the residual from the reduced form equation for the endogenous regressor (log total expenditure). The IV approach follows the lead of Newey and Powell (1998). This turns out to be more general but also more unstable in the sense that it is necessary to instrument an unknown function. We provide estimates from both methods.

For households with different demographic composition the natural approach to estimation is via the shape invariant regression model. The partially linear model that allows for demographic variation (see, for example, Robinson (1988)) has the very unattractive property that it reduces to Piglog demands (budget shares are linear in log total outlay) under homogeneity and symmetry. The shape invariant model of Härdle and Marron (1990), on the other hand, provides a theory consistent generalisation to the partially linear semiparametric method of pooling nonparametric Engel curves across households of different composition.

An empirical investigation of these ideas is presented using British Family Expenditure Survey data from 1974 to 1993. This long time series of cross-sections is used to estimate the associated nonparametric Engel curves for 22 goods, adjusted for endogeneity and demographic composition. We then examine whether revealed preference theory can be rejected for particular sub-periods of the data. From the asymptotic distribution theory for nonparametric regression we are able to provide a statistical structure within which to examine the consistency of data with revealed preference theory without imposing a global parametric structure to preferences. The approach adopted provides an alternative to the Afriat inefficiency measure explored in Famulari (1995) and Mattei (1994). The results suggest that GARP is not rejected for long periods of the data and for most income groups. A true cost of living index over the period and annual infla-
tion rates are computed. A comparison of these bounds is made to popular price index
numbers and to other nonparametric bounds. The new bounds are shown to provide
considerable improvements on classical revealed preference bounds.

A particular concern of this paper is on bounding ex-ante price and tax changes. For this there is no observed data for the new price or tax reform. There is therefore no chained Törnqvist (Divisia approximation) index. Banks, Blundell and Lewbel (1996) analyse this question using first order approximations and ask what is gained from demand system estimation. This paper makes a similar comparison but, by using nonparametric revealed preference theory, directly acknowledges the discrete nature of price and tax changes. The tightness of resulting bounds depend on the closeness of the new prices to the sets of previously observed prices. A method for achieving the tightest bounds is developed. This is then applied in the empirical analysis.

The remainder of the paper is as follows: In section 2 the necessary extensions
of revealed preference tests and bounds are developed. Section 3 uses this theory to
develop bounds on demand responses to price changes. The details of the nonparametric
estimator and the importance of the shape invariance model are described in Section 4.
In section 4 the alternative approaches to estimation subject to endogenous regressors in
this semiparametric framework and presented. The empirical analyses begin in section
6 with a description of the household data and some basic empirical properties of the
consumer expansion paths for different commodities. In section 7 the test of GARP is
implemented on this data set. Sections 8 and 9 present bounds on welfare measures and
on demand responses respectively. Section 10 concludes.

2. Individual Data and Revealed Preference

2.1. Revealed Preference and Observed Demands

Imagine that the objective is to test experimentally whether a particular agent has
‘rational’ and stable preferences. This could be done by facing the agent with a series
of prices and total expenditures and testing whether their demand responses satisfy the
Slutsky conditions. Specifically, suppose we have $T$ periods, $t = 1...T$, and we choose
$J$-vectors of (positive) prices $p_t$ and (positive) total expenditures $x_t$ for each period. We
assume that every agent responds with a unique positive demand for each price vector and outlay:

**Assumption 1.** For each agent there exists a set of demand functions \( q(p, x) : \mathbb{R}_{++}^{J+1} \rightarrow \mathbb{R}_{++}^J \) which satisfy adding-up: \( p'q(p, x) = x \) for all prices \( p \) and total outlays \( x \).

Thus we are implicitly assuming that preferences are strictly convex and locally non-satiated (but not necessarily transitive). For a given price vector \( p_t \) we denote the corresponding \( J \)-valued function of \( x \) as \( q_t(x) \) which we shall refer to as an *expansion path* for the given prices. We shall also have need of the following assumption:

**Assumption 2.** Weak normality: if \( x > x' \) then \( q_j(x) \geq q_j(x') \) for all \( j \) and all \( p_t \).

Thus increasing total outlay does not lead to a reduced demand for any good. Adding up and weak normality imply that at least one of the inequalities in this assumption is strict and that expansion paths are continuous.

For our hypothetical experiment we could observe the demands for the given prices and total outlays and test whether the resulting series of prices and demands satisfy revealed preference tests. To do this we need to define a variety of revealed preference relationships. We say that \( q_t(x_t) \) is *directly revealed weakly preferred* to \( q^* \) if the latter is affordable at period \( t \) prices and total expenditure \( x_t \): \( p_t'q_t(x_t) \geq p_t'q^* \) which we write as \( q_t(x_t) \overset{R^0}{\geq} q^* \). An alternative characterisation is that \( q^* \) is within the budget set defined by \((p_t, x_t)\). If the inequality in this condition is strict then we say that \( q_t(x_t) \) is *directly revealed strictly preferred* to \( q^* \) since the agent could have obtained the latter more cheaply (at the prices \( p_t \)) but chose not to. In this case, of course, \( q^* \) is in the interior of the budget set defined by \((p_t, x_t)\).

Now consider any sequence of prices and total outlays \((p_s, p_t, p_u, ..., p_v, p_w; x_s, x_t, x_u, ..., x_v, x_w)\). We say that the sequence of associated demand vectors \((q_s(x_s), q_t(x_t), q_u(x_u), ..., q_v(x_v), q_w(x_w))\) is *preference ordered* if \( q_s(x_s) \overset{R^0}{\geq} q_t(x_t), q_t(x_t) \overset{R^0}{\geq} q_u(x_u), ..., q_v(x_v) \overset{R^0}{\geq} q_w(x_w) \).

Thus a sequence of demands is preference ordered if each demand is directly revealed

---

We will denote a sequence by \((..)\) and a set by \(\{..\}\). We remind the reader that the order matters for a sequence (so that \((1, 2, 3)\) is different from \((3, 1, 2)\)) but not for sets (so that the sets \(\{1, 2, 3\}\) and \(\{2, 3, 1\}\) are the same).
at least as good as the next one. Given this, we say that \( q_s(x_s) \) is revealed weakly preferred to \( q_w(x_w) \) if there is a preference ordered sequence starting at the former and ending at the latter; we denote this by \( q_s(x_s) R q_w(x_w) \). Suppose now that we have \( q_s(x_s) R q_w(x_w) \) and that we also have that the final demand in the sub-sequence, \( q_w(x_w) \), is directly revealed strictly preferred to the first demand vector \( q_s(x_s) \) (that is, \( q_w(x_w) P^0 q_s(x_s) \)). In this case we say that this sub-sequence fails GARP. We shall say that a set of prices and demands fails GARP if any sub-sequence drawn from the set fails GARP. To illustrate, suppose that we have five time periods and that \( q_4(x_4) R^0 q_2(x_2) \), \( q_2(x_2) R^0 q_1(x_1) \) and \( q_1(x_1) P^0 q_4(x_4) \). Thus the sub-sequence \((q_4(x_4), q_2(x_2), q_1(x_1))\) fails GARP\(^3\) and consequently the set \( \{q_1(x_1), q_2(x_2), q_3(x_3), q_4(x_4), q_5(x_5)\} \) fails GARP.

2.2. Choosing a Path for Comparison Points

Below we take the sequence of (absolute) prices \((p_1, p_2, ..., p_T)\) that is given by our data set but we are free to choose the sequence of total expenditures used in the comparisons above. When considering how to do this, there is a well known problem with applying GARP tests to data to which Varian (1982) refers in his applied work. This problem arises since, particularly with time series data, income growth over time can swamp variations in relative prices (which are what we are interested in). This is because real income growth induces outward movements of the budget constraint and, combined with typically small period-to-period relative price movements, this means that budget lines may seldom cross. As a result, data often lacks power to reject GARP. Indeed, if we choose the \( x_t \)'s so that budget lines never cross then we can never violate the GARP conditions. Clearly then, with a given set of relative prices the power of a revealed preference test will depend critically on the choice of the outlay path \((x_1, x_2, ..., x_T)\).

One possible solution is to choose a sequence of constant ‘real’ total expenditures. Thus given \( x_1 \) and a set of price indices \( \{P_1(p_1), P_2(p_2), ..., P_T(p_T)\} \) we could choose \( x_t = x_1 P_t / P_1 \). Although superficially attractive this begs the question of what price index to

\(^3\)Note that this does not necessarily imply that the sub-sequence \((q_1(x_1), q_4(x_4), q_2(x_2))\) fails GARP.
use. More importantly, even if the series of demands generated in this way did satisfy GARP, we cannot be sure that any other series of total expenditures ‘starting’ from $x_1$ would also satisfy GARP. Instead of this, we present an algorithm for determining a sequence of demands which maximises the chance of finding a rejection given a particular preference ordering of the data.

Figure 2.1: Testing GARP with expansion paths

Consider any sub-sequence (taken to be of length 5 for illustrative purposes) of prices $(p_s, p_t, p_u, p_v, p_w)$. Now take an outlay $x_u$ in period $u$ with associated demand $q_u(x_u)$. We can construct a preference ordered sequence through $q_u(x_u)$ for this sequence of prices by using two recursive schemes, one forwards and the other backwards. For the backwards part (the set of demands that are at least as good as $q_u(x_u)$) we set total outlay in period $t$ so that the period $u$ quantity bundle is just affordable: $\tilde{x}_t = p' q_u(x_u)$. Thus $\tilde{q}_t = q_t(\tilde{x}_t)$ is the ‘lowest’ point on the period $t$ expansion path which is directly revealed at least as good as $q_u(x_u)$. Then set $\tilde{q}_s = q_s(p'_s \tilde{q}_t)$. Thus the sequence $(\tilde{q}_s, \tilde{q}_t, q_u(x_u))$ is preference ordered.

To construct the path of quantities to which $q_u(x_u)$ is weakly preferred, we first
solve for the value of outlay in period $v$ that satisfies $x_u = p'_v q_u(x_v)$, which we denote $\tilde{x}_v$, with demand $\tilde{q}_v = q_u(\tilde{x}_v)$.$^4$ This is constructed so that $\tilde{q}_v$ is the ‘highest’ demand on the period $v$ expansion path to which $q_u(x_u)$ is directly revealed weakly preferred. Then construct $\tilde{q}_w = q_w(\tilde{x}_w)$ by setting $\tilde{x}_v = p'_v q_w(\tilde{x}_w)$. By construction, the entire path $- (\tilde{q}_s, \tilde{q}_t, q_u(x_u), \tilde{q}_v, \tilde{q}_w)$ – is preference ordered. We term the path created in this way a sequential maximum power (SMP) path through $q_u(x_u)$. An SMP path is said to start (respectively, finish) at $q_u(x_u)$ if the latter is the first (respectively, the last) element in the sequence. Although we do not denote it explicitly it is important to recognise that an SMP path is always defined relative to a sequence of time indices (in this illustration $(s, t, u, v, w)$) and a point on an expansion path for one of these time periods (in this case, $q_u(x_u)$). For example, $(\tilde{q}_s, \tilde{q}_t, q_u(x_u))$ is an SMP path finishing at $q_u(x_u)$.

To illustrate why this gives maximal power for a particular sequence, consider the three period, two good example in figure 2.1. Here the order of the sequence is $(3, 2, 1)$ finishing at $q_1(x_1)$ so that $(q_3(\tilde{x}_3) R^0 q_2(\tilde{x}_2) R^0 q_1(x_1))$. In this figure the shaded part of the period 3 budget line gives the demands which result in a rejection of GARP. One can see that if we took any other preference ordered path of demands with the same sequence $(q_3(\tilde{x}_3) R^0 q_2(\tilde{x}_2) R^0 q_1(x_1))$ this would reduce the length of this segment. This is because any such path pushes out the period 3 budget line which reduces the chance of observing a GARP rejecting demand in period 3 (if demands are weakly normal)$^5$. More formally, we have:

**Proposition 1.** Suppose that the demand sequence

$$(q_s(x_s), q_t(x_t), q_u(x_u), ..., q_v(x_v), q_w(x_w))$$

rejects GARP. If demands are weakly normal then the SMP path for the same sequence of periods ending at $q_w(x_w)$:

$$(q_s(\tilde{x}_s), q_t(\tilde{x}_t), q_u(\tilde{x}_u), ..., q_v(\tilde{x}_v), q_w(x_w))$$

also rejects GARP.

$^4$Given continuity and weak normality of the expansion paths there always exists a unique outlay and demand that satisfies this condition.

$^5$This is valid for the true expansion path. In our empirical work below we use estimated expansion paths. For these, there is the possibility that the precision of the estimated path is such that although the length is reduced the probability of rejection is not.
Proof. See Appendix A.

Thus, if we test for GARP along a given SMP path finishing at $q_w(x_w)$ and we do not reject, then we can be confident that we would not reject for any other preference ordered path which finishes at the same demand and maintains the preference ordering implied by the SMP path. It is important to note that there may be other preference orderings that finish at $q_w(x_w)$ that do reject GARP so that our maximal power is always with respect to a particular sequencing of time periods. In our empirical work below we always take the chronological sequence finishing in period 1. It is important also to note that maintaining the ordering of demands but choosing a different end point – $q_w(x'_w)$ instead of $q_w(x_w)$ – will result in a different SMP path which may violate GARP, even if the SMP path finishing at $q_w(x_w)$ does not. To check this we take a number of quantile points in the $x$ distribution and apply the SMP procedure to demand sequences ending at $q_w(x)$ where $q_w(x)$ is evaluated at each of these outlays.

