Adaptive Dynamics in Games Played by Heterogeneous Populations

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Consider a population of agents who play a game through repeated interactions, and adapt their behavior based on information about other agents’ previous behavior. The standard way of modeling such a process is to assume that everyone in the population is governed by the same adaptive rule, e.g., best response, imitation, or the replicator dynamic. This paper studies heterogeneous populations of agents in which some agents are best responders, others are conformists (they do what the majority does), and still others are nonconformists (they do the opposite of what the majority does). Unlike deterministic best reply processes, which in $2 \times 2$ games converge to Nash equilibrium, these heterogeneous processes may have limit cycles; moreover limit cycles may exist even when the proportion of non best responders is arbitrarily small. We show how to analyze the asymptotic behavior of such processes through a suitable generalization of Bendixson stability theory combined with stochastic approximation theory. Journal of Economic Literature Classification Numbers: C44, C73, D83. © 2000 Academic Press

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1. INTRODUCTION

This paper is concerned with adaptive dynamics in games played by populations of boundedly rational agents. Previous work in this area has focused particularly on three types of dynamics: best reply, replicator, and imitation, either with or without perturbations. (For surveys of this literature see Weibull, 1995; Vega-Redondo, 1996; Samuelson, 1997; Fudenberg and Levine, 1998; Young, 1998.) The standard assumption, however, is that every agent in the population uses the same kind of adaptive rule. In this paper we depart from this assumption by analyzing a model in which the population is heterogeneous. In particular, we consider three types of agents. Best responders choose the action that maximizes one-period payoffs given their beliefs about what other side will choose. Conformists choose the most common action among members of their own population irrespective of expected payoffs. Non-conformists choose the least common action among members of their own population.

Even in relatively simple $2 \times 2$ games the resulting dynamics can be quite complex. In particular, unlike best reply dynamics, the process can cycle instead of converging to a point close to Nash equilibrium. We illustrate this by simulating the process for various choices of the parameters (Section 3). In Section 5, we introduce the mathematical machinery needed to study this and related types of adaptive learning processes. In particular, we show how to analyze the asymptotic behavior of such processes using an extension of Bendixson's criterion for cancelling cycles, combined with techniques from stochastic approximation theory. We apply this result to characterize the behavior of the heterogeneous dynamics for all $2 \times 2$ games (Section 4). Detailed proofs are contained in the Appendix.

2. THE LEARNING MODEL

We begin with a simple example. Imagine that there are two competing layouts for typewriter keyboards—Dvorak and Qwerty—and that there are two populations of agents who may adopt these technologies: employers and secretaries. Each secretary must decide which keyboard to become proficient on, and each employer buys just one kind of typewriter. The payoff to a secretary depends on the frequency of keyboard layouts among employers, and the payoff to an employer depends on the frequency of keyboard proficiencies among secretaries. This has the structure of a coordination game, such as the one shown below.
Secretaries

D  Q

D  5, 5  0, 0

Employers

Q  0, 0  4, 4

Assume that, in each time period, one secretary and one new employer make an adoption decision. A myopically rational secretary estimates the distribution of keyboard types by asking a random sample of \(s\) employers what keyboard they currently use; she then chooses the keyboard that maximizes her expected payoff against her estimate of the empirical distribution (which of course may be inaccurate if her sample is small). A conformist secretary, by contrast, makes her decision by asking \(s\) other secretaries at random and choosing the most popular keyboard; a nonconformist chooses the less popular keyboard. The object of the analysis is to study the asymptotic behavior of this process, and to ask whether or not it converges to something close to Nash equilibrium.

More generally, consider a two-person game \(G\) with payoff matrix

\[
\begin{pmatrix}
\alpha_{11}, \beta_{11} & \alpha_{12}, \beta_{12} \\
\alpha_{21}, \beta_{21} & \alpha_{22}, \beta_{22}
\end{pmatrix}.
\]

In what follows, we shall distinguish between two cases.

*Coordination games* have three equilibria—two pure and one mixed. Without loss of generality we may assume that the pure equilibria are \((0, 0)\) and \((1, 1)\), and the mixed equilibrium is \((\beta, \alpha)\), where

\[
\alpha = \frac{\alpha_{22} - \alpha_{12}}{\alpha_{11} - \alpha_{21} - \alpha_{12} + \alpha_{22}} \in (0, 1) \quad \beta = \frac{\beta_{22} - \beta_{21}}{\beta_{11} - \beta_{21} - \beta_{12} + \beta_{22}} \in (0, 1).
\]

*Noncoordination games* have one equilibrium, which is fully mixed and is represented by \((\beta, \alpha)\) as above.

There are two disjoint populations of agents: row players (\(R\)) and column players (\(C\)). In every time period \(t = 1, 2, \ldots\) one pair of players is drawn independently on the history of play at random from \(R \times C\) to play the game. The *state* at \(t\) is a vector of positive integers \((a_1^t, a_2^t, b_1^t, b_2^t)\), where \(a_1^t, a_2^t\) are the numbers of row players who have chosen strategies 1 and 2, respectively, up to and including time \(t\), and \(b_1^t, b_2^t\) are the numbers of column players who have chosen 1 and 2, respectively, up to and including time \(t\). The initial state can be more or less arbitrary, subject to the feasibility of samples used for decision making.

Let \(s\) be a positive integer. When it is his turn to play, an agent acquires information by drawing a random sample (without replacement) of
the actions taken in previous time periods. The sample size \( s \) is given exogenously and measures the amount of information that players have (Young, 1993a, b; Kaniaovski and Young, 1995). Using techniques similar to those described below, one can analyze situations in which agents have different sample sizes, but for notational simplicity we shall assume here that the same sample size applies to everyone.

A best responder is an agent who chooses a myopic best reply to the empirical frequency distribution of actions taken by the other population, as estimated from his sample. A conformist is an agent who chooses the most frequent action among members of his own population, as estimated from his sample. A nonconformist is an agent who chooses the least frequent action among members of his own population, as estimated from his sample. We shall always assume that the sample size \( s \) is odd and greater than one, so that conformists and nonconformists have uniquely determined choices. (In the case of best responders, we shall assume that ties are resolved in favor of action 1, a convention that is adopted purely for technical convenience.)

This generates a dynamic process in which agents are diverse in two respects: the information they have, and the way they respond to their information. The asymptotic behavior of this process depends on: (i) the type of game under consideration (coordination or noncoordination), (ii) the sample size, and (iii) the proportions of the various types of agents in the population. Let \( 1 - \epsilon \) denote the proportion of best responders in each of the Row and Column populations, and let \( \epsilon(1 - \delta) \) and \( \epsilon\delta \) be the proportion of conformists and nonconformists, respectively. A particular interesting case arises when both \( \epsilon \) and \( \delta \) are small, but for the moment we allow all values \( \delta, \epsilon \in [0, 1] \).

