“Reverse Bayesianism”: A Choice-Based Theory of Growing Awareness

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Abstract

This paper invokes the axiomatic approach to explore the notion of growing awareness in the context of decision making under uncertainty. It introduces a new approach to modeling the expanding universe of a decision maker in the wake of becoming aware of new consequences, new acts and new links between acts and consequences. The expanding universe, or state space, is accompanied by extension of the set of acts. The preference relations over the expanded sets of acts are linked by unchanging preferences over the satisfaction of basic needs. The discovery of new links between acts and consequences does not expand the state space, rather it renders nonnull events that were considered null before the discovery. The main results are representation theorems and corresponding rules for updating beliefs over expanding state spaces and null events that have the flavor of “reverse Bayesianism.”

Keywords: Awareness, unawareness, reverse Bayesianism

JEL classification: D8, D81, D83

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1 Introduction

According to the Bayesian paradigm, as new discoveries are made and new information becomes available, the universe shrinks: With the arrival of new information, events replace the prior universal state space to become the posterior state space, or universe of discourse. This process of “destruction” reflects the impossibility, in the Bayesian framework, of expanding the state space and of updating the probabilities of null events, coupled with the fact that conditioning on new information renders null events that, a-priori, were nonnull. Yet, experience and intuition alike contradict this view of the world. Becoming accustomed to things that were once inconceivable is part of history and our own life experience. There is a sense, therefore, in which our universe expands as we become aware of new opportunities.

In this paper we take a step toward modeling the process of growing awareness and expansion of the universe in its wake. To model the evolution of awareness, we invoke the theory of choice under uncertainty; borrowing its language and structure while modifying it to fit our purpose. In particular, we allow for new consequences and feasible acts to be introduced and for the discovery of new links among acts and consequences. The interpretation of the updating is somewhat different for the discovery of new feasible acts and consequences on the one hand and the discovery of new scientific links among feasible acts and consequences on the other. The discovery of new feasible acts and consequences represents growing awareness and leads to genuine expansion of the decision maker’s universe. The discovery of new scientific links between feasible acts and consequences results in rendering events that, prior to the discovery of the new links, were considered conceivable but unfeasible and, consequently, null, into nonnull events. This updating of zero probability events is part of the reverse Bayesianism nature of our model.

In this paper, a decision maker’s initial perception of the universe is determined by a primitive set of what he considers to be feasible acts and, corresponding to each feasible act, a potential set of consequences. The conceivable state space consists of all the mappings from the set of feasible acts to that of all consequences. Matching feasible acts with their

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2Here we follow the approach to defining a state space described in Schmeidler and Wakker (1987) and Karni and Schmeidler (1991).
potential consequences, taking into account the nature of the feasible set of acts and the
links between feasible acts and consequences that the decision maker consider possible,
defines a feasible state space. The discovery of new consequences and/or new feasible acts
expands both the conceivable and feasible state spaces, capturing the decision maker’s
growing awareness of the universe. By contrast, the discovery of new links among feasible
acts and consequences expands the feasible state space but not the conceivable state space,
thereby renderingnonnull events that were considered null before the discovery of the new
links. Within this framework, we axiomatize the evolution of beliefs in a way that can be
described as “reverse Bayesianism.”

We assume throughout that, within a given state space, decision makers’ choice behavior
is governed by the axioms of subjective expected utility theory. As the state space expands,
probability mass is shifted proportionally away from the nonnull events in the prior state
space to events created as a result of the expansion of the state space. When new act-
consequence links are established, null events become nonnull, requiring the shifting of
probability mass, equiproportionally, away from the prior nonnull events to the null events
that have now become nonnull. We note that the same process applies in the inverse
direction. The discovery that certain hypotheses about the connections among feasible
acts and consequences are invalid render some events null. This requires redistributing the
probability mass assigned to prior nonnull events, equiproportionally, among the remaining
nonnull events. This process amounts to Bayesian updating in Savage’s (1954) framework.

Preference relations under different levels of awareness are defined over different do-
mains. To link the preference relations across their corresponding domains, we assume
that decision makers have needs whose satisfaction determines their well-being. Choice
behavior is motivated by the desire to satisfy these needs. The “material” consequences
of acts are means by which these needs are satisfied. To model this concept of individual
behavior, we invoke an approach to consumer theory, due to Lancaster (1966) and Becker
(1965), according to which the material consequences are inputs in a “household production
function” generating characteristics that determine the decision maker’s well-being. In our
framework, these characteristics correspond to levels of satisfaction of diverse needs. We
assume that decision makers are fully conscious of their needs and that growing awareness
does not alter these needs.

In our model, awareness grows as a result of the discovery of new consequences or
new feasible acts. Such discoveries expand the state space, which represents the decision
maker’s perception of the universe in which he lives. The approach we employ also permits
the analysis of new scientific discoveries and technical innovations that establish new links
and/or eliminate existing links among acts and consequences.

This systematic evolution of beliefs makes it possible to predict, at least partially, the
decision maker’s behavior when something unforeseen occurs. With the discovery of a
contingency that he was unaware of, the decision maker’s prior conception – or “model”
– of the universe is falsified. When this happens, the decision maker’s prior model need
not be discarded; it can still provide some guidance for behavior in the “new” expanded
universe. In other words, decision makers can use their experiences and understanding of
the prior state space to guide their choices when their growing awareness enables them to
construct an expanded state space.

The exploration of the issue of unawareness in the literature has invoked at least three
different approaches. (a) the epistemic approach (see Fagin and Halpern [1988], Modica
and Rustichini [1999], Halpern [2001], Li [2009], and Hill [2010]); (b) the game-theoretic,
or interactive decision making, approach (see Heifetz, Meier, and Schipper [2006], Halpern
and Rego [2008], Grant and Quiggin [2009]); and (c) the choice-theoretic approach (see
Kochov [2010], Schipper [2010], Li [2008], Lehrer and Teper [2011]).

Our approach falls within the third category. However, unlike other studies that take
this approach, we do not take the state space as given. Instead, we construct the relevant
state space from the sets of feasible acts and consequences and the perceived links among
them. In so doing, we abstract from concrete interpretations of the states and treat them as
abstract resolutions of uncertainty. Consequently, decision makers’ unawareness concerns
feasible acts, feasible consequences, and/or their links.

Kochov (2010) considers a decision maker who knows that his perception of the universe
may be incomplete. He characterizes the collection of foreseen events and shows that the
result of the decision maker being aware of his incomplete perception of the environment
is that his beliefs are represented by a non-singleton set of priors, which he updates as his
perception of the environment becomes more precise.

Schipper (2010) focuses on detecting unawareness. Taking as primitive a lattice of
disjoint state spaces in the Anscombe and Aumann (1963) model and defining acts as
mappings from the union of these state spaces to the set of consequences, Schipper provides
conditions under which unawareness can be modeled as probability zero events in the
union of the disjoint state spaces in the lattice. He does not address the issue of updating
preferences in the wake of growing awareness.

Li (2008) takes as primitives a fixed set of consequences and two, exogenously given, state spaces that correspond to a decision maker being less than fully aware and fully aware. Decision makers are characterized by preference relations, conditional on the level of awareness, over Anscombe-Aumann acts on the corresponding state spaces. Li considers two types of unawareness: “pure unawareness,” depicting situations in which the decision maker’s perception of the environment is coarse, and “partial unawareness,” depicting situations in which the decision maker’s perception of the universe is a subset of the full state space. Partial unawareness has a flavor of unawareness of consequences or links between acts and consequences. However, since the set of consequences and states are given, Li’s model cannot accommodate the discovery of new consequences or new scientific links.

Lehrer and Teper (2011) model growing awareness due to the decision maker’s improved ability, in the wake of information acquisition, to distinguish among events. Unlike our approach in which the state space expands by adding states that are not elements of existing events, the expansion of the state space in the model of Lehrer and Teper takes the form of finer partitions of the existing state space.

2 The Meanings of Growing Awareness: Examples and Formalization

The examples below illustrate the sense in which a decision maker’s universe expands in the wake of his growing awareness.

