

NOTES AND COMMENTS

A NOTE ON PREFERENCE AGGREGATION<sup>1</sup>

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THE PURPOSE of this note is to give a short proof and somewhat stronger version of a result of J. H. Smith [1].

Let  $A$  be a finite set of  $n$  alternatives,  $\mathcal{R}$  the set of all partial, asymmetric transitive relations (subpreference orders) on  $A$ ,  $\mathcal{P}$  the set of all linear orders (strong preference orders) on  $A$ , and  $\hat{\mathcal{P}}$  the set of all weak orders (weak preference orders) on  $A$ . A profile,  $f$ , is a function from a finite subset of the nonnegative integers to  $\mathcal{P}$ . A subpreference function  $\varphi$  is a function from the set of all profiles to the set of relations  $\mathcal{R}$ . If  $\varphi(f) \in \hat{\mathcal{P}}$  for all  $f$ , we say that  $\varphi$  is a preference function. The image of  $f$  under  $\varphi$  will be denoted by the relation symbol  $>_f$ , and called the social preference. For all  $a, b \in A$  and  $f$ , write  $a \sim_f b$  ("a is socially indifferent to b") if not  $a >_f b$  and not  $b >_f a$ , and write  $a \succeq_f b$  if  $a >_f b$  or  $a \sim_f b$ .

A subpreference function  $\varphi$  is symmetric if it is anonymous on voters and neutral on alternatives. The domain of a symmetric  $\varphi$  can be identified with the set  $N^{m!}$  of all  $m!$ -tuples over the nonnegative integers (indexed by  $\mathcal{P}$ ). A subpreference function is consistent (in Smith's terminology, separable) if for any two profiles  $f$  and  $g$

- (1)  $a >_f b$  and  $a \succeq_g b$  imply  $a >_{f+g} b$ ,
- (2)  $a \sim_f b$  and  $a \sim_g b$  imply  $a \sim_{f+g} b$ .

This notion is analogous to one defined by this author for social choice functions [2 and 3].

Using a lengthy argument involving suprema, Smith shows that the only symmetric consistent preference functions are "generalized point systems," or in this author's terminology, scoring functions, i.e., there is a finite sequence  $s^1, s^2, \dots, s^k$  of vectors from  $R^m$  (scoring vectors) such that every voter gives score  $s_i^1$  to his  $i$ th most preferred alternative, and if alternative  $a$  gets a higher total score than  $b$ , then  $a$  is socially preferred to  $b$ . Ties relative to  $s^1$  are resolved using  $s^2$ , and so forth. In this note we shall use the separation theorem for convex sets to give a simple proof of a somewhat stronger statement—namely, that the only symmetric, consistent subpreference functions are the scoring functions. Thus, it is not necessary to assume the transitivity of social indifference—this follows from symmetry and consistency.

- (3) Let  $Q$  be the set of rational numbers. A subset  $S$  of  $Q^n$  is  $Q$ -convex if  $x, y \in S$  implies  $\lambda x + (1 - \lambda)y \in S$ , whenever  $\lambda \in Q$  and  $0 \leq \lambda \leq 1$ .

A basic result is that  $S \subseteq Q^n$  is  $Q$ -convex if and only if  $S = Q^n \cap \text{conv} S$ , where  $\text{conv} S$  is the real convex hull of  $S$ . Further, if  $S$  is  $Q$ -convex, then its closure,  $\bar{S}$ , is convex, and  $\bar{S} = \text{conv} S$ . (See [3, Lemmas (13) and (17)].)

**THEOREM:** A subpreference function is symmetric and consistent if and only if it is a scoring function.

**PROOF:** The "if" part is left to the reader. Conversely, let  $\varphi: N^{m!} \rightarrow R$  be symmetric and consistent. Letting  $e \in N^{m!}$  be the vector with "1" in every component, we may extend  $\varphi$  unambiguously to the domain  $Q^{m!}$  by defining  $\varphi(f - ne) = \varphi(f)$  and  $\varphi(f/n) = \varphi(f)$  for all positive integers  $n$ , and this extension is symmetric and consistent.

For  $m = 1$  the theorem is trivial, so let  $m \geq 2$  and fix distinct  $a, b \in A$ . For every profile  $f \in Q^{m!}$  define the linear map  $\alpha: Q^{m!} \rightarrow Q^m$  such that  $\alpha_i(f)$  is the number of times  $a$  occurs in rank  $i$  in  $f$  minus the number of times  $b$  occurs in rank  $i$  in  $f$ . Evidently  $\sum_{i=1}^m \alpha_i(f) = 0$  for all  $f$ , and the dimension of image  $\alpha$  is  $m - 1$ . Hence kernel  $\alpha = \{f \in Q^{m!}: \alpha(f) = 0\}$  has dimension  $m! - (m - 1)$ .

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For every  $p \in \mathcal{P}$ , let  $e_p$  denote the profile with "1" in coordinate  $p$ , "0" elsewhere. The set  $\mathcal{P}'$  of preferences in which neither  $a$  nor  $b$  has rank  $m$  has cardinality  $(m-1)(m-2)(m-2)!$ . With each  $p \in \mathcal{P}'$  associate the profile  $e_p + e_{\sigma(p)} + e_{\sigma^2(p)}$ , where  $\sigma$  permutes  $a, b$ , and the  $m$ th-ranked alternative in a 3-cycle. Consider the remaining preferences  $\mathcal{P} - \mathcal{P}'$  as the vertices of a graph in which  $p$  and  $q$  are adjacent if  $p$  is obtained from  $q$  by interchanging  $a$  with  $b$ , or by interchanging  $a$  with  $b$  and  $c$  with  $d$  for some  $c, d$  different from  $a, b$ . This graph has  $(m-1)$  connected components (one for each  $i < m$ ), so a maximal independent set  $I$  of edges has cardinality  $|\mathcal{P} - \mathcal{P}'| - (m-1) = 2(m-1)! - (m-1)$ . With each edge  $\{p, q\}$  in  $I$  associate the profile  $e_p + e_q$ . These profiles, together with the first type, form an independent set  $V$  of vectors in kernel  $\alpha$  having cardinality  $m! - (m-1)$ ; hence, they span kernel  $\alpha$ . Each  $f \in V$  is fixed by some permutation taking  $a$  to  $b$ , so symmetry and the transitivity of  $>$  imply that  $a \sim_f b$ ; hence, by (2),  $a \sim_f b$  for all  $f$  in kernel  $\alpha$ .