The learning process can be studied most conveniently by projecting it into the space of proportions \( \mathcal{X} = [0, 1] \times [0, 1] \). Let \( X^t \) and \( Y^t \) be the proportions of the Row and Column populations playing action 1 up through time \( t \):

\[
X^t = a_t^1/a^t \quad \text{and} \quad Y^t = b_t^1/b^t \quad \text{where} \quad a^t = a_1^t + a_2^t, \quad b^t = b_1^t + b_2^t.
\]

For each \((x, y)\in\mathcal{X}\), let \( \xi(t, (x, y)) \) and \( \psi(t, (x, y)) \) be mutually independent random variables with values 0 or 1. We let \( \xi(t, (X^t, Y^t)) = 1 \) if Row plays strategy 1 at time \( t \), and \( \psi(t, (X^t, Y^t)) = 1 \) if Column plays strategy 1 at \( t \). Then

\[
ad_{1}^{t+1} = a_{1}^{t} + \xi(t, X^t, Y^t) \quad \text{and} \quad b_{1}^{t+1} = b_{1}^{t} + \psi(t, X^t, Y^t).
\]

Then we may write

\[
X^{t+1} = X^t + (1/a^{t+1})[\xi(t, X^t, Y^t) - X^t], \quad t \geq 1, \quad X^1 = a_1^1/a^1, \tag{1}
\]

\[
Y^{t+1} = Y^t + (1/b^{t+1})[\psi(t, X^t, Y^t) - Y^t], \quad t \geq 1, \quad Y^1 = b_1^1/b^1.
\]
The distributions of $\xi^t(\cdot)$ and $\psi^t(\cdot)$ are mixtures, with weights $\epsilon$ and $1 - \epsilon$, of the probability that the number of successes exceeds a certain fraction of the number of draws, $s$, in Bernoulli trials, and another mixture, with weights $\delta$ and $1 - \delta$, of two binomial probabilities. The probability of success changes with each draw, because of our assumption that players sample without replacement. For example, at time $t$, the probability of success, that is, of Column drawing an instance of Row playing strategy 1, is $X^t$ on the second draw. Since the sampling is without replacement, a success on the second draw occurs with probability

$$\frac{a^t_1 - 1}{a^t - 1} = \frac{a^t_1}{a^t} \frac{a^t}{a^t - 1} - \frac{1}{a^t - 1} = X^t \left(1 + \frac{1}{a^t - 1}\right) - \frac{1}{a^t - 1},$$

if the first draw was a success, while it is

$$\frac{a^t_1}{a^t - 1} = X^t \left(1 + \frac{1}{a^t - 1}\right),$$

if the first draw was a failure. At the beginning of period $t$, there have been $t - 1$ plays, so $a^t = a^1 + t - 1$. Hence, the corrections to the probability of success are of the order $1/t$ as $t \to \infty$.

Let

$$f^s_\gamma(z) = \sum_{i \geq \gamma s} \binom{s}{i} z^i (1 - z)^{s - i}.$$

This is the probability that $s$ draws with replacement contain more than $\gamma s$ of the other side playing action 1, when the true proportion is $z$. Let $\delta^t_\gamma(\cdot, z)$ denote the correction term that arises because the sampling is, in fact, without replacement. Then we can write the distribution of the random variables $\xi^t(\cdot)$ and $\psi^t(\cdot)$ as follows:

**three-equilibria:**

$$P\{\xi^t(x, y) = 1\} = (1 - \epsilon) f^s_\alpha(y) + \epsilon[(1 - 2\delta) f^s_{1/2}(x) + \delta] + (1 - \epsilon) \delta^t_\alpha(b', y) + \epsilon(1 - 2\delta) \delta^t_{1/2}(a', x),$$

$$P\{\psi^t(x, y) = 1\} = (1 - \epsilon) f^s_\beta(x) + \epsilon[(1 - 2\delta) f^s_{1/2}(y) + \delta] + (1 - \epsilon) \delta^t_\beta(a', x) + \epsilon(1 - 2\delta) \delta^t_{1/2}(b', y);$$

**one-equilibrium:**

$$P\{\xi^t(x, y) = 1\} = (1 - \epsilon) f^s_\alpha(y) + \epsilon[(1 - 2\delta) f^s_{1/2}(x) + \delta] + (1 - \epsilon) \delta^t_\alpha(b', y) + \epsilon(1 - 2\delta) \delta^t_{1/2}(a', x),$$

$$P\{\psi^t(x, y) = 1\} = (1 - \epsilon) f^s_{1-\beta}(1 - x) + \epsilon[(1 - 2\delta) f^s_{1/2}(y) + \delta] + (1 - \epsilon) \delta^t_{1-\beta}(a', 1 - x) + \epsilon(1 - 2\delta) \delta^t_{1/2}(b', y).$$
Note that the correction terms satisfy
\[
\sup_{z \in [0,1]} |\delta^s_\gamma(n, z)| \leq c_{\gamma, s}/n.
\]

3. SIMULATED PATHS OF THE LEARNING PROCESS

As a benchmark case, let us first consider the qualitative properties of the process described above when everyone is a best responder \((\epsilon = 0)\). In this case, the process is a stochastic variant of fictitious play in which the players have incomplete information about the history, and the stochastic shocks arise from sampling variability. The asymptotics of this process have been treated by Kaniowski and Young (1995). The main result is the following: from any initial state, the process converges to a point which is close to a Nash equilibrium of the game. In the case of \(2 \times 2\) coordination games it converges with probability one to a pure Nash equilibrium. In \(2 \times 2\) noncoordination games it converges to a point that is close to the unique mixed Nash equilibrium; moreover, the larger the sample size, the closer this point lies to the Nash equilibrium. The complexity of the process is illustrated, however, by the following additional property that it has in coordination games: if all actions have positive probability in the initial state, then no matter how close this state lies to one of the coordination equilibria (in the space \(\mathcal{X}\) of proportions), there is a positive probability that the process will eventually converge to the other coordination equilibrium (Kaniowski and Young, 1995).

When conformists and nonconformists are added to the mix, the asymptotic behavior of the process remains essentially the same as if there were no such agents, provided we are dealing with a coordination game. In noncoordination games, however, the asymptotic behavior can change in important ways—in particular, the process can cycle around the interior Nash equilibrium.