2.1 Examples

A. Discovery of new consequences

The discovery of the New World. Columbus set out to discover a new sea route to India, presumably taking into account consequences such as ending the trip at the bottom of the ocean, having to turn back, losing some ships and crew members, reaching India, etc. He could not have included, among the set of consequences, the discovery of a new continent. This discovery expanded the universe for mankind.
The discovery of syphilis. The discovery of the New World ushered in its wake a new consequence of sexual intercourse. The risk of contracting venereal diseases was well known in the Old World. Syphilis, however, was new. Its discovery expanded the universe of the Europeans.

Discovery of a “new” consequence expands the state space and may affect the decision maker’s ordinal preferences over acts. In other words, two acts that agree on the “old” state space may become distinct when associated with new consequences; as a result, one of the newly defined acts may be strictly preferred over the other.

B. Discovery of new feasible acts

Artificial self-sustaining nuclear chain reaction. After the discovery of nuclear fission, Szilárd and Fermi discovered neutron multiplication in uranium, proving that a nuclear chain reaction by this mechanism was possible. On December 2, 1942, Fermi created the first artificial self-sustaining nuclear chain reaction, thus making it feasible to use nuclear energy, for peaceful and military purposes.

The invention of sound recordings. By making it possible to preserve sounds, the invention of sound recording devices expanded the state space to include future replays of currently produced sounds.

The invention of new financial instruments. The invention of option trading opened up new possibilities of creating portfolios and diversifying risks.

C. Discovery of new scientific links

Yellow fever. To prevent ants from crawling into hospitals’ beds, French doctors working in Panama during the French attempt to build the Panama Canal, placed the legs of the beds in bowls of water. These pools of water provided breeding grounds for the mosquitoes carrying yellow fever. Not being aware of the way the yellow fever was transmitted, the French did not conceive that their actions contributed to the propagation of the disease. Later, when the connection between stagnant water, mosquitoes, and yellow fever was understood, the Americans were able to eradicate yellow fever, eliminating a major stumbling point to the construction of the Panama Canal.

DDT. During World War II, soldiers sprayed themselves and their beds with DDT to kill bugs. The connection between DDT and genetic mutations in one’s offspring was not
discovered until later. The possibility of genetic mutations was known at the time, so it was not the consequence itself that was new but rather the discovery of the link between DDT and genetic mutation, which had implications for the use of DDT.

2.2 Growing awareness formalized

We introduce a unifying framework within which the different sources of growing awareness may be described and analyzed. We also illustrate how the different notions of growing awareness can be formalized in this framework.

States of nature, or *states* for short, are abstract representations of resolutions of uncertainty. To define the state space, we invoke the approach of Schmeidler and Wakker (1987) and Karni and Schmeidler (1991). According to this approach, there is a finite, nonempty set, $F$, of feasible acts, a finite, nonempty set, $C$, of feasible consequences and a correspondence, $\varphi : F \rightarrow C$, representing the decision maker’s beliefs about the possible links among feasible acts and consequences. In other words, to each $f \in F$, $\varphi (f) \subseteq C$ is the subset of consequences that the decision maker believes, are possible if he chooses the act $f$. A decision maker’s perception of the universe is bounded by his awareness of the sets of feasible acts and consequences and the links among feasible acts and their potential consequences. To lend this approach a formal meaning, we begin by defining a conceivable state space, $C^F$, whose elements depict the resolutions of uncertainty if all possible links between feasible acts and consequences are contemplated.

Next, observe that not all the states in $C^F$ are necessarily feasible. First, if for some $f \in F$, $\varphi (f)$ is a proper subset of $C$ then any $s \in C^F$ that assigns to $f$ a consequence in $C - \varphi (f)$ is not feasible. Moreover, it may well be the case that, because of the nature of the feasible acts, even if $s(f) \in \varphi (f)$ for all $f \in F$, the state $s$ may not be feasible. To grasp this consider the following example. An urn contains three balls, one red ball and two black balls. A ball is drawn at random and its color is observed. Let $C$ be a doubletone set, $\{x, y\}$, where $x$ and $y$ are real numbers representing dollar amounts, and $x > y$. Let there be two feasible acts: The act $f_R$, a “bet on red,” pays off $x$ dollars if a red ball is observed and $y$ dollars if a black ball is observed. The act, $f_B$, a “bet on black,” pays off $x$ dollars if a black ball is observed and $y$ dollars if a red ball is observed. The conceivable

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3See also Gilboa (2009) Chapter 11, for a detailed discussion and an ingenious use of this approach to formulating the state space as means of resolving Newcomb’s paradox.
state space consists of four states as described below:

\[
\begin{array}{cccc}
F \setminus C^F & s_1 & s_2 & s_3 & s_4 \\
\rho_R & x & x & y & y \\
\rho_B & x & y & x & y \\
\end{array}
\]

Moreover, the definition of the acts implies that \( \phi(\rho_R) = \phi(\rho_B) = C \). Hence, for all \( s \in C^F \) and \( f \in F \), \( s(f) \in \phi(f) \). Yet, not all of these states are feasible. To be specific, the description of the acts implies that the state \( s_2 \) corresponds to drawing the red ball and the state \( s_3 \) corresponds to drawing a black ball. Moreover, the definitions of the acts entail perfect negative correlation between their payoffs (that is, in the state in which the payoff of \( \rho_R \) is \( x \) the payoff of \( \rho_B \) must be \( y \), and vice versa). Hence, the states \( s_1 \) and \( s_4 \) are logically inconsistent with the definitions of the acts and are, therefore, non-feasible.

With this in mind, we define a payoff profile as an assignment of a unique consequence to each act. A payoff profile is feasible if it assigns to each act a consequence that is possible given the consequences it assigns to all the other acts. Formally, a payoff profile is a function \( \zeta : F \to C \). Denote by \( \zeta_{-f} \) the vector \( (\zeta(f'))_{f' \in F_{-f}} \), listing the consequence of all feasible acts other than \( f \). For each \( \zeta \in C^F \) and \( f \in F \), define the conditional correspondence \( \phi(f \mid \zeta_{-f}) : F \to C \) by \( \phi(f \mid \zeta_{-f}) = \{ c \in \phi(f) \mid c \text{ is possible given } \zeta_{-f} \} \). Then, the payoffs in the payoff profile \( \zeta \) are feasible if, for all \( f \in F \), \( \zeta(f) \in \phi(f \mid \zeta_{-f}) \).

A state is a payoff profile. Hence, \( s \in C^F \) is feasible if \( s(f) \in \phi(f \mid s_{-f}) \), for all \( f \in F \). We denote by \( S(F,C,\phi) \) the set of all such states and refer to it as the feasible state space. A feasible state indicates, for each feasible act, the resulting consequence that obtains, thereby completely resolving the uncertainty present in the decision maker’s perception of the universe. If, for some act \( f \), \( \phi(f \mid s_{-f}) \) is a proper subset of \( C \) for some \( s_{-f} \) then the conceivable but unfeasible event \( C^F - S(F,C,\phi) \) is presumably null. We lend this presumption a choice-theoretic meaning when we discuss the preference relation in Section 3.

Once the set of conceivable states is fixed, the set of feasible acts is expanded to include conceivable acts. The notion of conceivable acts captures the idea of acts that are imaginable given the feasible acts and consequences. The expansion of the set of acts includes two steps. First, conceivable new acts are formed by the association of feasible consequences to the existing states. By itself this allows the expansion of the set of acts from \( F \) to include all the functions from the set of conceivable states \( C^F \) to the set \( C \) of consequences, that
is,
\[ \bar{F} := \{ f : C^F \to C \} . \]  

(1)

Second, the decision maker may imagine acts whose outcomes are lotteries with consequences in \( C \) as prizes. Let the set of all such lotteries be denoted by \( \Delta(C) \). Then the set of acts may be enlarged to include the functions in the set

\[ \hat{F} := \{ f : C^F \to \Delta(C) \}, \]  

(2)

which we refer to as the set of conceivable acts. We identify \( c \in C \) with the degenerate lottery \( \delta_c \in \Delta(C) \) that assigns \( c \) the unit probability mass. Hence, \( F \subset \bar{F} \subset \hat{F} \).

