

An Axiomatization of Borda's Rule*

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Since Arrow's theorem [1], social choice theory for more than two alternatives has been concerned to a great degree with impossibility theorems, that is, statements to the effect that no desirable social decision functions exist satisfying certain proposed axioms. The point of view taken in this paper is that more positive results can be obtained by considering different types of axioms. In particular, we do not insist on the "independence of irrelevant alternatives." This allows us to consider various "scoring" methods, and in particular, Borda's rule. The object of this paper is to present a set of axioms that characterize Borda's rule uniquely.

We shall assume only that we want to find a social *choice* function, that is, a rule which specifies a set of winning alternatives for any specification of voters' preferences. This notion has been very successfully developed by Fishburn [4]. However, unlike Fishburn, we shall assume that a given social choice function is defined for any finite number of voters.

Let $A = \{a_1, a_2, \dots, a_m\}$ be the set of alternatives for election, and let the voters be named by elements of N , the set of natural numbers. We shall assume for the present that every voter's preferences can be expressed by a linear order on the alternative set A . (Later we shall see that this assumption can be relaxed.) We denote a linear order by a sequence $a_{i_1}, a_{i_2}, \dots, a_{i_m}$, where for $j < k$, a_{i_j} is preferred to a_{i_k} .

Let \mathcal{P} be the set of all linear orders on A . A *profile* is a function w from a finite subset $V \subset N$ to \mathcal{P} . A *social choice function* (abbreviated SCF) is a function f that assigns, to each profile, a nonempty subset of A called the *choice set* for that profile.

The following two non-bias conditions are generally considered desirable for any SCF f . f is *anonymous* if it depends only on the numbers of voters having each preference order, that is, $f(w)$ depends only on the

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numbers $|w^{-1}(P)|$ for all $P \in \mathcal{P}$. Any permutation σ of the alternative set A induces a permutation $\hat{\sigma}$ on the profiles in a natural way. We say that f is *neutral* if $f(\hat{\sigma}(w)) = \sigma f(w)$ for any such permutation σ and every profile w .

By allowing the number of voters to vary, we are led to consider certain consistency conditions which relate choices made by subsets of voters to choices made by their union. In particular, if two disjoint subsets of voters V and V' would choose the same alternative using f , then their union should also choose this alternative using f . Furthermore, any alternative not chosen by at least one of the subsets is in some sense not as "good" as an alternative chosen by both. Formally, f is *consistent* if whenever w, w' are profiles for disjoint voter sets V and V' , then

$$f(w) \cap f(w') \neq \emptyset \quad \text{implies} \quad f(w) \cap f(w') = f(w + w'), \quad (1)$$

where $w + w'$ is the profile on $V \cup V'$ that agrees with w on V and with w' on V' . Consistency is thus a kind of Pareto condition on subsets of voters. (Independently of this author, Smith [6] and Fine and Fine [3] define a notion similar to consistency for social ordering functions, and they prove some related results.)

The connection with the Pareto condition is illustrated by our next definition. A SCF is said to be *faithful* if "socially most preferred" and "individually most preferred" have the same meaning when society consists of a single individual; i.e., $f(w) = \{a_i\}$ when w represents a single individual whose most preferred alternative is a_i . If f is faithful and consistent, then clearly f is *Pareto*, that is, $f(w) = \{a_i\}$ for any profile w in which a_i is preferred to all other alternatives by every voter.

Black [2] gives an excellent survey of various types of social choice functions. We shall now mention several examples relevant to this discussion. The *Condorcet* SCF is the function that chooses those alternatives that tie or beat every other alternative under pairwise simple majority voting. Although the Condorcet function is not everywhere defined, i.e., may yield no choice for certain profiles, it is clearly consistent on its domain of definition. The *plurality* SCF is the function that chooses those alternatives with the most first-place votes. Clearly the plurality function is also consistent.