We illustrate this phenomenon by simulating the process for various values of the parameters \(\alpha, \beta, \epsilon,\) and \(\delta\). Instead of simulating (1) directly, however, we shall simulate the following continuous system of ordinary differential equations
\[
\frac{d}{dt} \tilde{z}^s = \tilde{g}^s(\tilde{z}^s),
\]
where \(\tilde{g}^s(\tilde{z}^s) = (g_1^s(\tilde{z}^s), g_2^s(\tilde{z}^s))\), \(\tilde{z}^s = (z_1^s, z_2^s)\). We use \(\tilde{z}^s\) instead of \((x^s, y^s)\) to distinguish between the deterministic and stochastic dynamics. The two cases are as follows:
three-equilibria:
\[
g_1^z(\tilde{z}) = (1 - \epsilon) f^z_\alpha(z_2) + \epsilon [(1 - 2\delta) f^z_{1/2}(z_1) + \delta] - z_1,
\]
\[
g_2^z(\tilde{z}) = (1 - \epsilon) f^z_\beta(z_1) + \epsilon [(1 - 2\delta) f^z_{1/2}(z_2) + \delta] - z_2;
\]

one-equilibrium:
\[
g_1^z(\tilde{z}) = (1 - \epsilon) f^z_\alpha(z_2) + \epsilon [(1 - 2\delta) f^z_{1/2}(z_1) + \delta] - z_1,
\]
\[
g_2^z(\tilde{z}) = (1 - \epsilon) f^z_{1-\beta}(1 - z_1) + \epsilon [(1 - 2\delta) f^z_{1/2}(z_2) + \delta] - z_2.
\]

The reason the deterministic, time homogeneous system (2) mimics the stochastic, time inhomogeneous system (1) is as follows. At time \( t \), let the process (1) be at the point \((z_1, z_2)\). Then the expected motion, given the current state \((z_1, z_2)\), is to the state
\[
(z_1 + (1/a^{t+1})[g^z_1(\tilde{z}) + O_1(t^{-1})]),
(z_2 + (1/b^{t+1})[g^z_2(\tilde{z}) + O_2(t^{-1})]),
\]
where \( O_i(t^{-1}) \) vanishes as \( t \to \infty \). Since \( a^{t+1} = a^1 + t \) and \( b^{t+1} = b^1 + t \), the long run behaviors of (1) and (2) are similar. A more formal argument can be found in the literature on stochastic approximation; see, for example, Benveniste et al. (1990). We remark that the idea of approximating a discrete stochastic evolutionary process by a continuous deterministic process is quite a standard technique, but there are many different ways of implementing it. In some evolutionary models, for example, the approach is to pass from a finite to an infinite population of players (see, for example, Binmore et al., 1995; and Boylan, 1995). In our model, by contrast, we keep the population finite but allow the weight of history to become unboundedly large.

First we shall study the limit behavior of this process using simulations; then we shall characterize it analytically. In particular, we shall study the limit properties of (2) for one-equilibrium games, which is the more interesting case. For all of the simulations given here we set \( \delta = 0 \). If \( \alpha = 0.41 \) and \( \beta = 0.45 \), then for all small enough \( \epsilon \) (including \( \epsilon = 0 \)) and all large enough \( s \), the process converges, as shown in Fig. 1. However, if \( \epsilon = 0.4 \), there is a limit cycle, as shown in Fig. 2. It turns out that when \( \alpha \) or \( \beta \) equals \( 0.5 \), the process cycles for all small \( \epsilon > 0 \) provided that \( s \) is sufficiently large. A typical picture in this case is given in Fig. 3, where \( \alpha = 0.5 \), \( \beta = 0.45 \), \( \epsilon = 0.4 \), and \( s = 31 \). On the other hand, if \( \epsilon = 0 \), the process always converges, as we remarked earlier; Fig. 4 illustrates this with \( \alpha = 0.5 \), \( \beta = 0.45 \), \( \epsilon = 0 \), and \( s = 31 \).

More complicated cycles arise when \( s \) increases without bound. Letting \( s \to \infty \), we obtain a system of form
\[
\frac{d}{dt} \tilde{z} = \tilde{g}(\tilde{z}) - \tilde{z},
\]

\( (3) \)
FIG. 1. $\alpha = 0.41$, $\beta = 0.45$, $\epsilon = 0.1$, $s = 51$.

FIG. 2. $\alpha = 0.41$, $\beta = 0.45$, $\epsilon = 0.4$, $s = 51$.

FIG. 3. $\alpha = 0.5$, $\beta = 0.45$, $\epsilon = 0.4$, $s = 31$. 
where $\tilde{g}(\cdot)$ is a piecewise constant function. The set $\mathcal{X}$ splits into nine domains $R_{ij}$, $i, j = 1, 2, 3$, inside which the function $\tilde{g}(\cdot)$ is constant, namely, $\tilde{g}_{ij} = (g_{1j}, g_{2j})$. The domains are given in Table I and the vectors $\tilde{g}_{ij}$ in Table II.

We shall see later that the solutions of this system include all possible limits for trajectories of (2) as $s \to \infty$. In particular, if cycles of (2) converge as sets in $R^2$ to a limit for $s \to \infty$, then this limit turns out to be a cycle of system (3). In Fig. 5 we see one such limit cycle ($\alpha = 0.41$, $\beta = 0.45$, $\epsilon = 0.2$, and $s = \infty$). Figure 6 exhibits a case with two limit cycles ($\alpha = 0.43$, $\beta = 0.48$, $\epsilon = 0.4$, and $s = \infty$).

Recall from our earlier discussion that the process with only best responders does not have periodic orbits (see Kaniovski and Young, 1995). The only way that a small proportion of conformists could lead to local instability of the steady state is if the term due to conformism in (2) has an

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**TABLE I**

The Partition of $\mathcal{X}$ Corresponding to the Limit System

<table>
<thead>
<tr>
<th>$R^{13}$</th>
<th>$R^{23}$</th>
<th>$R^{33}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0, \beta) \times (1/2, 1]$</td>
<td>$(\beta, 1/2) \times (1/2, 1]$</td>
<td>$(1/2, 1] \times (1/2, 1]$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$R^{12}$</th>
<th>$R^{22}$</th>
<th>$R^{32}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0, \beta) \times (\alpha, 1/2)$</td>
<td>$(\beta, 1/2) \times (\alpha, 1/2)$</td>
<td>$(1/2, 1] \times (\alpha, 1/2)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$R^{11}$</th>
<th>$R^{21}$</th>
<th>$R^{31}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0, \beta) \times [0, \alpha)$</td>
<td>$(\beta, 1/2) \times [0, \alpha)$</td>
<td>$(1/2, 1] \times [0, \alpha)$</td>
</tr>
</tbody>
</table>