Note that the state space is constructed using the set of feasible acts, relative to which the set of conceivable acts may seem quite abstract. Since (most of) these acts are non-feasible, there is a concern about the possibility of eliciting preferences over such acts. To see how this is done, consider again the example of the urn described above. The conceivable state space consists of four states and the feasible state space is \( S(F, C, \phi) = \{ s_2, s_3 \} \). Consider the following conceivable acts and their interpretation

\[
\begin{array}{cccccc}
F \setminus C^F & s_1 & s_2 & s_3 & s_4 \\
\hline
f_3 & x & x & x & x \\
f_4 & y & y & y & y \\
f_5 & y & y & x & x \\
f_6 & x & x & x & y \\
f_7 & y & x & y & y \\
f_8 & x & y & y & x \\
f_9 & y & y & x & y \\
\end{array}
\]

The constant acts \( f_3 \) and \( f_4 \) have the interpretation of “bet on \( R \cup B \)” and “bet against \( R \cup B \)”, respectively. The acts \( f_5 \) and \( f_9 \) are bets on black and the act \( f_7 \) is a bet on red. The acts \( f_6 \) and \( f_8 \) are bets on and against \( R \cup B \), respectively. Interpreted in this way, conceivable acts are meaningful, and preferences among such acts are not only possible to contemplate, but can be inferred from the preferences on the feasible set of acts.

“In practice, the distinction between feasible and conceivable acts is not always crucial, and in many applications the sets of states and consequences are taken as primitives.”

\[^4\text{To be clear, } \Delta(C) := \{ p \in [0,1]^{|C|} \mid \sum_{c \in C} p_c = 1 \} . \]
(Karni and Schmeidler (1991) p. 1766). In the present context the distinction between feasible and conceivable acts is crucial. It is the set of feasible acts, together with the feasible consequences and the links among feasible acts and consequences, that shape the decision maker’s vision of the universe.

Using this framework, we discuss the various types of unawareness with which we are concerned. We use the following notational convention throughout. We denote by \( F, C \) and \( \varphi \), respectively, the initial sets of feasible acts, consequences, and the correspondence representing the links between them. When new elements are introduced into each of these sets we denote the corresponding new sets by \( F' \) and \( C' \) and when new links are established we denote the resulting new correspondence by \( \varphi' \). When new consequences are discovered, the acts and the correspondence must be redefined. We denote the new set of acts by \( F^* \) and the new correspondence by \( \varphi^* \). When new links are discovered, the set of acts needs to be redefined. We denote the new set of acts by \( F^* \).

### 2.2.1 Discovery of new consequences

Let \( C \) denote the initial set of consequences and suppose that a new consequence, \( \bar{c} \notin C \), is discovered. The set of consequences of which the decision maker is aware then expands to \( C' = C \cup \{ \bar{c} \} \). The discovery of \( \bar{c} \) requires a reformulation of the initial model, incorporating the new consequence into the range of the feasible acts. Because ranges of the feasible acts rather than the acts themselves changed, we denote the set of feasible acts with extended range by \( F^* \) and the subset of consequences the decision maker now believes possible if he chooses \( f \) by \( \varphi^*(f) \). Then the new feasible state space is given by

\[
S(F^*, C', \varphi^*) := \{ s : F^* \to C' | s(f) \in \varphi^*(f | s_{-f}) \forall f \in F^* \},
\]

and the extended conceivable state space is \((C')^{F^*}\).

Define the corresponding expanded set of conceivable acts,

\[
\hat{F}^* := \{ f : (C')^{F^*} \to \Delta (C') \}.
\]

The event \((C')^{F^*} - C^F\) represents the expansion of the decision maker’s conceivable state space, while \(S(F^*, C', \varphi^*) - S(F, C, \varphi)\) represents the expansion of his feasible state space, as a result of his growing awareness of consequences.
As an illustration, suppose we start with two feasible acts, \( F = \{f_1, f_2\} \), two consequences, \( C = \{c_1, c_2\} \), and the links \( \varphi (f_1 \mid s_{-f_1}) = \varphi (f_2 \mid s_{-f_2}) = \{c_1, c_2\} \), for all \( s \in C^F \). The resulting feasible and conceivable state spaces coincide and are given by \( S(F, C, \varphi) = \{s_1, s_2, s_3, s_4\} = C^F \):

\[
F \setminus S(F, C, \varphi) \quad s_1 \quad s_2 \quad s_3 \quad s_4 \\
f_1 \quad c_1 \quad c_2 \quad c_1 \quad c_2 \\
f_2 \quad c_1 \quad c_1 \quad c_2 \quad c_2
\]

Suppose now that a new consequence, \( c_3 \), is discovered and that it is established that the feasible act \( f_1 \) may result in \( c_3 \). Specifically, suppose that, after the discovery, the conditional range of the act \( f_1 \) is \( \varphi^* (f_1 \mid s_{-f_1}) = \{c_1, c_2, c_3\} \) and that of \( f_2 \) remains unchanged, \( \varphi^* (f_2 \mid s_{-f_2}) = \{c_1, c_2\} \), for all \( s \in (C')^F \). The new conceivable state space consists of the 9 states in the set \((C')^F\). The new feasible state space is \( S(F^*, C', \varphi^*) = \{s_1, s_2, \ldots, s_6\} \):

\[
F^* \setminus S(F^*, C', \varphi^*) \quad s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_5 \quad s_6 \\
f_1 \quad c_1 \quad c_2 \quad c_1 \quad c_2 \quad c_3 \\
f_2 \quad c_1 \quad c_1 \quad c_2 \quad c_1 \quad c_2
\]

### 2.2.2 Discovery of new feasible acts

Suppose that a new act, say \( f_3 \), becomes feasible. Instead of \( F \), the set of feasible acts is now \( F' = \{f_1, f_2, f_3\} \). Assume further that the redefined correspondence depicting the links among feasible acts and consequences is \( \varphi^* (f_i \mid s_{-f_i}) = C = \{c_1, c_2\} \), \( i = 1, 2, 3 \), for all \( s \in C^{F'} \). The conceivable state space is expanded to \( C^{F'} \), which in this example coincides with the feasible state space, \( S(F', C, \varphi^*) \), after the discovery. It now consists of eight states:

\[
F' \setminus S(F', C, \varphi^*) \quad s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_5 \quad s_6 \quad s_7 \quad s_8 \\
f_1 \quad c_1 \quad c_2 \quad c_1 \quad c_2 \quad c_1 \quad c_2 \quad c_1 \quad c_2 \\
f_2 \quad c_1 \quad c_1 \quad c_2 \quad c_1 \quad c_1 \quad c_2 \quad c_2 \quad c_1 \quad c_2 \\
f_3 \quad c_1 \quad c_1 \quad c_1 \quad c_2 \quad c_1 \quad c_2 \quad c_2 \quad c_2 \quad c_2
\]

In general, the elements of the state space \( S(F', C, \varphi^*) \) constitute a finer partition of the state space \( S(F, C, \varphi) \). In other words, each state in \( S(F, C, \varphi) \) is a non-degenerate event in the expanded state space \( S(F', C, \varphi^*) \). For example, the state \( s_1 := (c_1, c_1) \in S(F, C, \varphi) \) is the event \( E = \{s_1, s_3\} \) in the state space \( S(F', C, \varphi^*) \). However, if \( \varphi^* (f_3) = \{c_1\} \), then
the number of states in the feasible state space remains the same (that is, it consists of the same number of states as in $S(F, C, \varphi)$ but each state is now redefined to include additional coordinates), but that of the conceivable state space is $C^{F'}$.

In either case, the decision maker’s original conception of the feasible state space is determined by the initial sets of acts and consequences he considers feasible and the links between acts and consequences he considers possible. The act $f_3$ was neither conceivable nor feasible before its discovery. Now that it has become feasible, it changes the decision maker’s conceptions of the feasible and conceivable state spaces.\(^5\)

Note that, unlike in the discovery of new consequences (and the discovery of new links among acts and consequences, as we shall see below), the discovery of new acts requires that the length of the vector of consequences defining each state increases. As we show later, this aspect of the evolving state space requires special treatment.

2.2.3 Discovery of new scientific links

Unlike the discovery of new feasible consequences and/or new feasible acts which expands the set of conceivable and feasible states, the discovery of new scientific links expands the set of feasible states but leaves the set of conceivable states intact. Consequently, the discovery of new feasible consequences and/or new feasible acts represent a genuine expansion of the decision maker’s universe while the discovery of new scientific links entails updating zero probability events.