A third consistent function that is well known is the so-called *Borda rule*. Given a profile w on m alternatives we assign a score of $m + 1 - 2i$ to an alternative every time it is i th most preferred by some voter. The Borda rule chooses the alternative(s) with highest total score, summed over all voters. Denote the total score of alternative a_k by $\beta_k(w)$. Also, for any distinct j and k , $1 \leq j, k \leq m$, let $\pi_{jk}(w)$ be the number of voters in w preferring a_j to a_k . Now an alternative is i th most preferred by a

voter if and only if it is preferred to $m - i$ others and $i - 1$ others are preferred to it. Hence for the whole profile w we have

$$\beta_k(w) = \sum_{\substack{1 \leq j \leq m \\ j \neq k}} \pi_{kj}(w) - \pi_{jk}(w). \quad (2)$$

We say that a SCF f has the *cancellation* property if any one voter's statement of binary preference—e.g., “ a_j is preferred to a_k ”—can be balanced or cancelled by any other voter's contrary statement, “ a_k is preferred to a_j .” Thus, if for all pairs (a_i, a_j) of alternatives the number of voters preferring a_i to a_j equals the number preferring a_j to a_i , then a tie between all alternatives should be declared. Formally then we say that f has the cancellation property if for any profile w

$$\pi_{ij}(w) = \pi_{ji}(w) \quad \text{for all } i \neq j \text{ implies } f(w) = A. \quad (3)$$

While the plurality function does not have this property, both the Condorcet and Borda functions do. One difficulty with the Condorcet function, however, is that it is not a SCF in the full sense of the word, since it is not everywhere defined.

Now we can state the principal result of this paper.

THEOREM 1. *For any fixed number m of alternatives, there is one and only one social choice function that is neutral, consistent, faithful, and has the cancellation property—namely, Borda's rule.*

Before proving Theorem 1 let us consider some particular consequences of the assumptions. We say that a SCF f is *based on pairwise comparisons* if f depends only on the net number of voters for or against in all votings between pairs of alternatives, that is, if $f(w)$ depends only on the numbers $\pi_{ij}(w) - \pi_{ji}(w)$ for $1 \leq i \neq j \leq m$. In other words, f is based on pairwise comparisons means that all assertions of preference between two alternatives carry equal weight, independent of the names of the voters and independent of their preferences on other pairs. This equal treatment of all binary assertions of preference seems intrinsically desirable for many applications.

Equation (2) shows at once that the Borda rule is based on pairwise comparisons. The Condorcet function is also based on pairwise comparisons, and, except for special tie-breaking rules, so is the so-called *sequential* method of voting used in the U.S. House of Representatives. In fact, the property of being based on pairwise comparisons is very desirable from a practical standpoint, since it reduces the amount of information needed to make a social choice to just the outcomes of all votings between pairs.

LEMMA 1. *If f is consistent and has the cancellation property, then f is based on pairwise comparisons.*

Proof. Let w_1 and w_2 be profiles on voter sets V_1 and V_2 respectively such that w_1 and w_2 yield the same net number of votes for and against in all pairwise comparisons,

$$\pi_{ij}(w_1) - \pi_{ji}(w_1) = \pi_{ij}(w_2) - \pi_{ji}(w_2) \quad \text{for all } i \neq j. \quad (6)$$

Since every voter either prefers a_i to a_j or a_j to a_i we also have

$$\pi_{ij}(w_1) + \pi_{ji}(w_1) = |V_1| \quad \text{for } i \neq j, \quad (7)$$

$$\pi_{ij}(w_2) + \pi_{ji}(w_2) = |V_2| \quad \text{for } i \neq j. \quad (8)$$

Adding (7) and (8) to (6) we obtain

$$2\pi_{ij}(w_1) + |V_2| = 2\pi_{ij}(w_2) + |V_1| \quad \text{for } i \neq j. \quad (9)$$