When considering tests of integrability, whether parametric or nonparametric, we must be careful to recognise that there are some alternatives against which both modes of test will have low power. To illustrate with a well known example, suppose we draw a large independent sample each period from a large population of agents. If each agent in each period chooses demands on their budget surface by drawing from a uniform distribution on the budget surface then in general no individual path of demands will be integrable. However the (population and sample) mean data will appear to be generated by a Cobb-Douglas utility function with weights equal to the inverse of the number of commodities (see Becker (1962) and Grandmont (1992)). Parametric and revealed preference tests are unlikely to reject the integrability conditions for such data but it is not clear that we would wish to characterise them as the outcome of a ‘rational’ procedure. Equally there will be paths of relative prices which lead to low power tests of the integrability conditions under certain alternatives. The extreme case is if we have no variation in relative prices in which case, of course, we cannot estimate price effects.
for parametric models and we have only one expansion path for our GARP tests.

To investigate this issue further Blundell, Browning and Crawford (2003a) consider three alternative generating processes that produce non-integrable demands: a random procedure, an integrable path with measurement error and a path generated by a slow adjustment model. All of the experiments suggest that this testing procedure is likely to have good power against an alternative in which there is failure of the integrability conditions. However, in the case where they only fail in the short run it seems unlikely that we will have good power.

2.3. Tight Bounds on Welfare Measures

Afriat (1977) showed how revealed preference restrictions can be used to provide information on the curvature of indifference surfaces in commodity space and then used to set bounds on the welfare effects of a price change. This is further developed in Varian (1982) and Manser and McDonald (1988). One problem with applying this procedure to the aggregate data that the latter use is that budget surfaces rarely cross so that the bounds from such data tend to be wide. Knowledge of expansion paths can greatly improve these bounds. Without loss of generality we consider an indifference surface passing through some base bundle \( \mathbf{q}_1 \) on the first expansion path \( \mathbf{q}_1(x) \). If GARP and weak normality hold then we shall show that we can partition each expansion path, \( \mathbf{q}_t(x) \), into three distinct segments. First, on any expansion path, there are the demands that can be shown to be weakly revealed preferred to \( \mathbf{q}_1 \). Second, we have the demands that we can show are weakly revealed dominated by \( \mathbf{q}_1 \). Finally there is an intermediate segment with demands that cannot be revealed preference ordered with respect to \( \mathbf{q}_1 \).

We then show how knowledge of these segments for each expansion paths allows us to construct tight bounds on the welfare costs of arbitrary price changes from the base.

\footnote{Varian (1983) and Manser and McDonald (1988) tighten the bounds using a maintained hypothesis of homotheticity, but this is problematic since much empirical evidence suggests that budget shares are not constant with respect to the total budget.}
price \( p_1 \). We first present an algorithm that we claim finds the ‘lowest’ point on each expansion path such that we can show \( q_t R q_1 \); we term this the weakly preferred set. We then show that if GARP and weak normality hold then this algorithm converges in a finite number of steps and the weakly preferred set has the claimed property.

**Algorithm A.1** Input: a base bundle \( q_1 \), price vectors \( p_t \) and expansion paths \( q_t(x) \) for \( t = 2, ..., T \). Output: \( Q_B(q_1) \).

1) Set \( s = 0 \) and \( F(s) = \{ q_1 , q_2(p_2^2 q_1) , ..., q_T(p_T^T q_1) \} \)
2) Set \( F(s+1) = \{ q_1 , q_2(\min_{q_t \in F(s)} \{ p_2^2 q_t \}) , ..., q_T(\min_{q_t \in F(s)} \{ p_T^T q_t \}) \} \).
3) If \( F(s+1) \equiv F(s) \) then set \( Q_B(q_1) = F(s) \) and stop. Else set \( s = s + 1 \) and go to (2).

The set \( Q_B(q_1) \) has \( T \) elements, one for each expansion path; we denote the \( t \)th element of \( Q_B(q_1) \) by \( q^B_t \). A discussion of this algorithm and the one following and an illustration can be found in Appendix B. Blundell, Browning and Crawford (2003a) prove the following proposition:

**Proposition 2.** If GARP and weak normality hold then:
A. algorithm A.1 converges in a finite number of steps.
B. \( (q_t \geq q^B_t) \iff (q_t R q_1) \).

**Proof.** See Blundell, Browning and Crawford (2003a).

The first part of the proposition assures that the algorithm is feasible (it in fact converges quite quickly in practice). The second part of the proposition verifies that the algorithm identifies the largest set of points on expansion paths that can be shown to be revealed preferred to \( q_1 \) with the data to hand.\(^7\)

We also have an algorithm that finds the ‘highest’ point on each expansion path such that \( q_1 \) can be shown to be revealed preferred to these points.

**Algorithm A.2** Input: a base bundle \( q_1 \) and price vectors \( p_t \) and expansion paths \( q_t(x) \) for \( t = 1, 2, ..., T \). Output: \( Q_W(q_1) \).

\(^7\)We could extend the weakly revealed preferred set to the whole commodity space by taking the convex hull of the points in \( Q_B(q_1) \) but this is not necessary for the welfare bounds we derive below.
1) Set $s = 0$ and $F^{(s)} = \{ q_1, q_2 : p'_1 q_2(x) = p'_1 q_1, \ldots, q_T(x) = p'_T q_T \}$. 
2) Set $F^{(s+1)} = \{ q_1, \max_{q_t \in F^{(s)}} (q_2 : p'_1 q_t = p'_1 q_2), \ldots, \max_{q_t \in F^{(s)}} (q_T : p'_1 q_t = p'_T q_T) \}$. 
3) If $F^{(s+1)} \equiv F^{(s)}$ then set $Q^W (q_1) = F^{(s)}$ and stop. Else set $s = s + 1$ and go to (2).

Denoting the $t$th element of $Q^W (q_1)$ by $q^W_t$ we have the following results for this algorithm:

**Proposition 3.** If GARP and weak normality hold then:

A. algorithm A.2 converges in a finite number of steps.
B. for any $x$ we have $(q^W_t \geq q_t) \Leftrightarrow (q_t R q_t)$

**Proof.** See Blundell, Browning and Crawford (2003a).

Finally we can show that for any $t$ we have $q^B_t \geq q^W_t$ so that the two points divide any expansion path into three connected segments (given weak normality).

Given the sets $Q^W (q_1)$ and $Q^B (q_1)$ we can derive bounds on the welfare effects of a price change. For example, suppose that we have a reference commodity level $q_1$ (on the expansion path $q_1(x)$) and an arbitrary absolute price vector $p_z$. The true cost-of-living index based at $q_1$ is given by:

$$\frac{c(p_z, q_1)}{c(p_1, q_1)}$$  \hspace{1cm} (2.1)

where $c(p_z, q_1)$ is the expenditure function giving the cost of attaining a bundle indifferent to $q_1$ at prices $p_z$. Bounds can be placed on this index using the two sets derived above:

$$\frac{\min_q \{ p'_t q | q \in Q^W (q_1) \}}{p'_1 q_1} \leq \frac{c(p_z, q_1)}{c(p_1, q_1)} \leq \frac{\min_q \{ p'_t q | q \in Q^B (q_1) \}}{p'_1 q_1}.$$  \hspace{1cm} (2.2)

**Computing Welfare Bounds:**

---

Note that there is the possibility of corner solutions with respect to the lower bound whereby the new price vector may cause one or more demands to fall to zero. To allow for this in the calculation of the cost-of-living index the lower bound set $Q^W (q_1)$ needs to be augmented in the following way

$$Q^W (q_1) = \max_j \left\{ p'_w q_w / p'_w : \forall q_w \in Q^W (q_1) \right\} \cup Q^W (q_1)$$

See Appendix B for an illustration.
Figure 2.2 illustrates the algorithm. In the first iteration step (1) begins with \( s = 0 \) and \( F^{(0)} = \{ q_1, q_2 (x'_2), q_3 (x'_3), q_4 \} \), where \( x'_2 = p'_2 q_1 \) and \( x'_3 = p'_3 q_1 \). Clearly \( q_4 \not\in Pq_2 \) and hence \( q_4 \not\in Pq_1 \). Clearly \( q_4 \not\in Pq_2 \) and hence \( q_4 \not\in Pq_1 \). In step (2) \( F^{(1)} = \{ q_1, q_2 (x'_2), q_3 (x'_3), q_4 (x'_4) \} \) since \( q_4 \not\in Pq_0 = Pq(x'_4) \). Because \( F^{(0)} \neq F^{(1)} \) we set \( s = s + 1 = 1 \) and go to step (2) at the second iteration. Now \( F^{(2)} = \{ q_1, q_2 (x'_2), q_3 (x'_3), q_4 (x'_4) \} \) and in step (3) the iteration ends defining \( Q_B (q_1) = \{ q_1, q_2 (x'_2), q_3 (x'_3), q_4 (x'_4) \} \). Algorithm A.2 proceeds in a similar way giving \( Q_W = \{ q_1, q_2 (x_2), q_2 (x_3), q_4 (x_4), x_4/p'_4, x_1/p'_1 \} \) but A.2 has the additional step which identifies the final two points on the \( q^1 = 0 \) and \( q^2 = 0 \) axes. The dashed lines marked ‘upper’ and ‘lower’ shows the bounds on \( c(p_z, q_1) \) given by \( \min \{ p'_z q_t | q_t \in Q_W (q_1) \} \) and \( \min \{ p'_z q_t | q_t \in Q_P (q_1) \} \) for some new set of relative prices \( p_z \). In section 5 we use these results together with nonparametric estimates of Engel curves to compute upper and lower bounds on the true fixed welfare base cost-of-living index over the period.
1974 to 1993 using British household budget survey data. These are then compared to standard cost-of-living index formulae and to alternative nonparametric and revealed preference bounds.

As well as being interested in fixed welfare base cost-of-living indices which span a period of, perhaps, several years, we are often even more interested in annual inflation rates and with these it is typical to update the welfare base in each period rather than let it get too out of date. For example the inflation rate between the adjacent years \( t \) and \( t+1 \) may be calculated as \( \left( \frac{c(p_{t+1}, q_t)}{c(p_t, q_t)} \right) - 1 \). Bounds can easily be derived by finding the bounds on the indifference curve through \( q_t \) — i.e. \( Q_W(q_t) \) and \( Q_B(q_t) \) — and by applying

\[
\min_q \left\{ \frac{p_{t+1}' q_t}{p_t q_t} \right\} \leq \frac{c(p_{t+1}, q_t)}{c(p_t, q_t)} \leq \min_q \left\{ \frac{p_{t+1}' q_t}{p_t q_t} \right\}.
\]

The inflation rate between \( t+1 \) and \( t+2 \) can be measured as \( \left( \frac{c(p_{t+2}, q_{t+1})}{c(p_{t+1}, q_{t+1})} \right) - 1 \) and a bound derived in an identical manner. In the empirical analysis below we present annual inflation bounds for 1975 to 1993 derived in this way.

3. Bounds on Demands Responses

3.1. Constructing Weak Bounds on Responses to New Prices

Given a sequence of J-vectors of prices \( \{p_1, p_2, \ldots, p_T\} \) and a sequence of total outlays \( \{x_1, x_2, \ldots, x_T\} \) suppose that we observe a sequence of demand vectors \( \{q_1, q_2, \ldots, q_T\} \). We have seen that the data \( \{(p_1, q_1), (p_2, q_2), \ldots, (p_T, q_T)\} \) satisfy the Generalised Axiom of Revealed Preference (GARP)\(^9\) if:

\[
p_s' q_s \geq p_s' q_t, p_t' q_t \geq p_t' q_u, \ldots, p_v' q_v \geq p_v' q_w \Rightarrow p_w' q_w \leq p_w' q_s \text{ for any set } (s, t, u, \ldots, v, w)
\]  

\(^9\)This definition is closer in spirit to the Afriat ‘cyclical monotonicity’ definition of GARP than the usual definition that uses explicit revealed preference conditions; see Varian (1982) for a proof of the equivalence.
Varian (1982) poses the question of how we can use such data to predict demands if we have a new price vector \( p_0 \) with total outlay \( x_0 \). This is given by the Varian support set defined as:

\[
S^V(p_0, x_0) = \left\{ \frac{q_0^\prime}{p_0^\prime} q_0 = x_0, q_0 \geq 0 \text{ and } \{ (p_0, q_0) : \{p_t, q_t\}_{t=1...T} \} \text{ satisfies GARP} \right\}
\]

Thus the Varian support set is the set of points on the budget surface that are utility consistent with the existing data. If the original set of demands do not satisfy GARP then this set is, of course, empty. Figure 3.1 reproduces the figure that Varian uses to illustrate this set if the original demands do satisfy GARP. One feature of these predictions is that GARP and the data only give tight bounds on hypothetical demands if the new budget line is ‘close’ to some original budget lines. At the extreme, if the new budget line does not intersect the old budget lines then we have no bounds on possible demands.
Without additional data there seem to be only a limited number of resolutions of the ‘wide bounds’ problem. One is to assume normality for all goods which rules out some end points for hypothetical budget lines. This leads to the definition of a new support set:

\[
S^N(p_0, x_0) = \left\{ q_0 : q_0 \in S^V(p_0, x_0) \text{ and } q_0 \geq \sum_{t=1}^{T} \lambda_t q_t \right\}
\]

for all \( \lambda_t \geq 0 \) with \( \sum_{t=1}^{T} \lambda_t = 1 \).