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**TABLE II**

The Values of $\tilde{g}(\cdot)$ on $R_{ij}$

<table>
<thead>
<tr>
<th>$\tilde{g}_{13}$</th>
<th>$\tilde{g}_{23}$</th>
<th>$\tilde{g}_{33}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1 - \epsilon(1 - \delta), 1 - \epsilon \delta)$</td>
<td>$(1 - \epsilon(1 - \delta), \epsilon(1 - \delta))$</td>
<td>$(1 - \epsilon \delta, \epsilon(1 - \delta))$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\tilde{g}_{12}$</th>
<th>$\tilde{g}_{22}$</th>
<th>$\tilde{g}_{32}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1 - \epsilon(1 - \delta), 1 - \epsilon(1 - \delta))$</td>
<td>$(1 - \epsilon(1 - \delta), \epsilon \delta)$</td>
<td>$(1 - \epsilon \delta, \epsilon \delta)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\tilde{g}_{11}$</th>
<th>$\tilde{g}_{21}$</th>
<th>$\tilde{g}_{31}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\epsilon \delta, 1 - \epsilon(1 - \delta))$</td>
<td>$(\epsilon \delta, \epsilon \delta)$</td>
<td>$(\epsilon(1 - \delta), \epsilon \delta)$</td>
</tr>
</tbody>
</table>
arbitrarily large first derivative at the steady state. This is, in fact, the case when $\alpha = 1/2$ or $\beta = 1/2$: since the most popular choice in a large sample is likely to be the one that is most popular in the whole population, the conformists' response function converges to a step function as the sample size increases. Thus the derivatives at the equal shares state become unboundedly large, while the derivatives converge to zero away from this state. Hence in a game with a single equilibrium, conformity reinforces deviation from equal shares, which is bad for the stability of the mixed equilibrium.
4. THE MAIN RESULTS

We are now in a position to state our main results on the asymptotic behavior of this class of adaptive processes. Convergence with probability one for coordination games with three equilibria is covered by the following.

**Theorem 1.** Let $G$ be a nondegenerate $2 \times 2$ game with three equilibria, the mixed one being $(\beta, \alpha)$. Let $s > 1$ be an odd number, $\epsilon \in (0, 1/2)$, $\delta \in [0, 1]$ and $\epsilon \delta < \min(\alpha, \beta, 1/2) \leq \max(\alpha, \beta, 1/2) < 1 - \epsilon \delta$. For all sufficiently large $s$, the adaptive process (1) converges with probability one as $t \to \infty$. The limit $(X^*, Y^*)$ is a random vector that assumes exactly two values, which are arbitrarily close to $(\epsilon \delta, \epsilon \delta)$ and $(1 - \epsilon \delta, 1 - \epsilon \delta)$ as $s$ becomes large. When $\delta = 0$, the limit values are $(0, 0)$ and $(1, 1)$ independently of $s$.

An initial state is said to be rich if both strategies in both populations are played with positive probability, so the motion is not deterministic. This condition holds if and only if $a_1^1 > \min(\beta, 1/2)s$, $a_2^1 > \min(1 - \beta, 1/2)s$, and $b_1^1 > \min(\alpha, 1/2)s$, $b_2^1 > \min(1 - \alpha, 1/2)s$. It turns out that, if the initial state is rich, then with probability one all subsequent states are also rich, that is, the process is strictly stochastic in each period. It can be shown that, if the initial populations are rich, the limit state $(X^*, Y^*)$ in Theorem 1 assumes both of its values with strictly positive probabilities, which depend on the initial state.

The message of Theorem 1 is that, when the probability of imitators is small, the outcome of the collective “learning” process is close to the dynamically stable Nash equilibria of the game, and the asymptotic behavior is similar to the case where agents always choose best replies. The situation is quite different, however, when the game has a unique, interior equilibrium.

**Theorem 2.** Let $G$ be a nondegenerate $2 \times 2$ game with a unique equilibrium $(\beta, \alpha)$, which is fully mixed. Let $s > 1$ be an odd number, $\epsilon \in (0, 1/2)$, $\delta \in [0, 1]$ and $\epsilon \delta < \min(\alpha, \beta, 1/2) \leq \max(\alpha, \beta, 1/2) < 1 - \epsilon \delta$. When $\delta \geq 1/2$ and $s$ is sufficiently large, the adaptive process (1) converges with probability one to a deterministic limit as $t \to \infty$. Moreover, the limit $(\tilde{x}^s, \tilde{y}^s)$ is arbitrarily close to $(\beta, \alpha)$ when $s$ is sufficiently large. When $\delta < 1/2$, for all sufficiently large $s$ we have the following possibilities:

(a) if $\beta \neq 1/2$ and $\alpha \neq 1/2$, $(X^t, Y^t)$ converges with positive probability to $(\tilde{x}^s, \tilde{y}^s)$ from every initial state $(X^1, Y^1)$, provided that the initial state is rich; moreover, if $\epsilon$ is sufficiently small, $(X^t, Y^t)$ converges to $(\tilde{x}^s, \tilde{y}^s)$ with probability one;

(b) if $\beta = 1/2$ or $\alpha = 1/2$ or $\beta = \alpha = 1/2$, then $(X^t, Y^t)$ converges to $(\tilde{x}^s, \tilde{y}^s)$ with probability zero from every initial state.
The proof is given in the Appendix. The striking feature here is that, if at least one of the coordinates of the mixed equilibrium equals 1/2, then, no matter how small $\epsilon$ is, the process does not converge to a neighborhood of the equilibrium with positive probability provided that the sample size is sufficiently large. The intuitive reason for this can be explained as follows. Recall that $\delta/(1 - \delta)$ is the "ratio" of nonconformists to conformists. If the population consists solely of these two types, the dynamics splits into two independent one-dimensional processes. If there are only conformists ($\delta = 0$), each of the processes converges almost surely to 0 or 1 (see Dosi et al., 1994). If there are only nonconformists ($\delta = 1$), the processes converge almost surely to 1/2. A value of $\delta$ between 0 and 1 measures the relative strength of these tendencies. In the case of a noncoordination game, if conformists outnumber nonconformists ($\delta < 1/2$) and best repliers are present in the population, then the pull towards $(0, 0)$ and $(1, 1)$ due to conformism is in opposition to the pull towards $(\bar{x}_p, \bar{y}_q)$ due to best repliers. These two conflicting forces upset almost sure convergence. The conflict is even stronger if one or both of the coordinates of the mixed equilibrium equals 1/2. In this case, with probability one, there is no convergence to a point. A formal explanation of this phenomenon follows from the stability results in Lemma 3 of Section 5 below. When there is no convergence, the behavior of the trajectories can be studied using numerical simulations. In this case, Bendixson's theory implies that only singular points, cycles, and phase polygons can be attracting sets for the trajectories of system (2). Consequently, if the trajectories of (1) do not converge to the (unique) singular point, they can become concentrated on the closed orbits of (2) as $t \to \infty$. In the case where $\beta, \alpha \neq 1/2$, the process may converge with probability less than one, and instead fall into a cycle of (2). Only by restricting $\epsilon$ (and $\delta$) can we eliminate these cycles and guarantee convergence with probability one.