Without loss of generality $n = |V_1| - |V_2| \geq 0$ and (9) implies n is even. Let t be any profile for a set T of n voters disjoint from $V_1 \cup V_2$ such that $n/2$ voters in T have preference order a_1, a_2, \dots, a_m and the $n/2$ remaining voters in T have preference order a_m, a_{m-1}, \dots, a_1 . Then $\pi_{ij}(t) = \pi_{ji}(t) = 1$ for $i \neq j$, so by (3), $f(t) = A$. By consistency,

$$f(w_2 + t) = f(w_2) \cap f(t) = f(w_2). \quad (10)$$

Also, by construction, $|V_2 \cup T| = |V_1|$ and

$$\pi_{ij}(w_2 + t) = \pi_{ij}(w_1) \quad \text{for all } i \neq j. \quad (11)$$

Let u, u' be isomorphic copies of w_1 defined on new voter sets U and U' , where $|U| = |U'| = |V_1|$, $U \cap U' = \emptyset$, and U, U' are disjoint from $V_1 \cup V_2 \cup T$. Let \bar{u}' be obtained from u' by inverting all preference orders. By the cancellation property, $f(u + \bar{u}') = f(w_1 + \bar{u}') = A$, hence by consistency,

$$\begin{aligned} f(u) &= f(u) \cap A = f(u + (w_1 + \bar{u}')) = f(w_1 + (u + \bar{u}')) \\ &= f(w_1) \cap A = f(w_1). \end{aligned}$$

Similarly, (11) implies $\pi_{ij}(w_2 + t + \bar{u}') = \pi_{ji}(w_2 + t + \bar{u}')$ for all $i \neq j$, so $f(w_2 + t + \bar{u}') = f(u + \bar{u}') = A$ and it follows as above that $f(u) = f(w_2 + t)$. Thus by (10),

$$f(w_2) = f(w_2 + t) = f(u) = f(w_1),$$

and the lemma is proved.

If f is based on pairwise comparisons then any profile w can be represented simply by a weighted, complete, directed graph having vertex set A , where the weight associated with (a_i, a_j) —the edge directed from a_i to a_j —is just $\pi_{ij}(w)$, the number of voters preferring a_i to a_j . (See Fig. 1.)

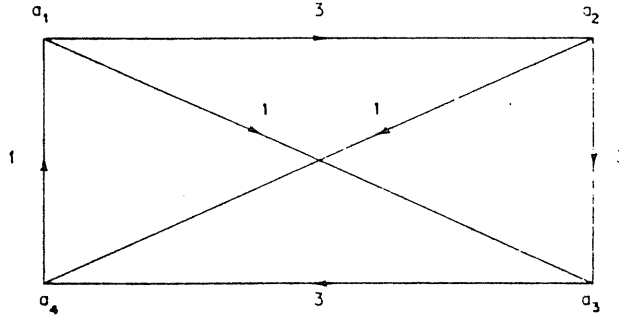


FIG. 1. Representation of the voting D .

Any such representation of a profile will be called a *voting*. For algebraic manipulation, we shall represent the voting obtained from w by the formal sum

$$\sum_{\substack{1 \leq i, j \leq m \\ i \neq j}} \pi_{ij}(w)(a_i, a_j). \tag{12}$$

$$D = 3(a_1, a_2) + (a_1, a_3) + (a_4, a_1) + 3(a_2, a_3) + (a_2, a_4) + 3(a_3, a_4)$$

Since f depends only on the net number of voters for any pair, we may identify (a_i, a_j) with $-(a_j, a_i)$ in any such expression, e.g.,

$$\pi_{ij}(w)(a_i, a_j) + \pi_{ji}(a_j, a_i) = (\pi_{ij}(w) - \pi_{ji}(w))(a_i, a_j).$$

(We can simply omit any terms with zero coefficients.)

The set \mathcal{D} of all sums $\sum_{i \neq j} q_{ij}(a_i, a_j)$, where the q_{ij} 's are rational and (a_i, a_j) is identified with $-(a_j, a_i)$, forms a dimension $\binom{m}{2}$ vector space over the field \mathbb{Q} of rationals, vector addition and scalar multiplication being defined in the obvious way. The set \mathcal{D}' of all votings is a subset of \mathcal{D} which is evidently closed under vector addition and scalar multiplication by integers. The vector $2(a_1, a_2)$ is in \mathcal{D}' , for it is the voting obtained, after cancellations, from any profile on two voters in which one voter has preference $a_1, a_2, a_3, \dots, a_{m-1}, a_m$ and the other voter has preference $a_m, a_{m-1}, \dots, a_3, a_1, a_2$. Similarly, we conclude that $2(a_i, a_j) \in \mathcal{D}'$ for all $i \neq j$.