The definition makes it clear that this new support set is a subset of the Varian support set. The use of the normality assumption is shown in figure 3.2. In some cases using the normality assumption can effect a considerable improvement (for example, take \( q_1 \) and \( q_2 \) to be closer to each other in the figure) but in other cases very little improvement is achieved by assuming normality.

Another possible resolution is to assume something about expansion paths. The constant budget shares assumption has, for example, been used extensively in the index.
number literature to give exact and superlative\textsuperscript{10} index number formulae (see Diewert (1976, 1981)). In this case we generally have tighter bounds, a fact that is exploited by Varian (198?) and Manser and McDonald (1988) who explicitly assume homotheticity. To illustrate this, consider figure 3.3 which simply adds linear expansion paths (denoted by $q_1(x)$ and $q_2(x)$) through the origin to figure 3.1. The resulting support set is $S^H(p_0, x_0)$.

As can be seen the only points on the new budget line that are consistent with GARP and the original data and the expansion paths is a strict subset of the Varian bounds. The problem with this approach is, of course, that preferences are not homothetic and the new expansion paths may not be anything like the true expansion paths. As far as we aware no one has suggested other ways of tightening the bounds on hypothetical

\textsuperscript{10}A superlative index number formula is one which is based upon utility function which is capable of providing a second order approximation to an arbitrary, twice differentiable, linearly homogeneous, utility function.
demands without better empirical information.

3.2. Expansion Path Based Bounds

Suppose now that rather than demands at specific total expenditures for each price regime we have expansion paths for each price regime. That is, we have a demand vector for prices $p_t$ for all $t$ and any total expenditure, denoted $q_t(x)$, with $q_t^j(x)$ denoting the demand for good $j$. We assume:

*The expansion paths $q_t(x)$ are unique valued, positive valued, continuous, normal (in the sense that $x > x'$ implies $q_t(x) > q_t(x')$) and satisfy adding-up $p'_t q_t(x_t) = x_t$.*

Given expansion paths we can define an alternative support set for $(p_0, x_0)$. This support set gives the largest set of demands on the new budget surface such that any element of this set and the existing data satisfy GARP. We proceed in a series of steps. We shall illustrate the steps by reference to figure 3.4 which presents the $T = 3$ case.

Step 1. The first step in the definition is to identify the $T$ points of intersection between the new budget surface and the expansion paths by solving the following implicit equation for $\tilde{x}_t$:

$$p'_t q_t(\tilde{x}_t(p_0, x_0)) = x_0$$

(3.2)

Usually we shall drop the explicit dependence on $(p_0, x_0)$ and simply write $\tilde{x}_t$. The continuity and normality assumptions given above ensure that for each expansion path $q_t(x)$ such a point exists and that it is unique. Using these values we have demands $q_t(\tilde{x}_t)$ for $t = 1, 2, ... T$ which we refer to as intersection points; these points are illustrated in figure 3.4.

Step 2. The second step in the construction of the support set is to test for GARP for the set of intersection points $(p_t, q_t(\tilde{x}_t))_{t=1,...,T}$. If GARP fails then we cannot give any bounds for the new prices and total outlay and so we set the support set to the empty set. In this case. If GARP passes then we move on to the third step.
Step 3. Consider any point $q_0$ on the new budget line and ask whether it could be a ‘rational’ demand at the new prices $p_0$. If it is, then $p'_t q_0 \geq p'_t q_t (\tilde{x}_t)$, otherwise we would have $p'_0 q_t (\tilde{x}_t) \geq p'_0 q_0$ (by the construction of $\tilde{x}_t$) and $p'_t q_t (\tilde{x}_t) > p'_t q_0$ which is a violation of GARP. This gives the following definition for the support set for $(p_0, x_0)$:

\[
S(p_0, x_0) = \{ q_0 : q_0 \geq 0, \quad p'_0 q_0 = x_0 \text{ and } \quad p'_t q_0 \geq p'_t q_t (\tilde{x}_t) \text{ for } t = 1, 2, \ldots T \}
\]

This is illustrated in figure 3.4 in which the shaded segment of the new budget line gives the support set. Thus the support set is the region on the new budget surface such that a point in this region and the set of intersection demands satisfy GARP.

Step 4. Finally, it is convenient to have some way of describing the support set succinctly. To do this we project onto each axis by finding the maximum and minimum values of each demand for the new prices and total outlay. For example, in figure 3.4,
the minimum predicted value of good \( a \) (the good on the ‘x’ axis) is at \( q_{a,\min}^t(=q_1(\tilde{x}_1)) \) and the maximum is at \( q_{a,\max}^t(=q_2(\tilde{x}_2)) \).

We have the following properties for the support set:

**Proposition 4.**

A. For any \((p_0, x)\), if the intersection demands \((p_t, q_t(\tilde{x}_t))_{t=1\ldots T}\) satisfy GARP then the support set is non-empty.

B. The set \( S(p_0, x_0) \) is convex.

C. For any point on the new budget line that is not in \( S(p_0, x_0) \), we have that the intersection demands and this point fail GARP.

**Proof.** See Blundell, Browning and Crawford (2003b).

Since the support set is empty if the intersection demands fail GARP, part A establishes that there are some predicted demands if and only the intersection demands satisfy GARP. Part B will be useful in the algorithm below. Part C states that the procedure we adopt yield the tightest bounds, given the data to hand. Nor can we effect improvements by imposing simple restrictions as in the previous sub-section. For example, we cannot improve the bounds by imposing normality since this is already imposed by the expansion paths being normal.

In practice we do not use a two step procedure to find the bounds, instead we use a linear programming procedure. Once we have found the intersection demands \( q_t(\tilde{x}_t) \) and tested that they pass GARP, we solve a series of linear programming problems:

\[
q_{j,\max}^t = \max_{q_0} q_{0,j}^t \text{ subject to } q_0^t \in S(p_0, x_0)
\]

\[
q_{j,\min}^t = \min_{q_0} q_{0,j}^t \text{ subject to } q_0^t \in S(p_0, x_0)
\]

A solution always exists since the support set is non-empty. This procedure is not computationally burdensome since we have only \( 2J \) problems (one maximum and one minimum for each good) each with \( J \) control variables, one equality constraint, \( J \) non-negativity constraints and \( T \) inequality constraints.

Given the convexity of the support set and the proposition above, it is trivial to show:
Proposition 5.

\[ q_0 \in S(p_0, x_0) \text{ if and only if } q_{j_{\min}}^j \leq q_0^j \leq q_{j_{\max}}^j \text{ for } j = 1\ldots J. \]

Proof. See Blundell, Browning and Crawford (2003b).

All of the above is for a given set of prices. We also have that if \( S(p_0, x_0) \) is the support set for \( T \) prices and expansion paths and we now add a new set of prices with an associated expansion path then the new support set (weakly) contains the old one. One trivial example of this is if the augmented \( T + 1 \) intersection demands fail GARP, in which case the new support set is the empty set. Conversely, the addition of a new expansion path that satisfies GARP may leave the support set unchanged. An obvious example is the addition of expansion path 3 to paths 1 and 2 in figure 3.4. We also have a closely related result which states that if we have an existing support set then we can always find a new set of prices and an expansion path such that the new support set is strictly smaller (assuming that any new expansion path satisfies GARP). Given that we are shrinking compact sets this means that we can shrink the support set (and the associated intervals on the axes) to a single point. This highlights clearly that the existence of bounds on predictions rather than point estimates is because we have discrete price information rather than continuous variation.

3.3. Allowing for some cost-inefficiency

A GARP test can be interpreted as a test of two sub-hypotheses\(^{11}\)

1. the consumer has rational preferences representable by a utility relation
2. the consumer is an efficient programmer

If the data violates GARP then the hypothesis can be modified. Afriat’s (1973) suggestion is that if (1) is not to be modified, then (2) must. Instead of requiring exact efficiency, a form of partial efficiency is allowed, denoted by \( e \) where \( 0 \leq e \leq 1 \). The

\(^{11}\)Afriat (1973)
consumer is now allowed to waste a fraction \((1 - e)\) of their budget. This is done by modifying the preference relation \(R^0\) to

\[ q_s R^0_e q_t \iff e \cdot p'_s q_s \geq p'_s q_t \]

This stiffens the requirement which reveals a preference in the sense that \(p'_s q_s\) must exceed \(p'_s q_t\) by more in order for \(q_s\) to be revealed preferred to \(q_t\) (\(p'_s q_s\) now has to be \(1/e\) times bigger than \(p'_s q_t\)). If \(e = 0\) then nothing can be revealed preferred to anything and the data contains no information on rankings of different bundles. This efficiency concept can be used to define a weaker consistency test:

\[ GARP (e) : q_s R_e q_t \Rightarrow \text{not } q_t \cdot p^0_e q_s \]

where \(q_t \cdot p^0_e q_s \iff e \cdot p'_t q_t \leq p'_t q_s\) and where \(R_e\) denotes the transitive closure of \(R^0_e\).

Note that if \(e = 1\) then \(GARP (e)\) is equivalent to \(GARP\) and that if \(e = 0\) then there is no restriction on behaviour.

The Varian support set can then be changed to allow for imperfect cost-efficiency:

\[ S_V^I \left( p_0, x_0, e \right) = \left\{ q_0 : p'_0 q_0 = x_0, q_0 \geq 0 \text{ and } \{p_0, p_t, q_0, q_t\}_{t=1}^{T} \text{ satisfies } GARP (e) \right\} \]

for some choice of \(e\). Similarly our support set may be modified to allow some cost-inefficiency, however, the definition of the intersection points also needs to be changed because, as it stands these are set such that

\[ x_0 = p'_0 q_t (\tilde{x}_t) \Rightarrow q_0 R^0_e q_t (\tilde{x}_t) \]

so for any \(e < 1\) this means

\[ e \cdot x_0 < p'_0 q_t (\tilde{x}_t) \not\Rightarrow q_0 R^0_e q_t (\tilde{x}_t) \]

and as a result for any \(e < 1\) there is no restriction on where the new demand vector may lie on the new budget hyperplane. Hence intersection demands need to be evaluated at at lower budget level such that

\[ e \cdot x_0 = p'_0 q_t (\tilde{x}_t) \Rightarrow q_0 R^0_e q_t (\tilde{x}_t) \]
Our support set is then

\[ S(p_0, x_0, e) = \left\{ q_0 : p_0 q_0 = x_0, q_0 \geq 0 \text{ and } \{p_t, q_t, q_0(x_t)\}_{t=1 \ldots T} \text{ satisfies } GARP(e) \right\} \]

### 3.4. Uncompensated and compensated demand curves.

We have already shown how to derive bounds for demand predictions given a new set of relative prices and total expenditure. We now show how to use this to give bounds on Marshallian (uncompensated) and compensated demands. We begin with Marshallian demands. Suppose we have a price vector \( p \) (which may or may not have been observed) and total outlay \( x \) and we wish to map out demands as we vary the price of good \( i \). We first evaluate \( S(p, x) \). We then take a change in the price of good \( i \), \( \Delta p_i \) and evaluate \( S(p^1, ..., p^i + \Delta p^i, ..., p^J, x) \). If we do this for series of changes (negative and positive) then we can map out the bounds on the demands for good \( i \) to give the own price uncompensated responses and the bounds on the demands for any good \( j \neq i \) to give uncompensated cross-price effects.

We also wish to display and use compensated demands. Since we do not have continuous price data nor a parametric functional form (to ‘fill in’ the missing predictions) we cannot derive Hicksian demands (which hold utility constant). Instead we work with Slutsky compensations. Thus whenever we change price by \( \Delta p \) we change total expenditure by \( \Delta x = \Delta p' q \) so that the original demand vector is just affordable at the new prices. To do this we are restricted to working with compensated demands that start from an observed price vector since only then do we have a point estimate of \( q(x) \). Denote demands for good \( j \) in period \( t \) given total expenditure \( x_t \) by \( q^j(p_t, x_t) \) and the vector with zeros everywhere except for position \( i \) by \( \Delta p^i \). We define the compensated demand for a discrete price change as:

\[ S_{ji}(p_t, x_t, \Delta p^i) = \frac{q^j(p_t + \Delta p^i, x_t + \Delta x) - q^j(p_t, x_t)}{\Delta p^i} \] (3.3)
where $\Delta x = \Delta p^i q^i (p_t, x_t)$. Adapting a conventional derivation (see, for example, Varian (1984), section 3.13) we have the Slutsky equation:

$$ S_{ji}(p_t, x_t, \Delta p^i) = \frac{q^i(p_t + \Delta p^i, x_t + \Delta x) - q^i(p_t, x_t + \Delta x)}{\Delta p^i} $$

$$ + \frac{q^j(p_t, x_t + \Delta x) - q^j(p_t, x_t)}{\Delta x} q^i(p_t, x_t) $$

(3.4)

The second term on the right hand side can be evaluated exactly given that we observe the expansion path for prices $p_t$. The first term on the right hand side is the Marshallian response; as we have seen we can derive bounds for this. Thus the bounds for the (Slutsky) compensated responses are simply the Marshallian bounds translated upwards by the income effect.