5. MATHEMATICAL ANALYSIS AND AN EXTENSION OF BENDIXSON'S THEORY

Standard methods from stochastic approximation theory deal mainly with systems having a single global attractor (Derievskii and Fradkov, 1974; Ljung, 1977; and Benveniste et al., 1990). Here we are dealing with systems having multiple attractors, which is the situation in a game with three equilibria. To analyze the asymptotic behavior of (1), let us compare it with (2). One can see that for every $\bar{z}_0 \in \mathcal{H}$ there is a single solution $\bar{z}(t), 0 \leq t < \infty$, of system (2) such that $\bar{z}(0) = \bar{z}_0$. Moreover, this solution belongs to $\mathcal{H}$, that is, $\bar{z}(t) \in \mathcal{H}$ for all $t \geq 0$. Thus, $\mathcal{H}$ is a positive invariant set for system (2). The proof of convergence to a point near Nash involves
four auxiliary results (see Kaniiovski and Young, 1995). One shows first that all singular points of (2) lie close to Nash equilibria of the game. Second, one establishes that the singular points are either linearly stable or linearly unstable. Third, one proves that all half-trajectories of (2) converge to its singular points; and, fourth, that the half-trajectories leading to linearly unstable points do not form phase polygons.

In the present case, difficulties arise with the third and fourth of these results. The theorem of Bendixson (see Hahn, 1967, p. 66) says that all half-trajectories of (2) belonging to a bounded domain in \( R^2 \) fall into one of the following three cases: (a) they approach a singular point (including the case when a half-trajectory coincides with a singular point); (b) they form cycles; (c) they spiral toward cycles or phase polygons. If we could exclude possibilities (b) and (c), it would follow that all half-trajectories converge to singular points of (2) and that they do not form phase polygons. To rule out (b) and (c), Kaniiovski and Young (1995) use Bendixson’s criterion (see Hahn, 1967, p. 66); namely, if

\[
\text{div} \vec{g}^s(\vec{z}) = \frac{\partial}{\partial z_1} g_1^s(\vec{z}) + \frac{\partial}{\partial z_2} g_2^s(\vec{z})
\]

preserves its sign in \( \mathcal{R} \), then all half-trajectories of (2) belonging to \( \mathcal{R} \) do not form cycles or phase polygons.

In the present situation we have (for both cases)

\[
\text{div} \vec{g}^s(\vec{z}) = -2 + \epsilon(1 - 2\delta)[\frac{d}{dz_1} f_{1/2}^s(z_1) + \frac{d}{dz_2} f_{1/2}^s(z_2)].
\]

(4)

The derivatives \( df_{1/2}^s(z)/dz \) are non-negative, and become arbitrarily large as \( s \to \infty \) (see the proof of Lemma 3 in Kaniiovski and Young, 1995); hence Bendixson’s criterion cannot be used to exclude cycles and phase polygons when \( \delta < 1/2 \). Consequently, we have to derive an extension of Bendixson’s criterion for this particular system, which will allow us to exclude cycles and phase polygons provided that \( s \) is sufficiently large. We show how to do this in the proof of Lemma 5 below. On the other hand, when \( \delta \geq 1/2 \), the term containing derivatives in (4) is non-positive, and we can still use Bendixson’s criterion. We formulate this latter observation as a separate statement.

**Lemma 1.** If \( \delta \geq 1/2 \), that is, the probability of imitating the majority is not larger than the probability of imitating the minority, then, for every \( \epsilon \geq 0 \), the system (2) does not have cycles (including limit cycles) or phase polygons. Hence, by the theory of Bendixson, its half-trajectories converge to singular points.

Let us consider now singular points of (2), that is, solutions of the following system of nonlinear equations:

\[
g_1^s(\vec{z}) = 0 \quad g_2^s(\vec{z}) = 0.
\]

(5)
We are going to show that, for three-equilibria games, system (5) has exactly three solutions: \((x_0^s, y_0^s), (x_\beta^s, y_\alpha^s),\) and \((x_1^s, y_1^s).\) Denote by \(J^s(\bar{z})\) the Jacobian of (2) at \(\bar{z},\)

\[
J^s(\bar{z})_{ij} = \frac{\partial g_i^s(\bar{z})}{\partial z_j}, \quad i, j = 1, 2.
\]

Let \(I\) designate the identity matrix in \(R^2.\) The following result is proved in the Appendix.

**Lemma 2.** Let \(G\) be a nondegenerate 2 \(\times\) 2 game with three equilibria, the mixed one being \((\beta, \alpha).\) Let \(s > 1\) be an odd number, \(\epsilon \in (0, 1/2), \delta \in [0, 1]\) and \(\epsilon \delta < \min(\alpha, \beta, 1/2) \leq \max(\alpha, \beta, 1/2) < 1 - \epsilon \delta.\) Then

(i) for all sufficiently large \(s,\) (5) has exactly three solutions, \((x_0^s, y_0^s), (x_\beta^s, y_\alpha^s),\) and \((x_1^s, y_1^s);\)

(ii) \(\lim_{s \to \infty}(x_0^s, y_0^s) = (\epsilon \delta, \epsilon \delta), \lim_{s \to \infty}(x_\beta^s, y_\alpha^s) = (\beta, \alpha),\) and \(\lim_{s \to \infty}(x_1^s, y_1^s) = (1 - \epsilon \delta, 1 - \epsilon \delta),\) with \((x_0^s, y_0^s) = (0, 0)\) and \((x_1^s, y_1^s) = (1, 1)\) for all \(s\) in the particular situation when \(\delta = 0;\)

(iii) \(\lim_{s \to \infty}J^s(x_i^s, y_i^s) = -I\) for \(i = 0, 1,\) so the matrices \(J^s(x_i^s, y_i^s)\) are stable for all sufficiently large \(s.\)

(iv) \(J^s((x_\beta^s, y_\alpha^s))\) is not a stable matrix for all large \(s;\) one of its real eigenvalues converges to \(\infty,\) and the other converges to \(-\infty\) as \(s \to \infty.\)

For one-equilibrium games there is a single solution \((\bar{x}_\beta^s, \bar{y}_\alpha^s)\) of (5), and the corresponding statement is as follows.