Now any vector $D = \sum_{i \neq j} q_{ij}(a_i, a_j)$ in \mathcal{D} can be written

$$D = \sum_{i \neq j} q'_{ij}(a_i, a_j) \quad \text{where } q'_{ij} \geq 0 \text{ for all } i \neq j.$$

For a suitably large integer n , nq'_{ij} are all nonnegative integers, hence $2nD \in \mathcal{D}'$ by the above. It follows that $\mathcal{D} = \{D/n: D \in \mathcal{D}', n \text{ any positive integer}\}$, and we may interpret \mathcal{D} as the set of votings that arise from profiles in which fractional numbers of voters are allowed. We will call these *generalized votings* and *generalized profiles*.

Suppose now that f is consistent. This is equivalent to saying that for any $D, D' \in \mathcal{D}'$,

$$f(D) \cap f(D') \neq \emptyset \quad \text{implies} \quad f(D) \cap f(D') = f(D + D'). \quad (13)$$

In particular, for any $D \in \mathcal{D}'$ and positive integer n , we have $f(nD) = f(D + D + \cdots + D) = f(D)$. We can therefore extend f to \mathcal{D} by defining $f(D/n) = f(D)$. This is unambiguous, because if $D/n = D'/n'$, then $n'D = nD'$ and so $f(D) = f(D')$. Moreover, this extension is consistent in that (13) is satisfied for all $D, D' \in \mathcal{D}$. We also have $f(\lambda D) = f(D)$ for all positive rational λ .

Proof of Theorem 1. It is left to the reader to verify that the Borda function is neutral, consistent, faithful, and has the cancellation property.

Conversely, if f satisfies these conditions then f is based on pairwise comparisons, and by the preceding discussion we may extend f in a natural way to the domain \mathcal{D} . Moreover, this extension is also neutral, consistent, and faithful. In particular, neutrality implies that for any permutation σ of A we have

$$f(\bar{\sigma}(D)) = \sigma(f(D)), \quad (14)$$

where $\bar{\sigma}$ is the permutation of the components of \mathcal{D} induced by σ .

Given a generalized voting $D = \sum_{i \neq j} q_{ij}(a_i, a_j)$ it follows from (2) and (12) that the Borda score for alternative k is

$$\beta_k(D) = \sum_{j \neq k} q_{kj} - q_{jk}, \quad 1 \leq k \leq m. \quad (15)$$

(15) defines a linear function β from the vector space \mathcal{D} to the vector space \mathbb{Q}^m :

$$\beta \left[\sum_{i \neq j} q_{ij}(a_i, a_j) \right] = \left(\sum_{j=1} (q_{1j} - q_{j1}), \sum_{j=2} (q_{2j} - q_{j2}), \dots, \sum_{j=m} (q_{mj} - q_{jm}) \right). \quad (16)$$

For each i , $1 \leq i \leq m$, let w^i be the generalized profile in which $\frac{1}{2}$ voter has preference order $a_i, a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_{m-1}, a_m$ and $\frac{1}{2}$ voter has preference order $a_i, a_m, a_{m-1}, \dots, a_{i+1}, a_{i-1}, \dots, a_2, a_1$. Since f is

faithful and consistent, f is Pareto, so $\{a_i\}$ is the choice set for w^i . The voting corresponding to w^i is

$$E^i = \sum_{\substack{j=1 \\ j \neq i}}^m 1(a_i, a_j),$$

so,

$$f(E^i) = \{a_i\}, \quad 1 \leq i \leq m. \quad (17)$$

We also have

$$\begin{aligned} \beta(E^1) &= (m-1, -1, \dots, -1) \\ \beta(E^2) &= (-1, m-1, \dots, -1) \\ &\vdots \\ \beta(E^m) &= (-1, -1, \dots, m-1). \end{aligned} \quad (18)$$