### 3.5. Examples

Here we present some simulations which illustrate the ideas on bounding demand responses introduced above and develop our intuition. It also serves as an introduction to the presentational methods we use in the empirical sections below. We start with the simple case of two goods ($a$ and $b$) and just two expansion paths (periods 1 and 2). We set the observed prices to be $p_{a1} = 0.5$, $p_{a2} = 2$ and $p_{b1} = p_{b2} = 1$, so that only the price of good $a$ changes. We compute the expansion paths for constant budget shares (Cobb-Douglas preferences) with fixed shares of one half for each good and we set $x_0 = 75$. To map out the demand bounds we vary the price of good $a$ from 0.4 to 2.5 in 200 equal steps. The results are illustrated in figure 3.5 which presents both bounds and the true demands as computed from the Cobb-Douglas demands.

The left panel shows the bounds on $q^a_0$ (the own price response) and the right the bounds on $q^b_0$ (the cross price response). In both cases the expansion path bounds are a point at the two observed prices (0.5 and 2) and elsewhere are given by ‘irregular boxes’ with these points at one vertex. The true responses (shown by the dashed line) are, of course, always inside the boxes and coincide with the predicted demands at observed...
prices. For prices between observed prices \((2 > p^a > 0.5)\) we have two sided bounds but not very precise ones. For example at the mean of the observed prices \((p^a = 1.25)\) the bounds are \((23.1, 42.9)\) as compared to a true value of 30. Moreover, even for prices very close to the observed prices, the bounds are wide; for example at \(p^a = 0.495\) we have bounds of \((75.4, 151.5)\). For prices above any observed price we cannot, of course, rule out that the agent stops buying good \(a\) altogether so that the lower bound is zero. The cross price responses are not very informative and would not rule out parametric forms that were initially decreasing and then increasing (as opposed to the true cross-price response which is, as shown, a horizontal line). This suggests that expansion path based bounds based on \(S(p_0, x_0)\) that are derived from a small set of price regimes are not likely to be very informative. We now present a series of ‘experiments’ to assess the robustness of this conclusion.

The first comparison we make is between the expansion path bounds and the Varian bounds. For the Varian bounds we utilised demands at observed total expenditures levels \(x_1 = 50, x_2 = 100\). We present only the own price response figure (see figure 3.6). The Varian bounds \((S^V(p_0, x_0))\) are shown by the dashed lines, and the expansion
path bounds by solid lines; where they coincide the solid line depicts the common bound. As can be seen the expansion path bounds for own prices responses are generally much tighter than the Varian bounds when we interpolate (that is, when we consider hypothetical prices that are in the convex hull of observed prices). Thus using expansion paths does seem to help somewhat.

We now investigate the value of having more data. Figure 3.7 shows the bounds using four price and expansion path observations (dashed lines) and the bounds from figure 3.5 derived from just two price/expansion path observations (solid outer bounds). We take the same environment as above with observed prices of $p_1 = 0.5$, $p_2 = 0.75$, $p_3 = 1$, $p_4 = 2$. As in figure 3.5 the left hand panels show own-price effects and the right-hand panels the cross price effects. As can be seen the bounds on both own-price and cross-price responses improve considerably.

In the next three simulations we look at the effects which the curvature of the indifference curve has on the accuracy of RP bounds. The closer to linear are true preferences, the more spread out expansion paths will be and hence (for a given set of observed prices) the wider the bounds our RP procedure will recover. In each case we
utilise the same price data as figure 3.7. We assume a CES functional form for utility \( u(q^a, q^b) = [(q^a)\rho + (q^b)^{\frac{1}{\rho}}]^{\frac{1}{\rho}} \) and vary the parameter \( \rho \). In figure 3.8 set \( \rho = 0 \) so that preferences are Cobb-Douglas. The bounds are exactly the same as the inner (four prices) bounds in figure 3.7, the dashed lines in each panel show the true demand curves. In figure 3.9 we set \( \rho = 1 \times 10^{-9} \) so that preferences are almost Leontief. As expected, little substitution means that the expansion paths lie close together in \( q \)-space and hence the bounds within the range of observed price variation are very tight (the true demand curves are hard to discern but run along the bottom (top) of the own-price (cross-price) bounds for \( p^a_0 < 0.5 \) and along the top (bottom) of the own-price (cross-price) bounds for \( p^a_0 > 2 \)). In the final set of simulations we set \( \rho = 0.999 \) so that preferences have almost linear indifference curves and the expansion paths are consequently spread out.

Finally, we examine the effects of an increase in the number of goods compared to the number of price observations. In figure 3.11 we add a third good to the data used in figure 3.7, with prices \( p^a_1 = 0.75, p^a_2 = 0.5, p^a_3 = 1, p^a_4 = 1.5 \) and, again, Cobb-Douglas preferences with all shares set at one third. The deterioration in the RP bounds compared to the inner (four period) bounds in figure 3.7 is reasonably clear, particularly
Figure 3.8: Cobb-Douglas preferences, and RP bounds

Figure 3.9: Almost Leontief preferences, RP bounds
for the cross-price effects.

Drawing together the results of the last sub-section it seems that three main lessons may be drawn. Firstly it will not be possible to derive tight bounds over a reasonable range unless prices are varying but ‘equally spaced’. Secondly, interpolation (in the sense of considering new price vectors which are inside the convex hull of the observed prices) will yield better results than extrapolation. Thirdly, it seems to be the case that the fewer the number of goods considered, for a fixed number of price observations, the looser the bounds are likely to be. It is worth pointing out that the circumstances under which this procedure will give loose bounds are generally those under which parametric approaches will be most fragile.

4. Semiparametric Estimation

4.1. Expansion Paths

To estimate the expansion paths for each price regime we employ nonparametric regression methods. Let \( \{(\ln x_i, w_{ij})\}_{i=1}^{n} \) represent a sequence of \( n \) household observations on the log of total expenditure \( \ln x_i \) and on the \( j \)th budget share \( w_{ij} \), for each household \( i \) facing the same relative prices. For each commodity \( j \), budget shares and
Figure 3.11: 3 goods, 4 periods, RP bounds
total outlay are related by the stochastic Engel curve

\[ w_{ij} = g_j(\ln x_i) + \varepsilon_{ij} \]  \hspace{1cm} (4.1)

where we assume that, for each household \( i \), the unobservable term \( \varepsilon_{ij} \) satisfies

\[ E(\varepsilon_{ij} | \ln x) = 0 \text{ and } \text{Var}(\varepsilon_{ij} | \ln x) = \sigma_j^2(\ln x) \forall \text{ goods } j = 1, \ldots, J \]  \hspace{1cm} (4.2)

so that the nonparametric regression of budget shares on log total expenditure estimates \( g_j(\ln x) \).\(^{12}\) In (4.1), if preferences are Piglog\(^{13}\), \( g_j \) is linear in \( \ln x \) for all goods \( j = 1, \ldots, J \).

In our empirical application we use the following unrestricted Nadaraya-Watson kernel regression estimator

\[ \hat{g}_j(\ln x) = \frac{\hat{r}^h_j(\ln x)}{\hat{f}^h(\ln x)} \equiv \hat{w}_j(\ln x) \]  \hspace{1cm} (4.3)

in which

\[ \hat{r}^h_j(\ln x) = \frac{1}{n} \sum_{l=1}^{n} K_h(\ln x - \ln x_l) w_{lj}, \]  \hspace{1cm} (4.4)

and

\[ \hat{f}^h(\ln x) = \frac{1}{n} \sum_{l=1}^{n} K_h(\ln x - \ln x_l), \]  \hspace{1cm} (4.5)

where \( h \) is the bandwidth and \( K_h(\cdot) = h^{-1}K(\cdot/h) \) for some symmetric kernel weight function \( K(\cdot) \) which integrates to one. We assume the bandwidth \( h \) satisfies \( h \to 0 \) and \( nh \to \infty \) as \( n \to \infty \). Under standard conditions the estimator (4.3) is consistent and asymptotically normal, see Härdle (1990) and Härdle and Linton (1994). Additionally, provided the same bandwidth and kernel are used to estimate each \( g_j(\ln x) \), adding-up across the share equations will be automatically satisfied for each \( \ln x \) and there is no efficiency gain from combining equations. This mirrors the invariance result for SURE systems with identical regressors (see Deaton (1983), for example).

\(^{12}\) Below we discuss how we allow for the endogeneity of \( \ln x \) in the Engel curve regression equation.

\(^{13}\) See Muellbauer (1975) and the empirical investigations by Working (1943) and Leser (1963). These are the preferences that underly the popular Translog and Almost Ideal demand systems.
4.2. Demographic Composition and Shape Invariance

Household expenditures typically display variation according to demographic composition. A fully nonparametric approach would be to stratify by each distinct household demographic type and estimate each Engel curve by nonparametric regression within each cell. Given that this would result in relatively small sample sizes within each cell, we choose to use a semiparametric specification to pool across household types.

Let \( z_i \) represent a vector of discrete household composition variables for each household observation \( i \). A simple semiparametric specification would be to assume partial linearity (see Robinson (1988) and Powell (1987))

\[
    w_{ij} = g_j(\ln x_i) + z_i' \gamma_j + \varepsilon_{ij}
\]

with

\[
    E(\varepsilon_{ij}|z_i, \ln x_i) = 0 \quad \text{and} \quad \text{Var}(\varepsilon_{ij}|z_i, \ln x_i) = \sigma_j^2(z_i, \ln x_i). \tag{4.7}
\]

in which \( \gamma_j \) represents a finite parameter vector of household composition effects for commodity \( j \) and \( g_j(\ln x_i) \) is some unknown function as in (4.1).

Although the partially linear model (4.6) motivates the semiparametric approach taken in this paper, consideration of the integrability conditions indicate that some modification is required. This is because the additive structure underlying (4.6) together with the Slutsky symmetry conditions

\[
    \frac{\partial w_j}{\partial \ln p_k} + w_k \frac{\partial w_j}{\partial \ln x} = \frac{\partial w_k}{\partial \ln p_j} + w_j \frac{\partial w_k}{\partial \ln x}, \tag{4.8}
\]

requires that \( g(.) \) be linear.

**Proposition 6.** Suppose that budget shares have a form that is additive in functions of \( \ln x \) and demographics

\[
    w_j(\ln p, \ln x, z) = m_j(\ln p, z) + g_j(\ln p, \ln x) \tag{4.9}
\]

If (i) Slutsky symmetry (4.8) holds and (ii) the effects of demographics on budget shares are unrestricted in the sense that \( m_j \) can vary in any way with \( z \) then \( g_j(.) \) is linear in \( \ln x \):
Proof. See Appendix A.

This proposition demonstrates that the additive form given in (4.9) will only be consistent with utility maximisation if we restrict the way in which demographics affect budget shares, or if preferences are Piglog. That is $g_j(\ln x)$ is linear in $\ln x$ for all $j$.

An alternative specification that we adopt which does not impose restrictions on the form of $g_j$ is the following extension of the partially linear model

$$w_{ij} = g_j(\ln x_i - \phi(z_i'\theta)) + z_i'\alpha_j + \varepsilon_{ij} \tag{4.10}$$

in which $\phi(z_i'\theta)$ is some known function of a finite set of parameters $\theta$. This function is common across share equations and can be interpreted as the log of a general equivalence scale for household $i$. Interestingly, the extended partially linear model (4.10) is precisely the shape invariant specification considered in the work on pooling nonparametric regression curves by Härdle and Marron (1990) and Pinske and Robinson (1995).

4.3. A Semiparametric Estimator

To examine the shape invariant restrictions implicit in (4.10) we define $s = 0, 1, \ldots, S$ distinct household types of group size $n_s$ and let $z^s$ represent the corresponding demographic structure for each group normalised such that for the base group $s = 0,

$$\phi(z^0_i'\theta) = z^0_i'\alpha_j = 0.$$ 

The share equation for the base group (e.g. a couple with no children) becomes

$$w_{ij}^0 = g_j^0(\ln x_i) + \varepsilon_{ij}^0. \tag{4.11}$$

---

14 Blundell, Duncan and Pendakur (1998) compare the semiparametric specification used here with this more general alternative and find that it provides a good representation of demand behavior for households in the British FES used in this study.

15 For example, we may choose $\phi(z_i'\theta) = \ln(z_i'\theta)$ where $\theta$ is the vector of corresponding equivalence scales. See Pendakur (1998), for example.