To see how a discovery of a new scientific link expands the feasible state space, consider the case in which there are two feasible acts, $F = \{f_1, f_2\}$ and two consequences, $C = \{c_1, c_2\}$. The conceivable state space $C^{F}$ consists of four states. Suppose that $\varphi(f_1 \mid s_{-f_1}) = C$ and $\varphi(f_2 \mid s_{-f_2}) = \{c_1\}$ for all $s \in C^{F}$, then the feasible state space is $S(F, C, \varphi) = \{s_1, s_2\}$, as described below

\[
\begin{array}{c|cc}
F \setminus S(F, C, \varphi) & s_1 & s_2 \\
\hline
f_1 & c_1 & c_2 \\
f_2 & c_1 & c_1 \\
\end{array}
\]

\(^5\)Ahn and Ergin (2010) model decision makers whose choice behavior depends on their perception of contingencies, represented by alternative partitions of a given state space. Unlike our work, in which the state space expands and is partitioned more finely as a result of the discovery of new acts, in Ahn and Ergin’s work new acts are defined as a consequence of finer partition of the state space. These acts represent growing alertness to possibilities that were always present and were simply ignored.
Suppose that a new scientific link between feasible acts and consequences is established. Let \( \phi \) denote the correspondence depicting the “old” links, and denote by \( \phi' \) the correspondence depicting the new links. In particular, suppose that it is discovered that \( f_2 \) may also result in \( c_2 \), (that is, after the new discovery, \( \phi'(f_i \mid s_{\not=f_i}) = C, i = 1, 2, \) for all \( s \in C^F \)). To indicate the fact that the range of consequences associated with (some) feasible acts is now larger, we denote the set of feasible acts by \( F^* \). The set of conceivable states remains intact, but the feasible state space, \( S(F^*, C, \phi') \), now consists of four states, \( s_1, ..., s_4 \), described in the following matrix:

\[
\begin{array}{cccc}
F^* \setminus S(F^*, C, \phi') & s_1 & s_2 & s_3 & s_4 \\
f_1 & c_1 & c_2 & c_1 & c_2 \\
f_2 & c_1 & c_1 & c_2 & c_2 \\
\end{array}
\]

Hence, following the discovery of the new (and final link) the feasible and conceivable state spaces coincide (that is, \( S(F^*, C, \phi') = C^F = C^F^* \)).

Before the discovery of the new link, the event \( \{s_3, s_4\} = C^F \setminus S(F, C, \phi) \) was null in the larger conceivable state space \( C^F \). Upon the discovery of the link, the decision maker realizes that his belief that it was impossible to obtain a particular consequence by implementing a particular feasible act is untenable. Before the discovery of the new link the event \( C^F \setminus S(F, C, \phi) \) was a conceivable but unfeasible and, hence, null. Following the discovery of the new link, \( C^F = S(F^*, C, \phi') \). The event \( C^F \setminus S(F, C, \phi) \), which was regarded as impossible before, became possible following the discovery of the new link. By the same logic, the discovery that a link that the decision maker believed possible is, in fact, impossible, results in nullifying an event that was initially nonnull.

What is a reasonable updating rule for probabilities of events that were considered impossible (null) and, as a result of scientific progress and growing understanding of the structure of the universe, become possible (nonnull)? Clearly, the Bayesian approach is useless for this purpose. Here we explore an alternative approach.

3 The Analytical Framework

A decision maker’s growing awareness of the feasibility of acts and consequences expands his perception of the universe and its structure. The discovery of new links among acts and consequences expands what he considers to be the feasible universe. How does the decision
maker’s growing awareness of the universe he lives in manifest itself in his choice behavior? In this section we introduce the analytical framework as well as some preliminary results used in the subsequent analysis.

3.1 Preferences, needs, and technology

Decision makers in our model are supposed to be able to express preferences among conceivable acts. Formally, let \( \mathcal{F} \) be a family of sets of conceivable acts corresponding to increasing levels of awareness from all sources (that is, from the discovery of new feasible acts, consequences, and links among them). Preferences are binary relations on \( \hat{F} \in \mathcal{F} \). Because the set of conceivable acts is a variable in our model, we denote the preference relation on \( \hat{F} \) by \( \succeq_{\hat{F}} \), and use the notation \( \succ_{\hat{F}} \) and \( \sim_{\hat{F}} \) to denote the asymmetric and symmetric parts of \( \succeq_{\hat{F}} \), respectively. When the state space expands in the wake of discoveries of new feasible consequences and/or new links among acts and consequences, the set of conceivable acts must be expanded and the preference relations must be redefined on the extended domain. For instance, if \( \hat{F}^* \) is the expanded set of conceivable acts, then the corresponding preference relation is denoted by \( \succeq_{\hat{F}^*} \). If the state space is expanded in the wake of the discovery of new feasible acts, then the new set of conceivable acts is denoted by \( \hat{F}' \) and the expanded preference relation by \( \succeq_{\hat{F}'} \).

Our main concern is how does the preference relation change when the decision maker’s universe expands as his awareness grows? To model the change of preferences resulting from increasing awareness, we employ a variation of the model proposed by Lancaster (1966) and Becker (1965). In particular, we assume that decision makers have needs, which they seek to satisfy by means of consumption of goods and services. Let \( N = \{1, \ldots, n\} \) be a list of needs (e.g., food, shelter, clothing, entertainment, social status, etc.). The trade-offs among the satisfaction of different needs are assumed to be a matter of personal taste. Let \( Z \subset \mathbb{R}^n \) be a set whose elements are levels of satisfaction of these needs. In other words, \( z \in Z \) is a vector whose \( j \)-th coordinate, \( z_j, j \in N \), indicates the degree to which the need \( j \) is satisfied. Let \( \Delta(Z) \) denote the set of simple probability measures on \( Z \), which we refer to as need-satisfaction lotteries.\(^6\) A decision maker’s well-being is determined by the satisfaction of his needs. Thus, at the basic level, a decision maker is characterized by a preference relation, \( \succeq \), on \( \Delta(Z) \).

\(^6\)A measure is simple if it has a finite support.
Let \( X \subset \mathbb{R}^m \) be a finite, nonempty set of feasible material outcomes, or outcomes, for short. For example, \( x \in X \) could be a bundle consisting of a lobster dinner, a two-bedroom apartment in an upscale neighborhood, and a James Bond movie. Let \( F \) be a finite set of feasible acts. For each \( f \in F \), denote by \( \varphi (f) \) the set of material outcomes that in the mind of the decision maker are possible if he chooses the act \( f \). Let \( S (F, X, \varphi) \) be the set of feasible states and \( X^F \) the set of conceivable states.

Denote by \( \Delta (X) \) the set of lotteries on \( X \). Then, following Anscombe and Aumann (1963), the set of conceivable acts, \( \tilde{F} \), consists of all the mappings from the set of states to the set of lotteries on outcomes. Formally, \( \tilde{F} := \{ f : X^F \to \Delta (X) \} \) is the set of conceivable acts. Henceforth, we indicate by \( \tilde{F} \) the set of conceivable acts corresponding to the universe depicted by \( X^F \).

Let \( t : X \to Z \) be a mapping representing the technology that generates needs satisfaction from material outcomes. Put differently, \( t \) is a “production function” that transforms material outcomes into need-satisfaction levels.\(^7\) In our example, the dinner, the apartment, and the movie allow, with the appropriate input of time, the attainment of some levels of satisfaction of the needs for nutrition, shelter, social status and entertainment. Given a technology \( t \), \( p \in \Delta (X) \) induces a lottery \( l_p \) in \( \Delta (Z) \) as follows: \( l_p (z) = p (t^{-1} (z)) \), for all \( z \in Z \).\(^8\)

Decision makers are characterized by a primitive preference relation \( \succ \) on need-satisfaction lotteries and preference relations \( \succ \tilde{F} \) on the sets of conceivable acts, for all \( \tilde{F} \in \mathcal{F} \). The connections between the preference relation on need-satisfaction lotteries, and the preference relations on sets of conceivable acts are at the core of our theory. They are defined and discussed in Section 4 below.

Growing awareness expands the sets of conceivable acts and states and thus alters the domain over which the corresponding sets of induced preference relations are defined. We postulate that the preference relations corresponding to different levels of awareness are linked by a primitive, unchanging, preference relation over need-satisfaction levels.