These vectors have dimension $m-1$ in \mathbf{Q}^m , and since $\sum_{k=1}^m \beta_k(D) = 0$ for all $D \in \mathcal{D}$, they span image β . Now the dimension of kernel $\beta = \{D \in \mathcal{D} : \beta(D) = (0, 0, \dots, 0)\}$ is $\dim \mathcal{D} - \dim(\text{image } \beta)$, so

$$\dim(\text{kernel } \beta) = \binom{m}{2} - m + 1. \quad (19)$$

A k -cycle ($k \geq 3$) is a voting of form

$$(a_{i_1}, a_{i_2}) + (a_{i_2}, a_{i_3}) + \dots + (a_{i_{k-1}}, a_{i_k}) + (a_{i_k}, a_{i_1}),$$

where i_1, i_2, \dots, i_k are distinct. We denote this particular k -cycle by $C_{i_1 i_2 \dots i_k}$. For $3 \leq k < m$ every k -cycle is a rational linear combination of $(k+1)$ -cycles, since for any index $i_l \neq i_1, i_2, \dots, i_k$ we have

$$(k-1) C_{i_1 i_2 \dots i_k} = C_{i_l i_1 \dots i_k} + C_{i_1 i_l \dots i_k} + \dots + C_{i_1 i_2 \dots i_{k-1} i_l i_k}.$$

By induction on k we conclude that every k -cycle is a rational linear combination of m -cycles. Since the set of all k -cycles ($k \geq 3$) has dimension $\binom{m}{2} - m + 1$ (see, for example, Harary [4]), this is also the dimension of the set of m -cycles. But every m -cycle is in kernel β , so by (19) the m -cycles span kernel β . Thus for any $D \in \text{kernel } \beta$, we have $D = \sum_r q_r C^r$ for some m -cycles C^r and rationals q_r , and by proper orientation of the cycles we may assume all $q_r > 0$. However, every m -cycle is invariant under a transitive cyclic permutation of A , hence by neutrality, $f(C^r) = A$ for all r . By consistency

$$f(D) = \bigcap_r f(q_r C^r) = \bigcap_r f(C^r) = A,$$

that is,

$$f(D) = A \quad \text{for all } D \in \text{kernel } \beta.$$

Suppose now that $\beta(D) = \beta(D')$ for $D, D' \in \mathcal{D}$. Then $D - D' \in \text{kernel } \beta$ so $f(D - D') = A$. Then by consistency again

$$f(D) = f(D' + (D - D')) = f(D') \cap A = f(D'),$$

from which we conclude that for any voting D , $f(D)$ depends only on the Borda scores $\beta(D)$.

To prove the theorem, it remains only to show that f depends on the Borda scores in the "right" way, i.e., that f choose the alternatives with highest Borda score. Given $D \in \mathcal{D}$, suppose without loss of generality that $\beta_1(D) \geq \beta_2(D) \geq \dots \geq \beta_m(D)$, and we will show that $f(D) = \{a_j: \beta_j(D) = \beta_1(D)\}$. Since the vectors $\beta(E^i)$ of (18) span image β , we have $\beta(D) = \sum_{j=1}^m \lambda_j \beta(E^j)$ for some rationals λ_j . Since $\sum_{j=1}^m \beta(E^j) = 0$ this can be rewritten as

$$\beta(D) = \sum_{j=1}^{m-1} (\lambda_j - \lambda_{j+1}) \sum_{i < j} \beta(E^i) = \beta \left[\sum_{j=1}^{m-1} (\lambda_j - \lambda_{j+1}) \sum_{i < j} E^i \right]. \quad (20)$$

By (18),

$$\begin{aligned} \beta_i(D) - \beta_j(D) &= \left[(m-1) \lambda_i - \sum_{k \neq i} \lambda_k \right] - \left[(m-1) \lambda_j - \sum_{k \neq j} \lambda_k \right] \\ &= m(\lambda_i - \lambda_j), \end{aligned}$$

hence by our hypothesis on $\beta(D)$,

$$\lambda_j - \lambda_{j+1} \geq 0 \quad \text{for } 1 \leq j \leq m-1. \quad (21)$$