16 In the remainder of this subsection we suppress the bandwidth parameter and use superscripts to represent the different demographic groups.
while for the remaining for \( s = 1, \ldots, S \) groups (e.g. couples with different numbers of children) the share equations become

\[ w_{ij}^s = g_j^s(\ln x_i - \phi(z_0^s \theta)) + z_i^s \alpha_j + \varepsilon_{ij}^s. \]

(4.12)

For any distinct household type \( z_i^s \) the shape invariance restrictions relative to the base group may be written

\[ g_j^s(\ln x_i) = g_j^0(\ln x_i - \phi(z_0^s \theta)) + z_i^s \alpha_j. \]

(4.13)

If the \( \alpha_j \) and \( \theta \) parameters for \( j = 1, \ldots, J - 1 \) were known then the shape restricted \( g_j \) could be estimated by kernel regression on the transformed data \( \ln x_i - \phi(z_0^s \theta) \) and \( w_{ij}^s - z_i^s \alpha_j \), pooled across the household types \( s = 0, 1, \ldots, S \). We replace the \( \alpha_j \) and \( \theta \) by \( \sqrt{n} \) consistent estimators and note that the asymptotic properties of the kernel regression estimates of \( g_j \) on the transformed data are unaffected. The choice of estimator for \( \alpha_j \) and \( \theta \) extends a method developed in the Härdle and Marron (1990) and Pinske and Robinson (1995) papers. The idea is to replace each \( g_j^s(\ln x_i) \) by its unrestricted Nadaraya-Watson kernel regression estimator and choose \( \alpha_j \) and \( \theta \) so as to minimise some weighted quadratic loss.\(^{17}\)

Define \( (\hat{\alpha}_j, \hat{\theta}) \) as the value of \( (\alpha_j, \theta) \) that minimises the integrated squared loss function

\[ L(\theta, \alpha) = \sum_{s=1}^{S} \sum_{j=1}^{J-1} \int_{\underline{x}}^{\bar{x}} (\Lambda_{js}(\ln x; \theta, \alpha_j))^2 \pi_x \omega_j(\ln x) d\ln x \]

(4.14)

where \( \alpha' = (\alpha'_1, \ldots, \alpha'_{J-1}) \) and where \( \underline{x} \) and \( \bar{x} \) are integration limits on the log of expenditure. The \( \Lambda_{js} \) term is given by

\[ \Lambda_{js}(\ln x; \theta, \alpha_j) = r_j^s f^0 - f_j^s(\ln x - \phi(z_0^s \hat{\theta})) - z_i^s \alpha_j f_j^s(\ln x - \phi(z_0^s \hat{\theta})) \]

(4.15)

\(^{17}\)In order to estimate these parameters there is no particular reason to use a kernel estimator for this shape invariant model. An attractive alternative semi-parametric estimator would be to adapt the sieve procedure in Ai and Chen (2000), for example.
where $\pi_s$ is a group specific weight ($n_s/n$ in our specification) and $\pi_j(\ln x)$ is an equation-specific weighting function. This choice is equivalent to using $(f^s f^0(\ln x - \phi(z_s' \theta)))^2$ as a weighting scheme for the Härdle and Marron (1990) estimator (4.14), and is precisely the estimator for random designs as suggested by Pinske and Robinson (1995). We apply this approach to our data using the predictions from the pooled model to estimate the $r^s_j$ terms in the loss function and implement a grid searching over a plausible range for $\phi$ to find the values for which the loss function attains a minimum within each year of our data.19

For the case where there are just two distinct groups $S = 1$ and one equation $J - 1 = 1$, Pinske and Robinson show $\sqrt{n}$-consistency and asymptotic normality of this estimator of $(\theta, \alpha)$. They also show that the first order asymptotic properties of the kernel regression estimator of $\tilde{g}$ under the shape invariant restrictions are unaffected by the use of $\tilde{\alpha}', \tilde{\theta}$ in place of $\alpha, \theta$.20 As noted above the latter result is particularly useful in our case as we are not directly interested in $\alpha, \theta$ but rather in $g_j$. Proposition 7 below extends their conditions for $\sqrt{n}$-consistency of $(b\alpha_0, b\theta)$ to the more general case of many groups and many equations. Given this result we can then proceed to estimate the nonparametric Engel curves pooled across household types using the transformed variables $(w_{ij} - z_s' \alpha_j)$ and $(\ln x_i - \phi(z_s' \theta))$.21

\[ g^s_j(\ln x) = z_s' \alpha_j + g^0_j(\ln x - \phi(z_s' \theta)) \iff \tilde{f}^0(\ln x - \phi(z_s' \theta)) r^s_j(\ln x) = \tilde{f}^* (\ln x) r^0_j(\ln x - \phi(z_s' \theta)) + \tilde{f}^* (\ln x) f^0(\ln x - \phi(z_s' \theta)) z_s' \alpha_j \quad (4.16) \]

for all $x$. To eliminate the random denominators in the kernel regression terms $g^s_j$ and $g^0_j$, the expression (4.15) can be weighted by the product of densities $f^* f^0$ where $f^*$ is evaluated at $\ln x_i$ and $f^0$ at $(\ln x_i - \phi(z_i' \tilde{\theta}))$.

We find that the optimum value for $\exp(\phi)$ lies in the range 1.2 to 1.5 over the period. We choose the value 1.29 which is very close to the average of our estimates, and is the OECD equivalence scale. See Blundell, Duncan and Pendakur (1988) for a further discussion of the estimation of this equivalence scale parameter.

In proving this result Pinske and Robinson (1995) allow for a different bandwidth, $nh^2 \rightarrow \infty$ and $h \rightarrow 0$ as $n \rightarrow \infty$, which is more than satisfied by our choice of bandwidth which is proportional to $n^{1/5}$.

21 An alternative estimator would be to adapt a minimum distance estimator for conditional moment
Proposition 7  Let \((\hat{\alpha}', \hat{\theta}')\) be the values of \((\alpha_j, \theta)\) that minimise the integrated squared loss function (4.14). Under assumptions A1 - A8 (see Appendix A), \((\hat{\alpha}', \hat{\theta}')\) is a \(\sqrt{n}\)-consistent estimator for \((\alpha_0, \theta_0)\).

Proof. See Appendix A. ■

One important requirement for this proposition to hold (Assumption A6 in Appendix A) is that
\[
\int f^s f^0 (\ln x - \phi(z^s \theta))
\]
is bounded away from zero at the true parameter value for \(\theta\). In our application we distinguish household types by family size with the base group being a couple without children and choose the log transformation for the equivalence scale function \(\phi\). Since the scale for children relative to a childless couple is assumed to be bounded between zero and one half for each child, condition A6 is preserved.

4.4. Unobserved Heterogeneity

We turn now to the relationship between nonparametric Engel curves and the average demands for a set of heterogeneous agents. For this discussion we omit dependence on observed characteristics \(z\). There are two alternative ways of interpreting the impact of heterogeneity on the average demands estimated from nonparametric Engel curve regression. We could assume individual demands are rational and then ask for conditions on preferences and/or heterogeneity that imply rationality for average demands. This is the approach of McElroy (1987), Brown and Walker (1991) and Lewbel (1996). Alternatively, we could make no rationality assumptions on individual demands and simply ask what conditions enable average demands to satisfy rationality properties. This is the approach of Becker (1962), Grandmont (1992) and Hildenbrand (1994).

Suppose for each good \(j\) we write average budget shares as
\[
E[w_j|\ln x, p] = g_j (\ln x, p)
\]
(4.17)
restrictions. This could also allow for endogeneity using an instrumental variable approach. A first attempt at this for the shape invariant Engel curve model is presented in Blundell, Chen and Kristensen (2001).
then, if we let $\epsilon$ represent a vector of unobserved heterogeneity terms with $E[\epsilon | \ln x, p] = 0$, a necessary condition for the average budget shares recovered by the nonparametric analysis discussed above to be equal to average budget shares is that

$$w_j = g_j (\ln x, p) + \phi_j (\ln x, p)' \epsilon.$$  

(4.18)

Notice this allows for quite different tastes across agents. In particular, the first-order price and income responses for agents can vary in any way. Thus a good may be a luxury for one person and a necessity for another.

The function $g_j (\ln x, p)$ gives mean responses to changes in prices conditional on a given level of total expenditure. Thus we can use this function for positive analysis, for example to recover the revenue implications from a change in taxes. Additionally, the utility function that is associated with an integrable set of demands $g_j (\ln x, p)$ is a prime candidate for use in equilibrium models that assume a representative agent. In our analysis below we apply the GARP tests to the mean function $g_j (\ln x, p)$. This averaging is very different to the standard aggregation structure in consumer theory developed by Gorman (1954) and Muellbauer (1976). In particular, we are not aggregating across different total budgets (incomes). Additionally, we are not assuming that individual demands are necessarily integrable; that is, for given $\epsilon$ we can have that the Slutsky conditions may fail for $w_j (\ln x, p, \epsilon)$. In this respect, our structure is closer to that of Hildenbrand (1994) and Grandmont (1992). However, their analysis shows conditions for average demands to satisfy the Weak Axiom of Revealed Preference (WARP, see Varian (1982)) but GARP requires more. GARP implies the Slutsky symmetry conditions. If we wish to impose integrability at the individual level then there are restrictions on the $\phi_j (x, p)$ and the distribution of the heterogeneity terms (see McElroy (1987), Brown and Walker (1989) and Brown and Matzkin (1995)). Indeed, Brown and Walker (1989) show that for Slutsky symmetry to hold $\phi_j (x, p)$ must be either a function of $x$ or $p$.

\footnote{If all preference parameters are to be heterogeneous then preferences are essentially restricted to the class of Piglog demands (see Lewbel (1996), for example).}
The reason that we are interested in testing for GARP using these mean responses is that without such a rationality condition holding, it is difficult to see how we would ever conduct coherent welfare analysis of non-marginal price changes. The heterogeneity conditions for using the mean function for the welfare analysis for consumers of a non-marginal price change are, however, stronger than the conditions given in (4.18) which suffice for positive analysis. In an important paper McElroy (1987) considers the case of estimating cost function and share equation parameters for production analysis. For consumer welfare measures these results need to be extended. Consider the welfare measure based on the second-order approximation\(^{23}\) of the log cost function for a non-marginal price change \(\Delta \ln p_j\)

\[
E \left[ \frac{\Delta \ln c}{\Delta \ln p_j} | x, p \right] = \left[ w_j | x, p \right] + \frac{1}{2} E \left[ S_{jj} | x, p \right] \Delta \ln p_j. \tag{4.19}
\]

where \(S_{jj}\) is the Slutsky substitution term

\[
S_{jj} = \frac{\partial w_j}{\partial \ln p_j} + \frac{\partial w_j}{\partial \ln x} w_j.
\]

Consequently in addition to the direct price effect on the share (4.19) includes the compensating income effect \(\frac{\partial w_j}{\partial \ln x} w_j\). This introduces a bias term additional to that considered in McElroy (1987). Using (4.18) the mean welfare measure (4.19) has the form

\[
E \left[ \frac{\Delta \ln c}{\Delta \ln p_j} | x, p \right] = g_j + \frac{1}{2} \left( \frac{\partial g_j}{\partial \ln p_j} + \frac{\partial g_j}{\partial \ln x} g_j \right) \Delta \ln p_j + \frac{1}{2} \frac{\partial g_j}{\partial \ln x} \Omega_x \phi_j \Delta \ln p_j. \tag{4.20}
\]

where \(E \{ \varepsilon \varepsilon' | x, p \} = \Omega_\varepsilon\). The first two terms on the right hand side of this expression can be computed using the mean function \(g_j(.)\) so that our mean function gives an exact first-order welfare effect. It also gives second order effects if the final bias term is zero.

\(^{23}\)See Banks, Blundell and Lewbel (1996), for example.
This will be the case if, for example, the heterogeneity term \( \phi(\ln x, p) \) is independent of total expenditure so that all households have the same marginal income effects.\(^{24}\)

In general the error term in (4.18) will represent measurement and optimisation error as well as preference heterogeneity so it would seem natural to work with local average demands. Averaging locally to each \( x \) eliminates unobserved heterogeneity, measurement error and (zero mean) optimisation errors in demands but preserves any nonlinearities in the Engel curve relationship for each price regime.

5. Endogeneity and Semiparametric Regression

There are both theoretical and empirical reasons why the total expenditure is likely to be endogenous. Suppose \( \ln x \) is endogenous in the sense that for each commodity \( j \)

\[
E(\varepsilon_{ij} | \ln x_i) \neq 0 \text{ or } E(w_{ij} | \ln x_i) \neq g_j(\ln x_i).
\]

(5.1)

In this case the nonparametric estimator will not be consistent for the function of interest. To be precise, it will not provide the appropriate counterfactual: how will expenditure share patterns change for some ceteris paribus change in total expenditure?

We consider two alternative, and non-nested, approaches to nonparametric estimation with endogenous regressors: the Control Function estimator and the Instrumental Variable estimator. These are reviewed in more detail in Blundell and Powell (2001).

5.1. A Control Function Approach

To adjust for endogeneity we adapt the control function or augmented regression technique (see Holly and Sargan (1982), for example) to the semiparametric Engel curve framework. Consider first the nonparametric Engel curve (4.1). Suppose \( \ln x \) is endogenous in the sense that for each commodity \( j \)

\[
E(\varepsilon_{ij} | \ln x_i) \neq 0 \text{ or } E(w_{ij} | \ln x_i) \neq g_j(\ln x_i).
\]

(5.2)

\(^{24}\)Note, however, that this condition is sufficient and not necessary; weaker assumptions suffice to make the bias term zero or small.
In this case the nonparametric estimator will not be consistent for the function of interest. To be precise, it will not provide the appropriate counterfactual: how will expenditure share patterns change for some \textit{ceteris paribus} change in total expenditure?

Suppose there exist instrumental variables $\zeta_i$ such that

\[
\ln x_i = \pi' \zeta_i + v_i \text{ with } E(v_i | \zeta_i) = 0. \tag{5.3}
\]

In the application below we take the log of disposable income as the excluded instrumental variable for log total expenditure, $\ln x$. Further, we make the following key assumptions

\[
E(w_{ij} | \ln x_i, \zeta_i) = E(w_{ij} | \ln x_i, v_i) \tag{5.4}
\]

\[
= g_j(\ln x_i) + \rho_j v_i \forall j. \tag{5.5}
\]

This implies the augmented regression model

\[
w_{ij} = g_j(\ln x_i) + \rho_j v_i + \tilde{\varepsilon}_{ij} \forall j \tag{5.6}
\]

with

\[
E(\tilde{\varepsilon}_{ij} | \ln x_i) = 0 \forall j. \tag{5.7}
\]

Note that $g_j(\ln x_i) = E(w_{ij} | \ln x_i) - E(v_i | \ln x_i)$ eliminating $g_j(\ln x_i)$ using (5.6) yields

\[
w_{ij} - E(w_{ij} | \ln x_i) = (v_i - E(v_i | \ln x_i))\rho_j + \tilde{\varepsilon}_{ij} \tag{5.8}
\]

which suggests a weighted instrumental variable estimator for $\rho_j$ by replacing the conditional means $E(w_{ij} | \ln x_i)$ and $E(v_i | \ln x_i)$ by their Nadaraya-Watson kernel regression estimators $\hat{w}(\ln x_i)$ and $\hat{v}(\ln x_i)$ respectively. Suitable instruments would be $I[\hat{f}(\ln x_i) > b_i]v_i$.