If  $\alpha(f) = \alpha(g)$ , then  $f - g$  is in kernel  $\alpha$ , so  $a \sim_{f-g} b$  and  $a >_f b$  if and only if  $a >_g b$ . Thus  $\varphi(f)$ , relative to  $a, b$ , depends only on  $\alpha(f)$ , and we may consider  $\varphi$  to have domain  $D = \{x \in \mathcal{Q}^m : \sum_{i=1}^m x_i = 0\}$ .

The sets  $D_1 = \{x \in D : a >_x b\}$  and  $D_2 = \{x \in D : b >_x a\}$  are  $\mathcal{Q}$ -convex (by consistency), and by the antisymmetry of  $>$  they are disjoint. Moreover (unless  $\varphi$  is trivial) they are nonempty. By (3),  $\bar{D}_1$  and  $\bar{D}_2$  are therefore nonempty convex sets. If  $\bar{D}_1 \cup \bar{D}_2$  is not the whole space  $\bar{D} = \{x \in \mathcal{R}^m : \sum_{i=1}^m x_i = 0\}$ , then  $\bar{D} - (\bar{D}_1 \cup \bar{D}_2)$  is open (in  $\bar{D}$ ) and contains a rational point  $x$ . For any  $y \in D_1$ , and sufficiently small rational  $\lambda > 0$ ,  $(1-\lambda)x + \lambda y$  is rational and in  $\bar{D} - (\bar{D}_1 \cup \bar{D}_2)$ , but  $a >_{(1-\lambda)x + \lambda y} b$ , a contradiction. Thus  $\bar{D} = \bar{D}_1 \cup \bar{D}_2$ . By symmetry,  $\bar{D}_1 = -\bar{D}_2$ , so they must both have nonempty interiors (relative to  $\bar{D}$ ). If interior  $\bar{D}_1$  meets interior  $\bar{D}_2$ , then their intersection contains a rational point  $x$ , and since interior  $\bar{D}_i = \text{interior}(cvx\bar{D}_i) = \text{interior}(cvxD_i)$  we have by (3) that  $x \in (cvxD_1 \cap cvxD_2) \cap D = D_1 \cap D_2$ , contradicting the disjointness of  $D_1$  and  $D_2$ .

Thus  $\bar{D}_1$  and  $\bar{D}_2$  are nonempty, disjoint, convex sets with disjoint interiors (relative to  $\bar{D}$ ), so the separation theorem for convex sets implies that there is a nonzero vector  $s^1 \in \bar{D}$  such that  $s^1 \cdot x \geq 0$  for all  $x \in \bar{D}_1$  and  $s^1 \cdot x \leq 0$  for all  $x \in \bar{D}_2$ .

If  $x \in D$  and  $s^1 \cdot x > 0$ , then  $x$  is in  $\bar{D}_1 - \bar{D}_2$ ; hence,  $x$  is the interior of  $\bar{D}_1$ , and since it is rational, it must be in  $D_1$ . Thus  $s^1 \cdot x > 0$  implies  $a >_x b$ . Similarly,  $s^1 \cdot x < 0$  implies  $b >_x a$ .

If  $D' = \{x \in D : s^1 \cdot x = 0\}$  contains points  $x$  such that  $a >_x b$ , apply the preceding argument to  $D'$  to obtain a nonzero scoring vector  $s^2$  such that  $s^2 \cdot x > 0$  implies  $a >_x b$  for all  $x \in D'$ , and so forth. This construction terminates in at most  $\dim D = m-1$  steps with a sequence of vectors  $s^1, s^2, \dots, s^k \in D$  that represent  $\varphi$  as a scoring function for the pair  $a, b$ . By symmetry, the same numbers must apply to all pairs from  $A$ , so  $\varphi$  is a scoring function (and in particular, a preference function). Q.E.D.

A subpreference function  $\varphi$  as in the theorem is *continuous* (Smith's *Archimedean*) if for every  $a, b \in A$  and sequence  $\{f_n\}$  from  $\mathcal{Q}^m$   $a \succ_{f_n} b$  and  $f_n \rightarrow f \in \mathcal{Q}^m$  implies  $a \succ_f b$ . If  $\varphi$  is continuous, then in the above proof we would have  $a \sim_x b$  for all  $x \in D'$ ; hence  $\varphi$  is determined by the single scoring vector  $s^1$ .

A theorem for social choice function analogous to the above theorem was proved independently by this author in [3]. That result says that if we only require consistency in *overall* comparisons between alternatives (i.e., for the "winners"), then it is still true that the only symmetric and consistent choice functions are the scoring functions. Actually, Smith's theorem can be obtained from [3] by noticing that any consistent preference function defines a consistent choice function in a natural way.

## REFERENCES

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- [3] ———: "Social Choice Scoring Functions," *SIAM Journal on Applied Mathematics*, forthcoming.