Moreover,

$$\{j: \beta_j(D) = \beta_1(D)\} = \{j: \lambda_j = \lambda_1\}. \quad (22)$$

Next, we prove by induction on $m-j$ that

$$f\left(\sum_{i < j} E^i\right) = \{a_1, a_2, \dots, a_j\}. \quad (23)$$

In case $m-j=0$, $f(\sum_{i < m} E^i) = A$, by neutrality. Suppose that (23) holds for $m-j-1$. $\sum_{i < j} E^i$ is invariant under the permutations $\bar{\sigma}$ and $\bar{\sigma}'$ induced by the cyclic permutations $\sigma = (a_1 a_2, \dots, a_j)$ and $\sigma' = (a_{j+1} a_{j+2}, \dots, a_m)$. If for some $k > j$, $a_k \in f(\sum_{i < j} E^i)$, then by applying σ' to $f(\sum_{i < j} E^i)$ we conclude by neutrality (14) that $a_{j+1} \in f(\sum_{i < j} E^i)$. But

then by (17) and consistency, $f(\sum_{i \leq j+1} E^i) = f(\sum_{i \leq j} E^i) \cap f(E^{j+1}) = \{a_{j+1}\}$ contradicting the induction hypothesis. Hence

$$k \leq j \quad \text{for all } a_k \in f\left(\sum_{i \leq j} E^i\right) \neq \emptyset.$$

Applying σ to $f(\sum_{i \leq j} E^i)$ we conclude that $f(\sum_{i \leq j} E^i) = \{a_1, a_2, \dots, a_j\}$, and (23) is proved.

By (20), $f(D) = f[\sum_{i=1}^{m-1} (\lambda_j - \lambda_{j+1}) \sum_{i \leq j} E^i]$, and since $\lambda_j - \lambda_{j+1} \geq 0$, consistency implies

$$\begin{aligned} f\left[\sum_{j=1}^{m-1} (\lambda_j - \lambda_{j+1}) \sum_{i \leq j} E^i\right] &= A \bigcap_{j: \lambda_j - \lambda_{j+1} > 0} \{a_1, a_2, \dots, a_j\} = \{a_j: \lambda_j = \lambda_1\} \\ &= \{a_j: \beta_j(D) = \beta_1(D)\}, \end{aligned}$$

the last by (22). This proves that f is the Borda rule. ■

The reader may verify by constructing examples that the four conditions of the theorem are independent. Theorem 1 and Lemma 1 can also be proved in much the same way if individual voters are allowed to express their preferences on the alternatives by weak orders, or, even more generally, by partial, antisymmetric relations. In these cases we simply define the Borda scores as in Eq. (2).

A particularly noteworthy aspect of Theorem 1 is that, while we only require a social *choice* function under our assumptions, the function we actually get (Borda's rule) in fact produces a (weak) social *ordering* of the alternatives for any profile. Indeed, it can be shown more generally that any social choice function which is anonymous, neutral, and consistent yields a weak social ordering of the alternatives. Specifically, we say that a SCF on m alternatives is a *scoring function* if there is a finite sequence s^1, s^2, \dots, s^k of vectors from \mathbf{R}^m such that every voter assigns score s_i^1 to his i th most preferred alternative, and the alternatives are "socially" ranked according to their total scores. Ties produced by the scoring vector s^1 are resolved using s^2 , and so forth. The choice set then consists of the top-ranked alternatives in the social order. In [7] it is shown that a social *choice* function is anonymous, neutral, and consistent if and only if it is a scoring function.

REFERENCES

1. K. J. ARROW, "Social Choice and Individual Values," 2nd ed., Wiley, New York, 1963.
2. D. BLACK, "The Theory of Committees and Elections," Cambridge University Press, Cambridge, England, 1958.

3. B. FINE AND K. FINE, Social choice and individual ranking, I and II, *Rev. Econ. Stud.*, to appear.
4. P. C. FISHBURN, "The Theory of Social Choice," Princeton University Press, Princeton, NJ., 1973.
5. F. HARARY, "Graph Theory," Addison-Wesley, Reading, MA, 1969.
6. J. H. SMITH, Aggregation of preferences with variable electorate, *Econometrica*, to appear.
7. H. P. YOUNG, Social choice scoring functions, *SIAM J. Appl. Math.*, to appear.