The resulting estimator of $g(\ln x_i)$ is given by

\[
\hat{g}(\ln x_i) = \hat{w}(\ln x_i) - \hat{v}(\ln x_i)\hat{\rho}_j. \tag{5.9}
\]
Note that the unobservable error component $v$ in (5.8) is unknown. In estimation $v$ is replaced with the first stage reduced form residuals

$$
\tilde{v}_i = \ln x_i - \hat{\pi}' \zeta_i
$$

(5.10)

where $\hat{\pi}$ is the least squares estimator of $\pi$. Since $\hat{\pi}$ and $\hat{\rho}$ converge at $\sqrt{n}$ the asymptotic distribution for $\hat{g}(\ln x_i)$ follows the distribution of $\hat{g}(\ln x_i) - \hat{\pi}(\ln x_i)\rho_j$. Moreover, a test of the exogeneity null $H_0 : \rho_j = 0$, can be constructed from this least squares regression.\(^{25}\) In application we apply this procedure by augmenting the semiparametric model (4.10).

For the shape invariant regression model (4.12)

$$
w_{ij}^s = g_j^s(\ln x_i - \phi(z_i^s \theta)) + z_i^s \alpha_j + \epsilon_{ij}^s \forall j.
$$

(5.11)

5.2. Nonparametric Instrumental Variables

Write the semiparametric shape invariant model as

$$
w_{ij} = g_j(\ln x_i - \phi(z_i \theta)) + z_i \alpha_j + \epsilon_{ij}
$$

With instruments $\zeta_i$ the moment condition underlying the IV estimator is given by

$$E[\epsilon_{ij} | z_i, \zeta_i] = 0, \quad j = 1, ..., J.
$$

First consider identification. Let $X_i = \{w_{ij}, \ln x_i, z_i, \zeta_i\}$ and $Z_i = \{z_i, \zeta_i\}$.

Define

$$\rho_j(X_i, \theta, \alpha_j) = w_{ij} - g_j(\ln x_i - \phi(z_i \theta)) - z_i \alpha_j
$$

\(^{25}\)This method can be viewed as a special case of the method proposed in Newey, Powell and Vella (1999). They adopt a series estimator for the regression of $w$ on $\ln x$ and $v$. This generalises the form of (5.3) and (5.6). We chose not to follow the fully nonparametric control function approach here for two reasons. First in Blundell, Duncan and Pendakur (1998) it is shown that adding additional terms makes little difference for estimating Engel curves on a sample from a single year of British Family Expenditure Survey data. Second, for the computations in this study we would also have to make this adjustment for each share equation in each time period and also to adjust the asymptotics accordingly.
and note the moment condition:

\[ E[\rho(X_{i}, \theta_{0}, \alpha_{o})|Z_{i}] = 0. \]

**Condition I (Identification):**

\[ E[w_{ij} - g_{j}(\ln x_{i} - \phi(z_{i}^{*}\theta)) - z_{i}^{*}\alpha_{j}|Z_{i}] = 0 \]

implies \( \theta = \theta_{o}, \alpha_{j} = \alpha_{oj} \) and \( g_{j} = g_{oj} \) a.s.

This is proved in Blundell, Chen and Kristensen (2001).

The suggested estimation method is similar to those in Newey and Powell (1989) for nonparametric IV regression, and Ai and Chen (1999) for semiparametric conditional moment restrictions. First we approximate the unknown functions \( g \) in some sieve space, that is, some finite-dimensional approximation spaces (e.g. Fourier series, orthogonal polynomials, splines, power series, wavelets, etc.) which become dense as sample size \( n \to \infty \). Then for each fixed value of the parameter vector we estimate the population conditional moment function \( m(x, \alpha) \). Consider the population conditional moment function \( m(Z, \alpha) : \)

\[ E[\rho(X_{i}, \theta, \alpha)|Z_{i} = Z] = m(Z, \theta, \alpha) \text{ for any } \theta, \alpha. \]

Estimate the \( \theta \) from

\[ \min_{\theta, \alpha \in \mathcal{A}_{n}} \frac{1}{n} \sum_{i=1}^{n} \hat{m}(Z_{i}, \theta, \alpha)'[\hat{\Sigma}(Z_{i})]^{-1}\hat{m}(Z_{i}, \theta, \alpha). \]

A semiparametric efficient estimator is obtained in three steps:

**Step 1.** Obtain an initial consistent estimator \( \hat{\alpha}_{n}, \hat{\theta}_{n} \)

\[ \min \frac{1}{n} \sum_{i=1}^{n} \hat{m}(Z_{i}, \theta, \alpha)'\hat{m}(Z_{i}, \theta, \alpha), \]

where \( \hat{m}(Z_{i}, \theta, \alpha) \) is a nonparametric consistent estimator of \( m(Z_{i}, \theta, \alpha) \) uniformly over \((Z_{i}, \theta, \alpha) \in \mathcal{X} \times \mathcal{A}_{n}.\)
Step 2. Obtain a consistent estimator $\hat{\Sigma}_o(Z)$ of the optimal weighting matrix $\Sigma_o(Z) \equiv Var[\rho(Z, \theta_o, \alpha_o)|Z]$ using $\hat{\alpha}_n, \hat{\theta}_n$ and any nonparametric regression procedures (such as kernel, nearest-neighbor or sieves).

Step 3. Obtain the optimally weighted estimator of $(\hat{\alpha}_n, \hat{\theta}_n, \hat{g}_n)$ by solving

$$\min_{\alpha \in A_n} \frac{1}{n} \sum_{i=1}^{n} \rho(X_i, \theta, \alpha)'[\hat{\Sigma}_o(Z_i)]^{-1} \rho(X_i, \theta, \alpha).$$

Under exogeneity of total expenditure we have

$$\min_{\alpha \in A_n} \frac{1}{n} \sum_{i=1}^{n} \rho(X_i, \theta, \alpha)'\hat{\Sigma}(X_i)^{-1} \rho(X_i, \theta, \alpha).$$

6. An Empirical Investigation on Repeated Cross-Sections

6.1. Data

The data were drawn from the repeated cross-sections of household-level data in the British Family Expenditure Survey (1974 to 1993). The FES is a random sample of around 7,000 households per year. From this we used a sub-sample of all the two-adult households both those with and those without children. The first and last percentiles of the within-year total expenditure distribution in this sub-sample was then trimmed out. This selection resulted in a sample size of 75,753 households (between 3,386 and 4,086 in each year). Expenditures on non-durable goods by these households were aggregated into 22 commodity groups and chained Laspeyres price indices for these groups were calculated from the sub-indices of the UK Retail Price Index giving 20 annual price points for each of our 22 commodity groups.

The commodity groups are non-durable expenditures grouped into: beer, wine, spirits, tobacco, meat, dairy, vegetables, bread, other foods, food consumed outside the home, electricity, gas, adult clothing, children’s clothing and footwear, household services, personal goods and services, leisure goods, entertainment, leisure services, fares,

\[26\] A further selection of households with cars was made in order to allow us to include motoring expenditures and, in particular, petrol as commodity groups.
motoring and petrol\textsuperscript{27}. Descriptive statistics for total nominal expenditure are given in Table B.1 of appendix D.

For a more detailed analysis of the data we turn to the total budget variable in our Engel curve analysis. This is typically transformed by the log transformation as total outlay is often supposed to have a normal cross section distribution. To see the power of the kernel method Figure 6.1 presents the (Gaussian) kernel density estimation using a group of around 1000 household from the UK Family Expenditure Survey. These are married couples with no children so as to keep a reasonable degree of homogeneity in the demographic structure.

The results are interesting showing that it is relatively difficult to tell apart the nonparametric density from the fitted normal curve which is also shown. The bivariate

\textsuperscript{27}More precise descriptions of components of the commodity groups are available from the authors.
kernel density plot in 6.2 indicate that the joint density of food expenditure share and log total expenditure seems close to bivariate normal, with strong negative correlation.

6.2. The Shape of Engel Curves (Expansion Paths)

To provide an idea of the importance of allowing flexibility in the shape of the Engel curve relationship, figures 6.3 and 6.4 present Kernel regressions for the Engel curves of two commodity groups in the FES, together with a quadratic polynomial regression. These curves are presented for a relatively homogeneous group of married women without children, although we will subsequently discuss how socio-demographic heterogeneity might be accommodated in kernel-based regression techniques.
Figure 6.3: Nonparametric Engel Curve: food share

Figure 6.4: Nonparametric Engel curve: alcohol share
6.3. A Graphical Analysis of the Shape Invariance Restrictions

For a graphical comparison of the alternative specifications we first consider the shape invariant restricted models without endogeneity correction Figures 6.5 - 6.10. The solid line in each figure is the reference curve (for exactly one child). The second (dotted) line is the shape invariant curve for families with two children. The third (hashed) line is the unrestricted equivalent kernel regression curve for families with two children. Note that shape invariant and unrestricted curves, at least for food shares, are quite comparable. For alcohol the case for the shape invariant transformation is less clear.

6.4. Engel Curves over Time

The three figures 6.11, 6.12 and 6.13 below are taken from Blundell, Browning and Crawford (2003a) and show the estimated Working-Leser Engel curves (budget share against log total nominal expenditure) for 3 of our 22 commodities, for 3 of our 20 periods (1975 (circles), 1980 (squares), 1985 (triangles)). These are at a more
Figure 6.6: shape invariant transformation: fuel share

Figure 6.7: shape invariant transformation: clothing share
Figure 6.8: shape invariant transformation: alcohol share

Figure 6.9: shape invariant transformation: transport share
disaggregated than the discussion of the shape of Engel curves above. They represent a typical necessity (bread), a luxury (entertainment) and beer which roughly displays a quadratic logarithmic Engel curve behaviour. As described above the nonparametric regression results are based on a Gaussian Nadaraya-Watson kernel estimation under the shape invariance restrictions.\(^\text{28}\) Adaptive kernel bandwidths\(^\text{28}\) were used throughout with the first round bandwidth chosen by cross-validation \([\text{cf. Härdle (1990)}]\).

On each Engel curve in Figures 6.11 - 6.13 we plot the points on the chronological SMP paths which correspond to the 1st, 10th, 25th, 50th, 75th, 90th and 99th percentile points in the base year (1974). Pointwise 95% confidence bands at these points are also drawn. Note that, as we would expect, the precision is much lower at the tails of the outlay distribution. The left to right drift of the Engel curves apparent in these figure illustrates the growth in nominal expenditure which took place between these periods.

\(^{28}\) The adaptive bandwidth is \(h = h \lambda\) where \(h\) is the pilot bandwidth and \(\lambda_i = \left[\frac{\hat{f}_i(h \ln x_i)}{\eta}\right]^{-\frac{1}{2}}\) where \(\eta = \exp\left[\sum_i \ln \hat{f}_i(h \ln x_i)\right]\) see Blundell and Duncan (1998).
Figure 6.11: The Engel curve for Bread

Figure 6.12: The Engel curve for Entertainment
7. Assessing the Validity of the Revealed Preference Restrictions

7.1. Using the Sequential Maximum Power Path

At each stage in the empirical analysis of the GARP conditions we will be comparing weighted sums of kernel regressions. The pairwise comparison \( p_t'q_t > p_t'q_s \) can be written

\[
x_t > \sum_{j=1}^{J} \frac{p_j^t}{p_j^s} \hat{g}_j(x_s)x_s \text{ for } s \neq t.
\]

(7.1)

where \( \hat{g}_j(x_s) \) is the estimated budget share in equation (4.1). Noting that adding-up implies \( \sum_{j=1}^{J} \hat{g}_j(x_t) \equiv 1 \) for all \( t \), condition (7.1) conveniently reduces to the comparison

\[
\delta_{ts} > \sum_{j=1}^{J-1} \gamma_{ts}^j \hat{g}_j(x_s),
\]

(7.2)

where \( \gamma_{ts}^j = \left( \frac{p_j^t}{p_j^s} - \frac{p_j^t}{p_t^s} \right) \) and \( \delta_{ts} = \left( \frac{x_t}{x_s} - \frac{p_t^j}{p_t^s} \right) \) are known constant weights in each price regime.

To test GARP we will need to evaluate the inequality (7.2) at particular points on an SMP path. Since the nonparametric Engel curve has a pointwise asymptotic normal
distribution we can evaluate the distribution of each \( \hat{g}_j^t(x) \) at any point \( x \).\(^{29}\) For (7.2) we need to find the distribution of the weighted sum of correlated kernel regression estimates \( \sum_{j=1}^{J-1} \gamma_{ts} \hat{g}_j^t(x) \). However, since on any SMP path in any period the \( \hat{g}_j^t(x) \) kernel estimates for each good \( j \) are to be evaluated using the same kernel smoother and the same bandwidth, the expression for the asymptotic variance of the weighted sum simplifies. In particular, the constants associated with the kernel function and the density \( f_h(x) \) itself will be common to all variance and covariance terms. Pointwise standard errors and confidence bands for expression (7.2) are therefore tractable and are used extensively in the empirical application below.