---

\(^7\)In Lancaster (1966) the technology transforms material goods into “characteristics” and is linear. We do not insist on linearity and identify characteristics with needs satisfaction.

\(^8\)Note that \( t^{-1} (z) \) is the preimage of \( z \) under the technology, representing an isoquant of the “household production function.” Formally, \( t^{-1} (z) := \{ x \in X \mid t (x) = z \} \).
3.2 Expected utility theory

Let $H$ be a convex set in a linear space and $\succeq$ a binary relation on $H$. The von Neumann-Morgenstern axioms applied to $\succeq$ are:

(A.1) (Weak order) The preference relation $\succeq$ is transitive and complete.

(A.2) (Archimedean) For all $p, q, r \in H$, if $p \succeq q$ and $q \succeq r$ then $\alpha p + (1 - \alpha) r \succeq q$ and $q \succeq \beta p + (1 - \beta) r$, for some $\alpha, \beta \in (0, 1)$.

(A.3) (Independence) For all $p, q, r \in H$ and $\alpha \in (0, 1]$, $p \succeq q$ if and only if $\alpha p + (1 - \alpha) r \succeq \alpha q + (1 - \alpha) r$.

The von Neumann-Morgenstern theorem states that $\succeq$ on $H$ satisfies (A.1) - (A.3) if and only if there exists a real-valued, affine function $U$ on $H$ that represents $\succeq$, and is unique up to positive linear transformations.

Since $\Delta(Z)$ is a convex set in a linear space, application of the von Neumann-Morgenstern theorem yields the expected utility theorem below:

**Theorem 1 (von Neumann-Morgenstern)** Let $\succeq$ be a binary relation on $\Delta(Z)$, then the following two conditions are equivalent:

(i) $\succeq$ satisfies (A.1), (A.2) and (A.3).

(ii) There exists a real-valued function, $u$, on $Z$, such that for all $l, l' \in \Delta(Z)$,

$$\sum_{z \in \text{Supp}(l)} u(z) l(z) \geq \sum_{z \in \text{Supp}(l')} u(z) l'(z).$$

Moreover, $u$ is unique up to positive linear transformations.

Consider the preference relation $\succ\tilde{F}$ on $\tilde{F}$. Note that $X^F$ is the domain of the acts in $\tilde{F}$. For any $f \in \tilde{F}$, $p \in \Delta(X)$, and $s \in X^F$, let $f_{-sp}$ be the act in $\tilde{F}$ obtained from $f$ by replacing its $s - th$ coordinate with $p$. A state $s \in X^F$ is said to be null if $f_{-sp} \sim_{\tilde{F}} f_{-sq}$ for all $p, q \in \Delta(X)$. A state is said to be nonnull if it is not null. Similarly, for all $f$ and $g$ in $\tilde{F}$ and $E \subset X^F$, we denote by $f_{-Eg}$ the act in $\tilde{F}$ obtained from $f$ by replacing its $s - th$ coordinate, $f(s)$, with $g(s)$, for all $s \in E$. We suppose that the event $K := X^F - S(F, X, \varphi)$ that consists of states that the decision maker regards as conceivable but infeasible, is null. Formally, henceforth we assume that $f_{-Kg} \sim_{\tilde{F}} f_{-Kh}$, for all $g, h \in \tilde{F}$.

The following axioms are due to Anscombe and Aumann (1963).
(A.4) **(State independence)** For all \( \hat{\mathcal{F}} \in \mathcal{F}, f \in \hat{\mathcal{F}}, p, q \in \Delta(X) \) and nonnull \( s, s' \in X^F \),
\[
  f - s p \succ_{\hat{\mathcal{F}}} f - s q \quad \text{if and only if} \quad f - s' p \succ_{\hat{\mathcal{F}}} f - s' q.
\]

(A.5) **(Nontriviality)** For all \( \hat{\mathcal{F}} \in \mathcal{F} \), \( \succ_{\hat{\mathcal{F}}} \neq \emptyset \).

For every given \( \hat{\mathcal{F}} \in \mathcal{F}, \) for all \( f, f' \in \hat{\mathcal{F}} \) and \( \alpha \in [0, 1] \), define the convex combination \( \alpha f + (1 - \alpha) f' \in \hat{\mathcal{F}} \) by
\[
  (\alpha f + (1 - \alpha) f') (s) = \alpha f(s) + (1 - \alpha) f'(s), \quad \forall s \in X^F.
\]

Then \( \hat{\mathcal{F}} \) is a convex set in a linear space.\(^9\) For future reference, we state below a version of the Anscombe-Aumann (1963) theorem.

**Theorem 2 (Anscombe-Aumann)** Let \( \succ_{\hat{\mathcal{F}}} \) be a binary relation on \( \hat{\mathcal{F}} \), then the following two conditions are equivalent:

(i) \( \succ_{\hat{\mathcal{F}}} \) satisfies (A.1)-(A.5).

(ii) There exists a real-valued, non-constant, affine function, \( U_{\hat{\mathcal{F}}} \) on \( \Delta(X) \), and a probability measure \( \pi_{\hat{\mathcal{F}}} \) on \( X^F \), such that for all \( f, f' \in \hat{\mathcal{F}}, \)
\[
  f \succ_{\hat{\mathcal{F}}} f' \Leftrightarrow \sum_{s \in X^F} U_{\hat{\mathcal{F}}} (f(s)) \pi_{\hat{\mathcal{F}}}(s) \geq \sum_{s \in X^F} U_{\hat{\mathcal{F}}} (f'(s)) \pi_{\hat{\mathcal{F}}}(s), \quad (6)
\]

Moreover, \( U_{\hat{\mathcal{F}}} \) is unique up to positive linear transformations,\(^{10}\) \( \pi_{\hat{\mathcal{F}}} \) is unique, and \( \pi_{\hat{\mathcal{F}}}(s) = 0 \) if and only if \( s \) is a null state.

**Remark:** Since \( X^F - S(F, X, \varphi) \) is a null event, \( \pi_{\hat{\mathcal{F}}} (X^F - S(F, X, \varphi)) = 0, \pi_{\hat{\mathcal{F}}} (S(F, X, \varphi)) = 1, \) and, for all \( f \in \hat{\mathcal{F}}, \)
\[
  \sum_{s \in X^F} U_{\hat{\mathcal{F}}} (f(s)) \pi_{\hat{\mathcal{F}}}(s) = \sum_{s \in S(F, X, \varphi)} U_{\hat{\mathcal{F}}} (f(s)) \pi_{\hat{\mathcal{F}}}(s).
\]

To simplify the exposition, henceforth we disregard the null event \( X^F - S(F, X, \varphi) \), and focus our attention on the feasible state space \( S(F, X, \varphi) \). Notice that all the events in \( S(F, X, \varphi) \) are nonnull.

---

\(^9\)Throughout this paper we use Fishburn’s (1970) formulation of Anscombe and Aumann (1963). According to this formulation, mixed acts, (that is, \( \alpha f + (1 - \alpha) f' \)) are, by definition, conceivable acts.

\(^{10}\)Hence, \( U_{\hat{\mathcal{F}}} (p) = \sum_{x \in \text{supp}(p)} u_{\hat{\mathcal{F}}} (x) p(x), \) where \( u_{\hat{\mathcal{F}}} (x) = U_{\hat{\mathcal{F}}} (\delta_x), \) for all \( x \in X. \)
4 The Main Results

To study the connections among the preference relations on expanding sets of conceivable acts, we assume that these preference relations are linked together by the properties of the unchanging preference relation, \( \succeq \), on the set of need-satisfaction lotteries. Using these connections we explore the impact of growing awareness on a decision maker’s choice behavior.

The analysis of the effects of growing awareness on choice behavior and the evolution of decision makers’ beliefs is divided into three parts. In the first part, we explore the implications of the discovery of new consequences. In the second part we explore the implications of the discovery of new feasible acts. In the third part, we explore the implications of the discovery of new acts-consequences links. In general, the discovery of new consequences increases the number of both conceivable and feasible states, but the “dimension” of each state is unchanged. By contrast, the discovery of new feasible acts increases the number of both conceivable and feasible states and, at the same time, changes the characterization of each state in such a way that what used to be a state before the discovery of the new act, is an event in the reconstructed state space following the discovery. The discovery of new acts-consequences links increase the set of feasible states without affecting the conceivable state space.