**Lemma 3.** Let \(G\) be a nondegenerate 2 \(\times\) 2 game with a unique equilibrium \((\beta, \alpha),\) which is mixed. Let \(s > 1\) be an odd number, \(\epsilon \in (0, 1/2), \delta \in [0, 1]\) and \(\epsilon \delta < \min(\alpha, \beta, 1/2) \leq \max(\alpha, \beta, 1/2) < 1 - \epsilon \delta.\) Then

(i) for all sufficiently large \(s,\) (5) has a single solution \((\bar{x}_\beta^s, \bar{y}_\alpha^s)\) and \(\lim_{s \to \infty}(\bar{x}_\beta^s, \bar{y}_\alpha^s) = (\beta, \alpha);\)

(ii) if \(\alpha \neq 1/2, \beta \neq 1/2,\) the Jacobian \(J^s((\bar{x}_\beta^s, \bar{y}_\alpha^s))\) is stable for all sufficiently large \(s,\) and the real parts of its eigenvalues converge to \(-1\) as \(s \to \infty;\)

(iii) if \(\alpha = 1/2,\) or \(\beta = 1/2,\) or \(\alpha = \beta = 1/2,\) then for all sufficiently large \(s,\) \(J^s((\bar{x}_\beta^s, \bar{y}_\alpha^s))\) is not stable for \(\delta < 1/2\) and is stable for \(\delta \geq 1/2.\) For \(\delta < 1/2\) the eigenvalues are either complex, in which case their real part converges to \(\infty\) as \(s \to \infty,\) or real, in which case they have opposite signs and converge to \(\pm \infty\) as \(s \to \infty.\) For \(\delta = 1/2\) the eigenvalues are complex, and their real parts converge to \(-1\) as \(s \to \infty.\) For \(\delta > 1/2,\) the eigenvalues are either complex, in which case their real parts converge to \(-\infty\) as \(s \to \infty,\) or real, in which case they converge to \(-\infty\) as \(s \to \infty.\)
This lemma is proved by essentially the same method as Lemma 2; we omit the argument. According to the plan of the proof outlined above, we now proceed to analyze the convergence properties of half-trajectories of (2) and demonstrate the absence of phase polygons. Since the case \( \delta \geq 1/2 \) is covered by Lemma 1, our interest here is in the case when \( \delta < 1/2 \).

In three-equilibria games there are three singular points of (2), exactly one of which is linearly unstable. Hence, the only type of phase polygons that can be formed by half-trajectories connecting linearly unstable points is 1-gons. The crucial point (as we shall show next) is that the coordinates of any trajectory of (2) are monotone functions on some interval of time adjoining infinity. Consequently, the trajectory must converge and the limit must be a singular point of (2). Another monotonicity argument implies that at least one coordinate of \( \bar{z}^s(t) \) is different when \( t \to \infty \) and \( t \to -\infty \). Hence, half-trajectories that converge to the linearly unstable singular point cannot form 1-gons. This result is summarized in the following lemma.

**Lemma 4.** Let \( G \) be a nondegenerate \( 2 \times 2 \) game with three equilibria. Let \( s > 1 \) be an odd number, \( \epsilon \in (0, 1/2) \) and \( \delta \in [0, 1] \). Every solution \( \bar{z}^s(t) \) of (2) belonging to \( \mathcal{K} \) for \( 0 \leq t < \infty \) converges to a limit as \( t \to \infty \), and the limit must be a singular point. If \( \bar{z}^s(t) \) belongs to \( \mathcal{K} \) for \( -\infty < t < \infty \), then either \( \lim_{t \to -\infty} \bar{z}^s(t) \) exists and \( \lim_{t \to -\infty} \bar{z}^s(t) \neq \lim_{t \to \infty} \bar{z}^s(t) \), or else \( \bar{z}^s(\cdot) \) is constant.

The detailed proof of this result is given in the Appendix. Here we remark that monotonicity is established by the following argument. At a time instant \( t \), each trajectory of (2) belongs to one of the following sets:

\[
\begin{align*}
\mathcal{K}_{++} &= \{ \bar{z} \in \mathcal{K}: g^1_1(\bar{z}) \geq 0, \ g^2_1(\bar{z}) \geq 0 \} \\
\mathcal{K}_{+-} &= \{ \bar{z} \in \mathcal{K}: g^1_1(\bar{z}) \geq 0, \ g^2_1(\bar{z}) \leq 0 \} \\
\mathcal{K}_{-+} &= \{ \bar{z} \in \mathcal{K}: g^1_1(\bar{z}) \leq 0, \ g^2_1(\bar{z}) \geq 0 \} \\
\mathcal{K}_{--} &= \{ \bar{z} \in \mathcal{K}: g^1_1(\bar{z}) \leq 0, \ g^2_1(\bar{z}) \leq 0 \}.
\end{align*}
\]

Observe that a trajectory of (2) leaving either \( \mathcal{K}_{+-} \) or \( \mathcal{K}_{-+} \) must enter \( \mathcal{K}_{++} \) or \( \mathcal{K}_{--} \), both of which are positive invariant sets for (2). By (viability) Theorems 1.2.1 and 1.2.3 in Aubin (1991, p. 27), the latter holds since \( \bar{g}^s(\bar{z}) \) belongs to the contingent cone \( T_{++}(\bar{z})(T_{--}(\bar{z})) \) to \( \mathcal{K}_{++}(\mathcal{K}_{--}) \) if \( \bar{z} \in \mathcal{K}_{++}(\mathcal{K}_{--}) \). Here the sets \( T_{++}(\bar{z}) \) and \( T_{--}(\bar{z}) \) are defined similarly, for example,

\[
T_{++}(\bar{z}) = \left\{ \tilde{v} \in \mathbb{R}^2 : \liminf_{\Delta \to +0} d(\bar{z} + \Delta \tilde{v}, \mathcal{K}_{++})/\Delta = 0 \right\},
\]

with \( d(\bar{x}, A) \) designating the Euclidean distance in \( \mathbb{R}^2 \) from a point \( \bar{x} \) to a set \( A \), that is, \( d(\bar{x}, A) = \inf_{y \in A} ||\bar{y} - \bar{x}|| \).
Now consider games with a unique (mixed) equilibrium. If \( \delta < 1/2 \), almost sure convergence holds only when \( \beta \neq 1/2 \) and \( \alpha \neq 1/2 \). Indeed, if either \( \beta = 1/2 \), or \( \alpha = 1/2 \), or \( \beta = \alpha = 1/2 \), then, by Lemma 3, the unique singular point of (2) is linearly unstable for all sufficiently large \( s \). Since (1) is a two-dimensional stochastic approximation process, \((X^t, Y^t)\) converges to \((x^*_\beta, y^*_\alpha)\) with probability zero (see Arthur et al., 1988, Lemma 2 or Pemantle, 1990, Theorem 1). Hence, there is no hope (when \( \delta < 1/2 \)) of proving convergence to \((x^*_\beta, y^*_\alpha)\) if \( \beta \) or \( \alpha \) equals \( 1/2 \).