When calculating demands on SMP paths we allow for the fact that the total expenditure levels in all periods except for the first are chosen on the basis of the estimated demands in the previous periods. For example, a SMP path constructed such that \( \tilde{x}_t = p'_t q_t = p'_t q_{t-1}(x_{t-1}) \), the expenditure level \( \tilde{x}_t \) is set such that \( \tilde{x}_t = \sum_{j=1}^{J} \frac{p'_j}{\pi'_t} \hat{g}_j^t(x_{t-1})x_{t-1} \) and therefore \( \tilde{x}_t \) depends on the estimate of \( \hat{g}_t(x_{t-1}) \) from the previous period. The test of \( q_t - P^0 q_t \) requires that we have an estimate of \( \text{Var}(\delta_{t-1,t} - \gamma_{t-1,t} \hat{g}_t) \) and that this takes into account that \( \tilde{x}_t \) is set according to estimates of \( g_{t-1}(x_{t-1}) \) (and likewise that \( \tilde{x}_s \) is set according to estimates of \( \hat{g}_r(x_r) \) etc.). This is derived using the standard delta-method approach applied sequentially.

\(^{29}\)Briefly, for bandwidth choice \( h \) and sample size \( n \) the variance can be well approximated at point \( x \) for large samples by

\[
\text{var}(g'(x)) \approx \frac{\sigma_j^2(x)c_K}{nhf_h(x)}
\]

where \( c_K \) is a known constant and \( f_h(x) \) is an (estimate) of the density of \( x \)

\[
\sigma_j^2(x) = n^{-1} \sum_{j=1}^{n} \omega_{jh}(x)(w_{ij} - g_j(x))^2
\]

with weights from the kernel function

\[
\omega_{jh}(x) = K_h(x - x_j)/f_h(x)
\]

see Härdele (1990).
To implement our procedure we need to choose a set of SMP paths along which to evaluate GARP. To do this we select the starting points for each path to be at the 1st percentile, 1st decile, 1st quartile, median, 3rd quartile, 9th decile and 99th percentile points in the $x$ distribution for 1974, the first year in our data set. The comparison points for the following years are chosen along the SMP path as described in section 2.2. By Proposition 1 we know that if this path passes GARP then no path which preserves the same preference ordering will violate GARP. The annual median and mean (non-SMP) paths are also computed for comparison.

Table 7.1 shows the number and pattern of rejections for the system of 22 goods. Each column provides a count of the total number of rejections according to inequality (7.2). In each case a one sided test of size $\alpha$ is used, based on the pointwise asymptotic distribution of $\sum_{j=1}^{J-1} \gamma_{ts}^j g_j^s(x_s)$. The column headed $\alpha = 1$ counts the number of rejects using inequality (7.2) directly without adjustment for estimation error in $g_j^s(x_s)$. In the remaining columns each inequality is adjusted by a one sided interval. From the
first of these columns GARP can be seen to be rejected for a large number of points, especially in the upper tail of the outlay distribution. However, these rejections are not statistically significant. Very little adjustment is needed to dramatically reduce the number of rejections.

It is also interesting to observe that there are no rejections, even in the raw data, for the median or mean (non-SMP) paths. This is consistent with the observation which arises in tests of GARP on aggregate data that if the budget constraint is allowed to shift much either way between comparison points, as it does for median or mean total expenditure, then there is little chance of being able to find demands that cannot be rationalised.

7.2. Sub-Periods Which Satisfy GARP

Using the same set of SMP comparison points as in table 7.1, table 7.2 presents the continuous sub-periods of the data that satisfy GARP. For example, the chronological SMP path which starts at median total budget in 1974 runs into a violation of GARP when 1986 is added to the sequence. In this case it is the pair of years 1985 and 1986 which fail to satisfy GARP: the SMP path is constructed to reflect the ordering \( q_{1986} R^0 q_{1985} \) but we find that \( q_{1985} P^0 q_{1986} \), giving the violation.

Interestingly, the table also shows the largest continuous sub-period in which we are able to bound the indifference curve. For example, using the reference demand bundle at median total outlay in 1974 we are able to bound a curve using the expansion paths and price data for 1974 to 1985 inclusive (we are also able to bound curves using reference demands at any within-year median total expenditure level or reference demands at any

---

30 Our interest is primarily in the points commonly used in the analysis of income distributions, i.e. interdecile points, interquartile points and the median. We include the 1st and the 99th percentile points for completeness.

31 We have not attempted to compute the size of the implicit joint test.
Table 7.2: Continuous periods that satisfy GARP.

<table>
<thead>
<tr>
<th>Periods</th>
<th>74</th>
<th>75</th>
<th>76</th>
<th>77</th>
<th>78</th>
<th>79</th>
<th>80</th>
<th>81</th>
<th>82</th>
<th>83</th>
<th>84</th>
<th>85</th>
<th>86</th>
<th>87</th>
<th>88</th>
<th>89</th>
<th>90</th>
<th>91</th>
<th>92</th>
<th>93</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10th</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25th</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50th</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>75th</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>90th</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>99th</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

point on the chronological SMP path between 1974 and 1985). However, if we add 1986 to the set of admissible periods the algorithm fails to converge. We then start again using the 1986 point on the median SMP path as our starting point. In all, for the median we find the entire period separates down into two sub-periods within which we are able to bound an indifference curve. Similarly the 1st and 9th decile paths break into two and four sub-periods respectively, while the 99th percentile breaks down into five.

7.3. Imposing GARP Restrictions

Let \( \hat{W} \) be the \((K \times T)\) matrix of estimated budget shares with element \( \hat{w}_{kt} \), let \( P \) be the corresponding \((K \times T)\) matrix of prices. Denote the true budget shares by \( W \) with element \( w_k \). Let \( e(P, W, x) \) be a procedure which returns the maximal Afriat efficiency parameter such that the data \( \{P, W, x\} \) satisfies GARP, where \( x \) is a \((1 \times T)\) row vector of budgets.

\[
e(P, W, x) = \max \{ e \mid q_t R_e q_s \text{ implies } e p'_t q_s \leq p'_t q_t \text{ for all } s, t \}
\]

where \( q_t \) is a \((K \times 1)\) quantity vector with element \( q^k_t = w^k_t x_t p^{-1}_t \) and where \( R_e \) denotes the transitive closure of \( R^0_e \) and \( q_t R^0_e q_s \Leftrightarrow e p'_t q_t \geq p'_t q_s \). When \( e(P, W, x) = 1 \) the GARP restrictions are satisfied.
From this we can also derive the test statistic:

$$\chi = \min_{\mathbf{W}} \left\{ \text{vec}(\mathbf{W} - \mathbf{W})' \text{vec}(\mathbf{W} - \mathbf{W}) | 0 \leq w_i^k \leq 1, \sum_{k=1}^{K} w_i^k = 1, e(P, W, x) = 1 \right\}.$$ 

We can use this knowledge of periods in which GARP is satisfied in a number of ways. To illustrate two of them we present bounds on the base-period reference cost-of-living index, and bounds on year-to-year inflation rates.

8. Bounds on Cost-of-living Indices and Inflation

8.1. Cost of Living Indices for Different Income Groups

Table 7.2 shows that preferences on the SMP paths starting at the 10th and 90th percentile points, the quartiles and the median of the base period total budget distribution all satisfy integrability at least up until 1985. We use the data for this period and the algorithms described in section 2.4 to bound the true cost-of-living index $c(p_{85}, q_{74})/c(p_{75}, q_{74})$ for a reference demand bundle at each of these points in the 1974 total budget distribution. Figure 8.1 shows the bounds for each reference budget in 1985, with 1974=1000. It is interesting to note that the bounds for 10th and 90th percentile points do not overlap and indicate greater rise on the cost of living for poorer, compared to richer, households over this period.

We also compare the performance of the GARP bounds for the true index with other nonparametric bounds and other popular price index formulae over a longer period. This is shown in table 8.1. The first panel shows the price index numbers for the Paasche, Laspeyres and the chained Törnqvist. These indices can also be thought of as corresponding exactly to true indices under various assumptions regarding the precise form of preferences\footnote{The Paasche and Laspeyres, for example, are exact for Leontief preferences, the Törnqvist is exact for translog.}. The second panel in table 8.1 shows various nonparametric bounds on the true index referenced at $q_{74}$ where $q_{74} = q_{74}(x)$ evaluated at 1974 median total.
budget. The bounds provided by Lerner (1935-36) are simply a reflection of the idea that the true index (being a weighted average of price changes) must lie somewhere between the maximum and the minimum ratio of the price changes of all goods: i.e.

$$\min_j \left\{ \frac{p_j^t}{p_j^{74}} : j = 1, ..., J \right\} \leq \frac{c(p_t, q_{74})}{c(p_{74}, q_{74})} \leq \max_j \left\{ \frac{p_j^t}{p_j^{74}} : j = 1, ..., J \right\}.$$  

Pollak (1971) improves this by linking Lerner’s result with the original Konüs (1924) result that the Laspeyres index approximates the true base-referenced cost of living index from above, i.e.

$$\min_j \left\{ \frac{p_j^t}{p_j^{74}} : j = 1, ..., J \right\} \leq \frac{c(p_t, q_{74})}{c(p_{74}, q_{74})} \leq \frac{p_t^t q_{74}}{p_{74} q_{74}}.$$  

The bounds from classical revealed preference restrictions of the type used by Varian (1982) and calculated using the demands in each period at median within-period total budget are also reported (labelled classical revealed preference (RP)). None of these nonparametric solutions have any trouble in providing bounds for the entire period.
The classical bounds for example, do not violate GARP for the reasons explained above. However, the bounds derived by our method must take account of the break between 1985 and 1986. This is because when we seek to derive the bounds using the data from both 1985 and 1986 the algorithms do not converge (convergence requires GARP as shown in propositions 2 and 3). Instead we bound the indifference curves using prices and expansion paths from all periods excluding 1986. We then use these to bound the cost-of-living index using all of the price data (including 1986) as described in 2.2.

Table 8.1: Popular price indices, nonparametric and GARP bounds, 1974 to 1993.

<table>
<thead>
<tr>
<th>Year</th>
<th>Price Indices</th>
<th>Nonparametric/RP bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>P</td>
<td>L</td>
</tr>
<tr>
<td>74</td>
<td>1000</td>
<td>1000</td>
</tr>
<tr>
<td>75</td>
<td>1215</td>
<td>1232</td>
</tr>
<tr>
<td>76</td>
<td>1516</td>
<td>1530</td>
</tr>
<tr>
<td>77</td>
<td>1762</td>
<td>1787</td>
</tr>
<tr>
<td>78</td>
<td>1931</td>
<td>1957</td>
</tr>
<tr>
<td>79</td>
<td>2086</td>
<td>2119</td>
</tr>
<tr>
<td>80</td>
<td>2463</td>
<td>2514</td>
</tr>
<tr>
<td>81</td>
<td>2780</td>
<td>2841</td>
</tr>
<tr>
<td>82</td>
<td>3093</td>
<td>3189</td>
</tr>
<tr>
<td>83</td>
<td>3260</td>
<td>3381</td>
</tr>
<tr>
<td>84</td>
<td>3408</td>
<td>3558</td>
</tr>
<tr>
<td>85</td>
<td>3551</td>
<td>3733</td>
</tr>
<tr>
<td>86</td>
<td>3700</td>
<td>3911</td>
</tr>
<tr>
<td>87</td>
<td>3825</td>
<td>4035</td>
</tr>
<tr>
<td>88</td>
<td>3922</td>
<td>4163</td>
</tr>
<tr>
<td>89</td>
<td>4130</td>
<td>4379</td>
</tr>
<tr>
<td>90</td>
<td>4406</td>
<td>4669</td>
</tr>
<tr>
<td>91</td>
<td>4723</td>
<td>5044</td>
</tr>
<tr>
<td>92</td>
<td>4996</td>
<td>5437</td>
</tr>
<tr>
<td>93</td>
<td>5177</td>
<td>5650</td>
</tr>
</tbody>
</table>

Notes: P = Paasche, L = Laspeyres, T = Chained Törnqvist/Divisa
We find, confirming the results in Varian (1982) and Manser and McDonald (1988), that classical non-parametric/revealed preference bounds based on the median demand data gives little additional information on the curvature of the indifference curve through commodity space and hence the bounds on the true index are wide. However, by the use of expansion paths we can dramatically improve these bounds. This is illustrate in figure 8.2 in which the GARP bounds are represented by the solid lines and the classical revealed preference bounds by the dashed line.

Comparing the GARP bounds on the true, fixed base cost of living index to the three price index number formulae we see that the chained Törnqvist. performs the best as an empirical approximation to the true index\textsuperscript{33}. This is despite the fact that, as an index in which reference utility is updated in each period, the Törnqvist. cannot strictly be compared to a fixed base true index. The Laspeyres, which is a first order approximation to the true index in question, understates the true increase in the cost of living by between about 3% and 5% by the end of the period.