We assume throughout that a decision maker’s preference relation over the set of need-satisfaction lotteries, representing his basic tastes, does not change as his awareness grows.

For every given \( \succ \) on \( \hat{F} \) and \( s \in S(F,X,\varphi) \), define a conditional preference relation, \( \succeq_s \), on \( \Delta(Z) \) induced by \( \succ \), as follows:

\[
l_p \succeq_s l_q \text{ if } f_{s}p \succ f_{s}q, \text{ for all } p,q \in \Delta(X) \text{ and } f \in \hat{F}.
\]

The next axiom requires that the conditional preference relations, \( \{\succeq_s\}_{F \in \mathcal{F}} \), and unconditional preference relation, \( \succeq \), on need-satisfaction lotteries agree. Put differently, it asserts that a decision maker’s preferences regarding his basic needs and his risk attitudes toward these needs are independent of the particular process (that is, acts) by which the need-satisfaction lotteries are obtained and by the specificity of the manner by which the uncertainty is resolved. Formally,

(A.6) (Taste consistency) For all \( \hat{F} \in \mathcal{F} \) and \( s \in S(F,X,\varphi) \), \( \succeq_s = \succeq \).
4.1 The discovery of new consequences

The following axiom requires that, as the decision maker’s awareness of consequences grows and his universe expands, his preferences conditional on the prior perception of the universe remain intact. In other words, the discovery of new consequences does not alter the preference relation conditional on the original set of feasible states. To formalize this idea, let \( S(F^*, X', \varphi^*) \), where \( X \subset X' \), and \( F^* \) and \( \varphi^* \) denote, respectively, the new set of acts and the extended correspondence needed to accommodate the new consequences.

\[
(A.7) \quad \text{(Awareness consistency)} \quad \text{For all } \hat{F}, \hat{F}^* \in \mathcal{F}, \text{ if } S(F^*, X', \varphi^*) \supset S(F, X, \varphi) \text{ and } f', g' \in \hat{F}^*, f' = f \text{ and } g' = g \text{ on } S(F, X, \varphi) \text{ and } f' = g' \text{ on } S(F^*, X', \varphi^*) - S(F, X, \varphi) \text{ then } f \succ_{\hat{F}} g \text{ if and only if } f' \succ_{\hat{F}^*} g'.
\]

4.2 Representation of preferences when growing awareness reflects the discovery of new consequences

Our first result describes the evolution of a decision maker’s beliefs in the wake of discoveries of new consequences. Specifically, a decision maker whose preferences have the structure depicted by the axioms above is a subjective expected utility maximizer. Moreover, when he becomes aware of new consequences, the decision maker updates his beliefs in such a way that likelihood ratios of events in the original state space remain intact. That is to say, probability mass is shifted away from states in the prior state space to the posterior state space, proportionally. We refer to this property as “reverse Bayesianism.”

**Theorem 3.** For each \( \hat{F} \in \mathcal{F} \), let \( \succeq_{\hat{F}} \) be a binary relation on \( \hat{F} \) then, for all \( \hat{F}, \hat{F}^* \in \mathcal{F} \), the following two conditions are equivalent:

(i) Each \( \succeq_{\hat{F}} \) satisfies (A.1) - (A.6) and, jointly, \( \preceq_{\hat{F}} \) and \( \succeq_{\hat{F}^*} \) satisfy (A.7).

(ii) There exists a real-valued, non-constant, affine function, \( U \) on \( \Delta(X) \) and, for each \( \hat{F} \in \mathcal{F} \), there is a probability measure \( \pi_{\hat{F}} \) on \( S(F, X, \varphi) \), such that for all \( f, f' \in \hat{F} \),

\[
\begin{align*}
    f \succ_{\hat{F}} f' & \iff \sum_{s \in S(F, X, \varphi)} U(f(s)) \pi_{\hat{F}}(s) \geq \sum_{s \in S(F, X, \varphi)} U(f'(s)) \pi_{\hat{F}}(s). \\
\end{align*}
\]

(7)

\[11 \text{This axiom is reminiscent of Savage’s (1954) sure thing principle in that it requires that preference between acts be independent of the aspects on which they agree.}\]
Moreover, \( U \) is unique up to positive linear transformations, \( \pi_{\hat{F}} \) is unique and

\[
\frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(s')} = \frac{\pi_{\hat{F}^*}(s)}{\pi_{\hat{F}^*}(s')},
\]

for all \( \hat{F}, \hat{F}^* \in \mathcal{F} \) and \( s, s' \in S(F, X, \varphi) \).

4.3 The discovery of new feasible acts

The introduction of new feasible acts may or may not increase the number of states. In either case, however, it increases the number of coordinates defining a state. If it also increases the number of states, the newly defined states constitute a finer partition of the original state space. Thus, if \( F \subset F' \) then \( S(F, X, \varphi) \cap S(F', X, \varphi^*) = \emptyset \), and for each \( s \in S(F, X, \varphi) \) there corresponds an event \( E(s) \subset S(F', X, \varphi^*) \) defined by \( E(s) = \{ s' \in S(F', X, \varphi^*) \mid P_{S(F', X, \varphi^*)}(s') = s \} \), where \( P_{S(F', X, \varphi^*)}(\cdot) \) is the projection of \( S(F', X, \varphi^*) \) on \( S(F, X, \varphi) \). Using these notations we state the next axiom, which is analogous to axiom (A.7). The axiom requires that if two acts on the original state space disagree on two states, then the preference ranking of these acts is the same as that of two acts that disagree, in the same way, on the corresponding events in the expanded state space.

\[(A.8) \text{ (Projection consistency)} \]

For all \( \hat{F}, \hat{F}' \in \mathcal{F} \) such that \( F \subset F' \), \( p, q, \bar{p}, \bar{q} \in \Delta(X) \), \( h \in \hat{F}, h' \in \hat{F}' \), \( s, s' \in S(F, X, \varphi) \) and \( E(s), E(s') \subset S(F', X, \varphi^*) \), \( \left( h - s_{E(s)} p \right) - s_{E(s')} \bar{p} \) \( \succeq_{\hat{F}} \left( h' - s_{E(s') \bar{q}} \right) \) if and only if \( \left( h'_{-E(s)q} \right) - s_{E(s')} \bar{q} \) \( \succeq_{\hat{F}'} \left( h_{-E(s)} p \right) - s_{E(s')} \bar{q} \).

4.4 Representation of preferences when growing awareness is due to the discovery of new feasible acts

The representation theorem below describes how a decision maker’s beliefs evolve as he becomes aware of new feasible acts. As before, the decision maker is a subjective expected utility maximizer. When he becomes aware of new feasible acts, the decision maker updates his beliefs in a way that the likelihood ratios of events in the original state space remain

\[\text{Suppose that } |F| = r \text{ and } |F'| = k > r. \text{ Let } s = (c_1, ..., c_k) \in S(F', X, \varphi^*), \text{ then } P_{S(F, X, \varphi)}(s) = (c_1, ..., c_r) \in S(F, X, \varphi). \]

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intact. Because of the difference in the evolution of the state space, probability mass is shifted from states in the prior state space to the corresponding events the posterior state space, in such a way as to preserve the likelihood ratios of the events in the posterior state space and their corresponding projected states in the prior state space.13

Theorem 4 For each $\hat{F} \in \mathcal{F}$, let $\succsim_{\hat{F}}$ be a binary relation on $\hat{F}$. Then for all $\hat{F}, \hat{F}' \in \mathcal{F}$, the following two conditions are equivalent:

(i) Each $\succsim_{\hat{F}}$ satisfies (A.1) - (A.6) and, jointly, $\succsim_{\hat{F}}$ and $\succsim_{\hat{F}'}$ satisfy (A.8).