Let us therefore consider the situation where \( \beta \neq 1/2 \) and \( \alpha \neq 1/2 \). According to Lemma 3, the unique singular point of system (2) is stable in this case. Hence there cannot be phase polygons, and if there are closed orbits they must involve a limit cycle. We need the following generalization of Bendixson's criterion, which excludes cycles around asymptotically stable singular points.

**Lemma 5.** Consider a system of ordinary differential equations

\[
\frac{d}{dt} \tilde{x} = \tilde{f}(\tilde{x}),
\]  

(7)

where the right hand side is continuously differentiable on \( \mathcal{K} \). Let

\[ \mathcal{K}_+ = \{ \tilde{x} \in \mathcal{K} : \text{div } \tilde{f}(\tilde{x}) \geq 0 \} \quad \text{and} \quad \mathcal{K}_- = \{ \tilde{x} \in \mathcal{K} : \text{div } \tilde{f}(\tilde{x}) \leq 0 \}. \]

Suppose that \( \tilde{x}^* \in \mathcal{K} \) is a singular point of (7), \( S \) is a set such that \( S \subseteq \mathcal{K}_+ \) (\( S \subseteq \mathcal{K}_- \)), and

\[
\int_S \text{div } \tilde{f}(\tilde{x}) d\tilde{x} > \int_{\mathcal{K}_-} \text{div } \tilde{f}(\tilde{x}) d\tilde{x} \quad \left( \left| \int_S \text{div } \tilde{f}(\tilde{x}) d\tilde{x} \right| > \int_{\mathcal{K}_+} \text{div } \tilde{f}(\tilde{x}) d\tilde{x} \right).
\]

Then there exists no cycle of (7) belonging to \( \mathcal{K} \) that encircles \( \tilde{x}^* \) and contains \( S \).

**Proof.** The argument is straightforward. Assume to the contrary that (7) has a cycle \( \mathcal{L} \) belonging to \( \mathcal{K} \), encircling \( \tilde{x}^* \), and containing \( S \). In Hahn (1967, p. 66) it is shown that if \( L \) is the closed domain bounded by \( \mathcal{L} \), then

\[
\int_L \text{div } \tilde{f}(\tilde{x}) d\tilde{x} = 0.
\]  

(8)

Let \( S \subseteq \mathcal{K}_+ \) be the set hypothesized in the lemma. Then

\[
\int_L \text{div } \tilde{f}(\tilde{x}) d\tilde{x} = \int_{L \cap \mathcal{K}_+} \text{div } \tilde{f}(\tilde{x}) d\tilde{x} + \int_{L \cap \mathcal{K}_-} \text{div } \tilde{f}(\tilde{x}) d\tilde{x}
\]

\[
\geq \int_S \text{div } \tilde{f}(\tilde{x}) d\tilde{x} - \int_{\mathcal{K}_-} \text{div } \tilde{f}(\tilde{x}) d\tilde{x} \quad > 0,
\]

which contradicts (8). If \( S \subseteq \mathcal{K}_- \) the argument is analogous.
Note that this result makes use of the same idea as Bendixson’s criterion: if there is a cycle, then the integral of the divergence of the vector field over the domain bounded by the cycle must be zero. To exclude cycles, it therefore suffices to show that this integral is not zero. However, unlike Bendixson’s criterion, Lemma 5 exploits the geometry of the situation requiring that the sign of the integral of the divergence over the closed set bounded by the cycle coincides with the sign of this integral over some set $S$.

To find a set $S$ for system (2) we notice that by (4) the divergence in $\mathcal{H}$ is $-2$ plus a vanishing term as $s \to \infty$. This is true except on two narrow strips around the intervals $\{1/2\} \times [0, 1]$ and $[0, 1] \times \{1/2\}$. However, we can show that the integral of the divergence over the strips does not exceed $2\epsilon (1 - 2\delta)$, plus a term that converges to zero as $s \to \infty$. Using the piece-wise linearity of the right hand side of (3), we can identify a polygon set $S$ which lies inside any cycle of (3) should a cycle exist. Consequently, this set is inside any cycle of (2) for all sufficiently large $s$. The integral of the divergence over $S$ is $-2\mu(S)$ plus a vanishing term, where $\mu(S)$ designates the Lebesgue measure of $S$ in $R^2$. Hence, the inequality required in Lemma 5 reads $\mu(S) > \epsilon (1 - 2\delta)$ plus a negligible term as $s \to \infty$. To simplify the estimation of $\mu(S)$, let us restrict our attention to the situation where $\beta < 1/2$ and $\alpha < 1/2$. (The same analysis applies to other combinations of values of $\beta$ and $\alpha$.)

**Lemma 6.** Let $G$ be a nondegenerate $2 \times 2$ game with a unique equilibrium $(\beta, \alpha)$, which is fully mixed. Let $\epsilon \in (0, 1/2)$, $\delta \in [0, 1/2)$, $\max(\beta, \alpha) < 1/2$ and $\epsilon \delta < \min(\alpha, \beta)$. If

$$\epsilon (1 - 2\delta) < \min \left[ \left( \frac{1}{2} - \beta \right)^2 \frac{\alpha - \epsilon \delta}{1 - 2\epsilon \delta}, \left( \frac{1}{2} - \alpha \right)^2 \frac{1 - \beta - \epsilon (1 - \delta)}{1 - 2 \epsilon \delta} \right],$$

then for all sufficiently large $s$ the solutions of (2) belonging to $\mathcal{H}$ do not form cycles.

This lemma is proved in the Appendix. We remark that the upper bound on $\epsilon$ and $\delta$ for excluding cycles presented here is not the sharpest possible one. A discussion in the Appendix shows how one can improve on it.

We conclude with several further remarks about the cycling behavior of the process when the game has a unique interior Nash equilibrium. Consider first the case where $\alpha \neq 1/2$ and $\beta \neq 1/2$. One can show that there are locations of the mixed equilibrium, and numbers $\epsilon \in (0, 1/2)$ and $\delta \in [0, 1/2)$, such that system (2) exhibits cycles for all sufficiently large $s$; moreover the diameter of these cycles is at least $\min(1/2 - \alpha, 1/2 - \beta)$. Consequently these cycles are of nonvanishing size. Below some critical $\epsilon$ they disappear. On the other hand, if either $\alpha = 1/2$, or $\beta = 1/2$, or
\( \alpha = \beta = 1/2 \), then for all sufficiently small \( \epsilon > 0 \) and all sufficiently large \( s \) there are cycles whose sizes go to zero as \( \epsilon \to 0 \).