\textsuperscript{33}Comparisons with other price indices are available from the authors.
8.2. Bounds on Inflation Rates

As well as deriving bounds on fixed base cost-of-living indices, revealed preference restrictions can also be calculated on annual inflation rates in which the reference demand bundle is updated in each period. The 1990 annual inflation bound rate for example is calculated from the bound on the 1989-based cost-of-living index $c(p_{90}, q_{89}) / c(p_{89}, q_{89})$. The results are shown in Table 8.2 where the improvement afforded by the GARP bounds over the previously available nonparametric bounds is apparent. Indeed the tightness of the GARP bounds is remarkable. Again the Törnqvist performs the best of the index number formulae followed by the Laspeyres which is often close to the top of the GARP bounds. Note that the inflation rate for the year to 1986 is missing from the GARP bounds because of the GARP violation between these two years.

9. Bounds on Price Responses

9.1. Own and Cross Demands

In this section (to be completed) the nonparametric expansion paths and revealed preference theory are used to bound demand curves. We focus on the Alcohol and Tobacco items as a group. Below we use this analysis to bound the responses and welfare impact of an indirect tax reform to these commodities.
Table 8.2: Annual inflation rates for popular price indices, nonparametric and GARP bounds, 1975 to 1993.

<table>
<thead>
<tr>
<th>Year</th>
<th>Price Indices</th>
<th>Nonparametric/RP bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>P</td>
<td>L</td>
</tr>
<tr>
<td>75</td>
<td>21.48</td>
<td>23.16</td>
</tr>
<tr>
<td>76</td>
<td>24.60</td>
<td>25.27</td>
</tr>
<tr>
<td>77</td>
<td>16.54</td>
<td>16.89</td>
</tr>
<tr>
<td>78</td>
<td>9.85</td>
<td>10.05</td>
</tr>
<tr>
<td>79</td>
<td>8.13</td>
<td>8.31</td>
</tr>
<tr>
<td>80</td>
<td>18.29</td>
<td>18.74</td>
</tr>
<tr>
<td>81</td>
<td>12.92</td>
<td>13.11</td>
</tr>
<tr>
<td>82</td>
<td>11.53</td>
<td>12.18</td>
</tr>
<tr>
<td>83</td>
<td>5.95</td>
<td>6.18</td>
</tr>
<tr>
<td>84</td>
<td>4.80</td>
<td>4.91</td>
</tr>
<tr>
<td>85</td>
<td>4.65</td>
<td>4.72</td>
</tr>
<tr>
<td>86</td>
<td>4.77</td>
<td>4.73</td>
</tr>
<tr>
<td>87</td>
<td>2.89</td>
<td>3.04</td>
</tr>
<tr>
<td>88</td>
<td>3.06</td>
<td>3.04</td>
</tr>
<tr>
<td>89</td>
<td>5.07</td>
<td>5.11</td>
</tr>
<tr>
<td>90</td>
<td>6.59</td>
<td>6.65</td>
</tr>
<tr>
<td>91</td>
<td>7.78</td>
<td>7.81</td>
</tr>
<tr>
<td>92</td>
<td>7.07</td>
<td>7.24</td>
</tr>
<tr>
<td>93</td>
<td>3.28</td>
<td>3.34</td>
</tr>
</tbody>
</table>

Notes: P = Paasche. L = Laspeyres, T = Törnqvist/Divisa

Figure 9.1: The Demand for Alcohol and Tobacco
Figure 9.2: Bounds on the Cross Price Effect: Food-Out
9.2. Bounds on Responses to Tax Policy Reforms

Finally we turn to constructing bounds on demand responses from tax reforms for individuals at different points in the income distribution. The figures below present the compensated demands. The higher solid curve in each is the ‘no response’ curve - this uses the pre-response demand level to weight the price change. The bounds from the revealed preference analysis using the estimated nonparametric expansion paths are given by the ‘crosses’. These are relatively tight and show the no response exaggerates the demand response especially for those on higher incomes. The second figure includes an additional curve which describes the ‘Törnqvist’ first order approximation using the pre-response budget share as a weight and corresponds to homothetic Cobb-Douglas preferences.

Figure 9.3: Bounds on the Cross Price Effect: Services
Figure 9.4: Tax Reform Responses Across the Income Distribution: An Increase in Duties

Figure 9.5: Tax Reform Responses: A First Order Approximation
10. Summary and Conclusions

These lectures have described three new contributions to the empirical analysis of consumer responses to price and income changes. First it presented a powerful test of integrability conditions that is achieved completely within a nonparametric framework involving nonparametric estimation and nonparametric revealed preference theory. Semi-parametric expansion paths estimates (Engel curves) were shown to massively improve the power of revealed preference tests. Second, using this nonparametric framework, tight bounds on the welfare costs of relative price and tax changes were derived. Third, tight bounds on demand responses to price and tax changes were derived. This analysis was then applied successfully to detailed household data from the UK. It was assumed that there was a finite set of discrete relative price or tax regimes and individual household level data was available on incomes and expenditures and demographic composition. New methods for dealing with shape invariant semiparametric regression models and for allowing for endogenous regressors in semiparametric regression were presented.

Appendices

A. Appendix: Proofs of Lemmas and Propositions

Proof of Proposition 1
Without loss of generality we take the GARP rejecting preference ordered sub-sequence to be \((q_s(\hat{x}_s), q_t(\hat{x}_t), q_u(\hat{x}_u))\). We have:

1. \(\hat{x}_s = p_s q_s(\hat{x}_s) \geq p_s q_t(\hat{x}_t)\) and
2. \(\hat{x}_t = p_t q_t(\hat{x}_t) \geq p_t q_u(\hat{x}_u)\) and
3. \(\hat{x}_u = p_u q_u(\hat{x}_u) > p_u q_s(\hat{x}_s)\).

We consider the SMP path for this preference ordered sub-sequence and show that it too rejects GARP. The SMP path \((q_s(\bar{x}_s), q_t(\bar{x}_t), q_u(\bar{x}_u))\) has:

4. \(\bar{x}_t = p_t^0 q_t(\bar{x}_t) = p_t q_u(\bar{x}_u)\) and
5. \(\bar{x}_s = p_s^0 q_s(\bar{x}_s) = p_s q_t(\bar{x}_t)\).

By construction this is a preference ordered sub-sequence \((q_t(\bar{x}_t)R^0 q_u(\bar{x}_u)\) and \(q_u(\bar{x}_s)R^0 q_t(\bar{x}_t)\)) so that this sub-sequence rejects GARP if \(q_u(\bar{x}_u)P^0 q_s(\bar{x}_s)\); that is, if:

6. \(p_s^0 q_u(\bar{x}_u) > p_t^0 q_u(\bar{x}_s)\).

Conditions (2) and (4) imply \(p_t^0 q_t(\hat{x}_t) \geq p_t^0 q_t(\bar{x}_t)\) which implies \(\hat{x}_t \geq \bar{x}_t\).
This and conditions (1) and (5) give:

\[ p'_s q_s(\hat{x}_s) \geq p'_t q_t(\hat{x}_t) \geq p'_s q_s(\hat{x}_s) = p'_t q_t(\hat{x}_s) \]

which implies \( \hat{x}_s \geq \hat{x}_s \). Finally, condition (3) and normality imply \( p'_0 q_s(\hat{x}_u) > p'_0 q_s(\hat{x}_s) \geq p'_0 q_s(\tilde{x}_s) \) which is condition (6); hence GARP is rejected for this sub-sequence. ■

**Proof of Proposition 6.**

Given the budget share form of the Slutsky equation (4.8) and the additive structure in (4.9) we have by differentiating both side of (4.8) with respect to \( \ln{x} \) then with respect to \( z \) gives

\[ m^k g^i_{xx} = m^i g^k_{xx}. \]

If \( m^k \) and \( m^i \) are unrestricted this must hold for any values of \( m^k \) and \( m^i \). If either \( m^k \) or \( m^i \) are allowed to be zero then this implies \( g^i_{xx} = g^k_{xx} = 0 \). ■

**Proof of Proposition 7.**

**Assumptions:**
A1: \( \varepsilon_{ji} \) are assumed mutually independent and have finite second moments
A2: \( E(\varepsilon_{ji} | \ln{x}, z^s) = 0 \)
A3: \( \ln{x_i} \) is independently distributed with density \( \hat{f}^s(\cdot) \) that is two times boundedly differentiable.
A4: \( \hat{f}^s(\cdot)(\hat{f}^s(\cdot))^2 \) are two times boundedly differentiable functions.
A5: \( (\alpha', \theta') \) is in a bounded and open set.
A6: The twice boundedly differentiable weight function \( \varpi \), is non-negative and positive only on the interior of a compact interval \( \Xi_x \). For all points \( x \in \Xi_x \) we have that \( f^s(\ln{x}) > 0 \) and that for all \( (\pi', \theta'), x, z \in \Theta \times \Xi \) that \( f'(\ln{x} - \phi(z' \theta)) > 0 \).
A7: No parameter vector \( (\alpha', \theta') \neq (\alpha'_0, \theta'_0) \) exists such that for some \( j \), \( g^i_j(\ln{x}) = z'^{\theta'} \alpha'_j + g^i_j(\ln{x} - \phi(z' \theta')) \) almost all \( x \in \Xi_x \).
A8: The same kernel is used for all \( s = 0, 1, \ldots, S \) groups with bandwidth \( n_s h^5 \rightarrow \infty \), \( n_s h^6 \rightarrow 0 \) as \( n_s \rightarrow \infty \).

With assumptions A1 - A8 in place, Proposition 5 follows directly from Lemmas 1-6 and Theorem 1 in Pinske and Robinson (1995).

■

**B. Data Appendix**
Table B.1: Total nondurable nominal expenditure: Annual descriptive statistics.

<table>
<thead>
<tr>
<th>Year</th>
<th>No. of Obs</th>
<th>Mean</th>
<th>Std Dev.</th>
<th>10%</th>
<th>50%</th>
<th>90%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1974</td>
<td>3386</td>
<td>39.11</td>
<td>17.95</td>
<td>20.41</td>
<td>35.19</td>
<td>62.93</td>
</tr>
<tr>
<td>1975</td>
<td>3696</td>
<td>47.17</td>
<td>21.17</td>
<td>24.83</td>
<td>42.36</td>
<td>75.92</td>
</tr>
<tr>
<td>1976</td>
<td>3553</td>
<td>52.79</td>
<td>24.20</td>
<td>27.75</td>
<td>47.23</td>
<td>84.15</td>
</tr>
<tr>
<td>1977</td>
<td>3683</td>
<td>60.94</td>
<td>27.71</td>
<td>31.87</td>
<td>54.83</td>
<td>98.65</td>
</tr>
<tr>
<td>1978</td>
<td>3583</td>
<td>67.84</td>
<td>31.33</td>
<td>35.34</td>
<td>60.78</td>
<td>108.76</td>
</tr>
<tr>
<td>1979</td>
<td>3476</td>
<td>79.18</td>
<td>37.04</td>
<td>40.36</td>
<td>71.42</td>
<td>127.72</td>
</tr>
<tr>
<td>1980</td>
<td>3717</td>
<td>92.84</td>
<td>43.07</td>
<td>47.67</td>
<td>82.77</td>
<td>152.70</td>
</tr>
<tr>
<td>1981</td>
<td>4072</td>
<td>102.63</td>
<td>47.94</td>
<td>52.78</td>
<td>91.29</td>
<td>169.21</td>
</tr>
<tr>
<td>1982</td>
<td>3974</td>
<td>108.89</td>
<td>50.10</td>
<td>56.83</td>
<td>98.15</td>
<td>175.15</td>
</tr>
<tr>
<td>1983</td>
<td>3749</td>
<td>117.11</td>
<td>54.40</td>
<td>60.33</td>
<td>105.69</td>
<td>190.41</td>
</tr>
<tr>
<td>1984</td>
<td>3755</td>
<td>124.71</td>
<td>59.71</td>
<td>62.81</td>
<td>110.22</td>
<td>206.58</td>
</tr>
<tr>
<td>1985</td>
<td>3775</td>
<td>132.56</td>
<td>64.68</td>
<td>64.94</td>
<td>117.65</td>
<td>219.00</td>
</tr>
<tr>
<td>1986</td>
<td>3826</td>
<td>143.35</td>
<td>71.64</td>
<td>69.35</td>
<td>126.01</td>
<td>240.79</td>
</tr>
<tr>
<td>1987</td>
<td>3962</td>
<td>150.49</td>
<td>74.20</td>
<td>72.42</td>
<td>134.40</td>
<td>249.69</td>
</tr>
<tr>
<td>1988</td>
<td>4003</td>
<td>163.01</td>
<td>83.09</td>
<td>75.71</td>
<td>145.68</td>
<td>274.40</td>
</tr>
<tr>
<td>1989</td>
<td>4086</td>
<td>173.93</td>
<td>86.57</td>
<td>83.38</td>
<td>155.14</td>
<td>292.80</td>
</tr>
<tr>
<td>1990</td>
<td>3772</td>
<td>191.01</td>
<td>95.95</td>
<td>91.15</td>
<td>169.15</td>
<td>320.19</td>
</tr>
<tr>
<td>1991</td>
<td>3886</td>
<td>199.59</td>
<td>99.41</td>
<td>96.19</td>
<td>177.71</td>
<td>332.81</td>
</tr>
<tr>
<td>1992</td>
<td>3909</td>
<td>205.58</td>
<td>97.29</td>
<td>101.02</td>
<td>185.86</td>
<td>339.20</td>
</tr>
<tr>
<td>1993</td>
<td>3800</td>
<td>219.84</td>
<td>111.99</td>
<td>105.47</td>
<td>192.97</td>
<td>363.91</td>
</tr>
</tbody>
</table>

References