(ii) There exists a real-valued, non-constant, affine function, $U$ on $\Delta(X)$ and, for each $\hat{F} \in \mathcal{F}$, there is a probability measure $\pi_{\hat{F}}$ on $S(F, X, \varphi)$, such that for all $f, f' \in \hat{F}$,

$$ f \succsim_{\hat{F}} f' \iff \sum_{s \in S(F, X, \varphi)} U(f(s)) \pi_{\hat{F}}(s) \geq \sum_{s \in S(F, X, \varphi)} U(f'(s)) \pi_{\hat{F}}(s). $$

Moreover, $U$ is unique up to positive linear transformations, $\pi_{\hat{F}}$ is unique, and if $F \subset F'$ then

$$ \frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}'}(s')} = \frac{\pi_{\hat{F}'}(E(s))}{\pi_{\hat{F}'}(E(s'))}, $$

for all $s, s' \in S(F, X, \varphi)$ and $E(s), E(s') \subset S(F^*, X, \varphi^*)$, where $E(s)$ and $E(s')$, are the projections of $s$ and $s'$ on $S(F^*, X, \varphi^*)$.

4.5 Discovery of new acts-consequences links

The discovery of new acts-consequences links or the discovery that some links that were believed to exist are, in fact, nonexistent, do not affect the conceivable state space. Rather such discoveries expand or contract only the feasible state space. To model this, fix $C$ and $F$, and let $\varphi$ denote the original correspondence describing the links the decision maker believes to exist between $F$ and $C$. Suppose that a new link is established, and denote by $\varphi'$ the new correspondence, where $\varphi(f \mid s_{-f}) \subset \varphi'(f \mid s_{-f})$, for some $f \in F$ and $s \in C^F$. We denote by $F^*$ the set of acts redefined to take into account the new possible ranges of

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13 This is “reverse Bayesianism” applied to the present context. Li (2008) conjectures an axiomatization of the link between preferences under full awareness and those under pure unawareness and states a proposition linking the evolution of beliefs. This is in the spirit of Theorem 4. Li’s axiom neither implies, nor is it implied by, our projection consistency axiom.
\( f \in F \). Note that since the number of acts is the same in \( F \) and \( F^* \), \( C^{F^*} = C^F \), that is, the conceivable state space is unchanged. Hence, we can write \( C^F \) in place of \( C^{F^*} \) in Theorem 5 below.

Using these notations we restate axiom (A.7) as follows:

(A.7') (Updating consistency) For all \( \hat{F}, \hat{F}^* \in F \), if \( S(F^*, X, \varphi') \supset S(F, X, \varphi) \) and \( f', g' \in \hat{F}^* \), \( f' = f \) and \( g' = g \) on \( S(F, X, \varphi) \) and \( f' = g' \) on \( S(F^*, X, \varphi') - S(F, X, \varphi) \) then \( f \succ_{\hat{F}} g \) if and only if \( f' \succ_{\hat{F}^*} g' \).

Similarly, if the feasible state space is contracted due to the nullification of a link that was supposed to exist, (that is, \( \varphi'(f | s_{-f}) \subset \varphi(f | s_{-f}) \), for some \( f \in F \) and \( s \in C^F \)), then Axiom (A.7') can be restated as:

(A.7'') (Bayesian updating) For all \( \hat{F}, \hat{F}^* \in F \), if \( S(F^*, X, \varphi') \subset S(F, X, \varphi) \) and \( f, g \in \hat{F} \), \( f = f' \) and \( g = g' \) on \( S(F^*, X, \varphi') \) and \( f = g \) on \( S(F, X, \varphi) - S(F^*, X, \varphi') \), then \( f' \succ_{\hat{F}^*} g' \) if and only if \( f \succ_{\hat{F}} g \).

4.6 Representation of preferences in the wake of changing acts-consequences links

We show next that the process of updating the zero probability events in the wake of discovery of new links among acts and consequences is the exact counterpart of Bayesian updating in the wake of discovery that some links that were presumed to exist, in fact do not exist.

**Theorem 5.** For each \( \hat{F} \in F \), let \( \succ_{\hat{F}} \) be a binary relation on \( \hat{F} \) then, for all \( \hat{F}, \hat{F}^* \in F \), the following two conditions are equivalent:

(i) Each \( \succ_{\hat{F}} \) satisfies (A.1) - (A.6) and, jointly, \( \succ_{\hat{F}} \) and \( \succ_{\hat{F}^*} \) satisfy (A.7') and (A.7'').

(ii) There exists a real-valued, non-constant, affine function, \( U \), on \( \Delta(X) \) and, for each \( \hat{F} \in F \), there is a probability measure \( \pi_{\hat{F}} \) on \( C^F \) such that \( \pi_{\hat{F}}(C^F - S(F, X, \varphi)) = 0 \) and, for all \( f, f' \in \hat{F} \),

\[
\frac{f \succ_{\hat{F}} f'}{\sum_{s \in S(F, X, \varphi)} U(f(s)) \pi_{\hat{F}}(s)} \geq \frac{\sum_{s \in S(F, X, \varphi)} U(f'(s)) \pi_{\hat{F}}(s)}{11}
\]
Moreover, $U$ is unique up to positive linear transformations, $\pi_\hat{F}$ is unique, and

$$\frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(s')} = \frac{\pi_{\hat{F}^*}(s)}{\pi_{\hat{F}^*}(s')}$$

for all $s, s' \in S(F, X, \varphi) \cap S(F^*, X, \varphi')$.\(^{14}\)

The proof of Theorem 5 is based on the same argument as that of Theorem 3 and is omitted.

## 5 Concluding Remarks

The model presented in this paper predicts the relative likelihoods of events in the original state space, but is silent about the absolute levels of these probabilities. In other words, our theory does not predict the probability of the new events in the expanded state space. This may appear as a serious limitation of our approach. We claim, however, that this appearance is misleading. In fact, the relation between the prior and posterior probabilities in our model is not essentially different from the Bayesian model.

To grasp this claim, consider the Bayesian model. In that model, new information shrinks the state space by rendering null events that were assigned positive prior probabilities. Furthermore, given the prior probability of an event that has been rendered null, the Bayesian model predicts the absolute levels and, consequently, the likelihood ratios, of the posterior probabilities of all the events in the original algebra. These predictions, however, are predicated on the prior, about which the Bayesian model is silent. In Savage’s (1954) model, the prior is derived from a primitive preference relation over acts.

Our approach is analogous. Rather than being silent on the prior, it is silent on the posterior probabilities of the newly discovered events. If we proceed analogously to Savage (1954), the posterior is derived from a primitive preference relation on the acts defined over the expanded state space. Given the posterior, our model predicts the absolute probabilities and, consequently, the likelihood ratios, of all the events in the original algebra, including those between newly discovered and previously known events.

\(^{14}\)Notice that $S(F, X, \varphi) \cap S(F^*, X, \varphi') = S(F, X, \varphi)$ or $S(F, X, \varphi) \cap S(F^*, X, \varphi') = S(F^*, X, \varphi')$. 

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6 Proofs

6.1 Proof of theorem 3.

(Sufficiency) By Theorems 1 and 2, for all \( \hat{F} \in \mathcal{F} \), \( f \in \hat{F} \), and \( p, q \in \Delta (X) \),

\[
f_{-s}p \succ_{\hat{F}} f_{-s}q \iff \sum_{x \in \text{Supp}(p)} u_{\hat{F}}(x) p(x) \geq \sum_{x \in \text{Supp}(q)} u_{\hat{F}}(x) q(x) .
\]  

(13)

By definition of \( \succeq_{\hat{F}} \) and axiom (A.6), for every \( \hat{F} \in \mathcal{F} \), \( f \in \hat{F} \), and \( p, q \in \Delta (X) \),

\[
f_{-s}p \succ_{\hat{F}} f_{-s}q \iff l_p \succeq l_q .
\]  

(14)

By Theorem 1,

\[
l_p \succeq l_q \iff \sum_{z \in \text{Supp}(l_p)} u(z) l_p(z) \geq \sum_{z \in \text{Supp}(l_q)} u(z) l_q(z) .
\]  

(15)

But \( l_p(z) := p(t^{-1}(z)) \) and \( l_q(z) := q(t^{-1}(z)) \), for all \( z \in \mathbb{Z} \). In particular, if \( p = \delta_x \) then \( l_p(z) = \delta_z \), where \( z = t(x) \).

Fix \( z \in \mathbb{Z} \) and let \( x, x' \in t^{-1}(z) \). Then, \( \delta_{t(x)} = \delta_{t(x')} \) and, by Theorem 1, \( u(t(x)) = u(t(x')) = u(z) \). By (14), this implies that \( f_{-s}\delta_x \sim_{\hat{F}} f_{-s}\delta_{x'} \), for all \( \hat{F} \in \mathcal{F} \), \( f \in \hat{F} \), and \( x, x' \in t^{-1}(z) \). Thus, by the representation (13), \( u_{\hat{F}}(x) = u_{\hat{F}}(x') \), for all \( x, x' \in t^{-1}(z) \), and \( \hat{F} \in \mathcal{F} \). We denote this fact by defining \( u_{\hat{F}}(t^{-1}(z)) := u_{\hat{F}}(x) \), for \( x \in t^{-1}(z) \).

Using these notations, the representation (13) may be written as follows:

\[
f_{-s}p \succ_{\hat{F}} f_{-s}q \iff \sum_{z \in \text{Supp}(l_p)} u_{\hat{F}}(t^{-1}(z)) p(t^{-1}(z)) \geq \sum_{z \in \text{Supp}(l_q)} u_{\hat{F}}(t^{-1}(z)) q(t^{-1}(z)) .
\]  

(16)

But (14), (15), and (16) imply that

\[
\sum_{z \in \text{Supp}(l_p)} u_{\hat{F}}(t^{-1}(z)) p(t^{-1}(z)) \geq \sum_{z \in \text{Supp}(l_q)} u_{\hat{F}}(t^{-1}(z)) q(t^{-1}(z))
\]  

(17)

if and only if

\[
\sum_{z \in \text{Supp}(l_p)} u(z) l_p(z) \geq \sum_{z \in \text{Supp}(l_q)} u(z) l_q(z) .
\]  

(18)

Since, \( l_p(z) := p(t^{-1}(z)) \), for all \( z \in \mathbb{Z} \), the equivalence of (17) and (18), and the uniqueness of the von Neumann-Morgenstern utility function imply that \( u_{\hat{F}}(t^{-1}(z)) = bu(z) + a \),
Let \( f' \preceq_{\hat{F}^*} g \) if and only if
\[
\sum_{s \in S(F,X,\varphi)} U(\hat{F}, s) \pi_{\hat{F}^*}(s) \geq \sum_{s \in S(F,X,\varphi)} U(g, s) \pi_{\hat{F}^*}(s),
\]
which, by the definition of \( f' \) and \( g' \), is equivalent to
\[
\sum_{s \in S(F,X,\varphi)} U(f, s) \pi_{\hat{F}^*}(s) \geq \sum_{s \in S(F,X,\varphi)} U(g, s) \pi_{\hat{F}^*}(s).
\]
But Axiom (A.7) implies
\[
f \preceq_{\hat{F}} g \iff f' \preceq_{\hat{F}^*} g'.
\]
By Theorem 2 and the representation (20),
\[
f \preceq_{\hat{F}} g \iff \sum_{s \in S(F,X,\varphi)} U(f, s) \pi_{\hat{F}}(s) \geq \sum_{s \in S(F,X,\varphi)} U(g, s) \pi_{\hat{F}}(s).
\]
We thus have that the expressions in (22) and (24) are equivalent. Now, by the uniqueness of the probabilities in Theorem 2,
\[
\frac{\pi_{\hat{F}^*}(s)}{\sum_{s \in S(F,X,\varphi)} \pi_{\hat{F}^*}(s)} = \pi_{\hat{F}}(s), \text{ for all } s \in S(F,X,\varphi).
\]
(Necessity) The necessity of (A.1)-(A.5) is an implication of Theorem 2. The necessity of (A.6) and (A.7) is immediate.

The uniqueness part is an implication of the uniqueness of the utility and probability in Theorem 2.
6.2 Proof of Theorem 4.

(Sufficiency) By Theorem 2 and since $X^F - S(F, X, \varphi)$ is a null event, for all $\hat{F} \in \mathcal{F}$, and $f, g \in \hat{F}$,

$$f \succ_{\hat{F}} g \iff \sum_{s \in S(F, X, \varphi)} U_{\hat{F}}(f(s)) \pi_{\hat{F}}(s) \geq \sum_{s \in S(F, X, \varphi)} U_{\hat{F}}(g(s)) \pi_{\hat{F}}(s),$$

(26)

where $U_{\hat{F}}$ is affine.

Let $u$ be the von Neumann-Morgenstern utility function representing $\succsim$ on $\Delta(Z)$. Then, by the same argument as in the proof of Theorem 3, and invoking axiom (A.6), $u_{\hat{F}}(t^{-1}(z)) = bu(z) + a$, for all $z \in Z$ and $\hat{F} \in \mathcal{F}$. Let $U(f(s)) := \sum_{z \in \text{Supp}(l(z))} u(z) l_f(z)$, for all $f \in \hat{F}$ and $s \in S(F, X, \varphi)$. Then, for all $\hat{F} \in \mathcal{F}$, and $f, g \in \hat{F}$,

$$f \succ_{\hat{F}} g \iff \sum_{s \in S(F, X, \varphi)} U(f(s)) \pi_{\hat{F}}(s) \geq \sum_{s \in S(F, X, \varphi)} U(g(s)) \pi_{\hat{F}}(s).$$

(27)

Let $\hat{F}, \hat{F}' \in \mathcal{F}$ and, without loss of generality, suppose that $S(F', X, \varphi)$ is a refinement of the partition $S(F, X, \varphi)$.

Take $(h_{-E(s)}^{-} p)_{-E(s')}^{-} \bar{p}) \in \hat{F}'$ and $(h_{-E(s)}^{-} q)_{-E(s')}^{-} \bar{q}) \in \hat{F}'$ as defined in Axiom (A.8). For these acts, (27) is equivalent to

$$\left( (h_{-E(s)}^{-} p)_{-E(s')}^{-} \bar{p}) \succ_{\hat{F}'} (h_{-E(s)}^{-} q)_{-E(s')}^{-} \bar{q}) \right) \iff (h_{-s}^{-} p)_{-s'}^{-} \bar{p}) \succ_{\hat{F}} (h_{-s}^{-} q)_{-s'}^{-} \bar{q}).$$

(28)

if and only if

$$U(p) \pi_{\hat{F}'}(E(s)) + U(\bar{p}) \pi_{\hat{F}'}(E(s')) \geq U(q) \pi_{\hat{F}}(E(s)) + U(\bar{q}) \pi_{\hat{F}}(E(s')).$$

(29)

By Axiom (A.8),

$$\left( (h_{-E(s)}^{-} p)_{-E(s')}^{-} \bar{p}) \right) \succ_{\hat{F}'} \left( (h_{-E(s)}^{-} q)_{-E(s')}^{-} \bar{q}) \iff (h_{-s}^{-} p)_{-s'}^{-} \bar{p}) \succ_{\hat{F}} (h_{-s}^{-} q)_{-s'}^{-} \bar{q}).$$

(30)

By (27),

$$((h_{-s}^{-} p)_{-s'}^{-} \bar{p}) \succ_{\hat{F}} ((h_{-s}^{-} q)_{-s'}^{-} \bar{q})$$

if and only if

$$\sum_{s \in S(F, X, \varphi)} U(((h_{-s}^{-} p)_{-s'}^{-} \bar{p}) \pi_{\hat{F}}(s) \geq \sum_{s \in S(F, X, \varphi)} U(((h_{-s}^{-} q)_{-s'}^{-} \bar{q}) \pi_{\hat{F}}(s),$$

---

15 Hence, $F \subset F'$.  

---
which, since common terms cancel out, is equivalent to

$$U(p)\pi_F(s) + U(\bar{p})\pi_F(s') \geq U(q)\pi_F(s) + U(\bar{q})\pi_F(s').$$  \hspace{1cm} (31)

By (30), the expressions (28) and (31) are equivalent, which holds for all \(p, \bar{p}, q, \bar{q} \in \Delta(X)\), if and only if

$$\frac{\pi_F(s)}{\pi_F(s')} = \frac{\pi_{F'}(E(s))}{\pi_{F'}(E(s'))},$$  \hspace{1cm} (32)

for all \(s, s' \in S(F, X, \varphi)\) and \(E(s), E(s') \subset S(F', X, \varphi^*)\), where \(E(s)\) and \(E(s')\) are the projections of \(s\) and \(s'\) on \(S(F', X, \varphi^*)\).

(Necessity) The necessity of (A.1)-(A.6) is an implication of Theorem 2. The necessity of (A.8) is immediate.  \hspace{1cm} ■
References