We remark that, for \( \delta < 1/2 \), the vector field governing the deterministic dynamics (2) is not area decreasing; nevertheless, we are able to establish convergence. Consequently, the arguments used here are sharper than standard approaches in the literature (see, for example, Benaim and Hirsh, 1996). We also remark that the ideas given here can be applied to more general two-dimensional stochastic approximation processes evolving on a compact set. Namely, the reasoning used in Lemma 4 applies to any situation when the analogs of \( K_{++} \) and \( K_{--} \) are positive invariant, and the argument applied in Lemma 5 covers the case where the square of any hypothetical cycle can be estimated a priori.

APPENDIX

Let us establish the statements concerning singular points of (2). In games with three-equilibria, Eqs. (5) look as follows:

\[
f_\delta^s(z_2) - T_{\epsilon, \delta}^s(z_1) = 0 \quad f_\beta^s(z_1) - T_{\epsilon, \delta}^s(z_2) = 0,
\]

where

\[
T_{\epsilon, \delta}^s(x) = \frac{x - \epsilon[(1 - 2\delta)f_{1/2}^s(x) + \delta]}{1 - \epsilon}.
\]

We need the following auxiliary result, which is a refinement of Lemma 2 in Kaniowski and Young (1995). In general, let \( [x] \) denote the integer part of \( x \) and let \( N(\cdot) \) be the standard Gaussian distribution function.

**Proposition 1.** For every \( \gamma \in (0, 1) \)

1. \( \lim_{s \to \infty} f_\gamma^s(x) = f_\gamma(x) \) and this convergence is uniform on every closed set \( K \subset [0, 1] \) which does not contain \( \gamma \), where

\[
f_\gamma(x) = \begin{cases} 
0 & \text{if } x \in [0, \gamma), \\
1/2 & \text{if } x = \gamma, \\
1 & \text{if } x \in (\gamma, 1];
\end{cases}
\]

2. for every closed set \( K \subset [0, 1] \) not containing \( \gamma \),

\[
\lim_{s \to \infty} \max_{x \in K} \frac{d}{dx} f_\gamma^s(x) = 0;
\]

3. for all sufficiently large \( s \), \( \frac{d}{dx} f_\gamma^s(\cdot) \) is positive and increasing on \((0, \frac{[\gamma]}{s-1})\), and is positive and decreasing on \((\frac{[\gamma]}{s-1}, 1)\); hence \( f_\gamma^s(\cdot) \) is convex on \((0, \frac{[\gamma]}{s-1})\) and concave on \((\frac{[\gamma]}{s-1}, 1)\) being an increasing function on \([0, 1] \);
(4) for every $T > 0$

$$\lim_{s \to \infty} \sup_{t \in [-T, T]} |f_s^\gamma(y + t/\sqrt{s}) - 1 + N(-t/\sqrt{\gamma(1-\gamma)})| = 0;$$

(5) for every $T > 0$

$$\lim_{s \to \infty} \sup_{t \in [-T, T]} \left| \frac{1}{\sqrt{s}} \frac{d}{dx} f_s^\gamma(x)_{|x = y + t/\sqrt{s}} - \frac{\exp(-t^2/2)}{\sqrt{2\pi}\gamma(1-\gamma)} \right| = 0.$$  

Notice that, when $s$ is odd, $[\frac{1}{2}s] = \frac{s-1}{2}$ and hence $[\frac{s}{2}]_{s-1} = 1/2$. Let

$$T_{\epsilon, \delta}(x) = \begin{cases} 
\frac{x - \epsilon\delta}{1 - \epsilon} & \text{if } x \in [0, 1/2) \\
1/2 & \text{if } x = 1/2 \\
\frac{x - \epsilon(1-\delta)}{1 - \epsilon} & \text{if } x \in (1/2, 1]. 
\end{cases}$$

Proposition 1 implies the following result.

**Proposition 2.** Let $s > 1$ be an odd number, $\epsilon \in (0, 1/2)$, and $\delta \in [0, 1]$.

(1) $\lim_{s \to \infty} T_{\epsilon, \delta}(x) = T_{\epsilon, \delta}(x)$ and the convergence is uniform on every closed set $K \subset [0, 1]$ that does not contain $1/2$;

(2) $\lim_{s \to \infty} \frac{d}{dx} T_{\epsilon, \delta}(x) = \frac{1}{1-\epsilon}$ for $x \neq 1/2$ and the convergence is uniform on every closed set $K \subset [0, 1]$ that does not contain $1/2$;

(3) if $\delta < 1/2$, $T_{\epsilon, \delta}(\cdot)$ is concave on $(0, 1/2)$ and convex on $(1/2, 1)$; if $\delta = 1/2$, $T_{\epsilon, \delta}(\cdot)$ is linear; if $\delta > 1/2$, $T_{\epsilon, \delta}(\cdot)$ is convex on $(0, 1/2)$ and concave on $(1/2, 1)$. In the two latter cases $T_{\epsilon, \delta}(\cdot)$ is increasing on $[0, 1]$;

(4) $T_{\epsilon, \delta}(x) \leq T_{\epsilon, \delta}(x)$ for $0 \leq x \leq 1/2$, and $T_{\epsilon, \delta}(x) \geq T_{\epsilon, \delta}(x)$ for $1/2 \leq x \leq 1$;

(5) $\lim_{x \to 1/2-} T_{\epsilon, \delta}(x) = \frac{1}{2} + \frac{\epsilon}{1-\epsilon} (\frac{1}{2} - \delta)$ and $\lim_{x \to 1/2+} T_{\epsilon, \delta}(x) = \frac{1}{2} + \frac{\epsilon}{1-\epsilon} (\delta - \frac{1}{2})$;

(6) $\lim_{s \to \infty} \sup_{t \in [-L, L]} |T_{\epsilon, \delta}(1/2 + t/\sqrt{s}) - \frac{1-2(1-\delta)\epsilon}{2(1-\epsilon)} - \frac{\epsilon}{1-\epsilon} (1 - 2\delta)N(-2t)| = 0$ for every fixed $L \in (0, \infty)$.

The following lemma shows that, when $s$ is sufficiently large, (5) has exactly three solutions, $(x_0^s, y_0^s)$, $(x_\beta^s, y_\beta^s)$, and $(x_1^s, y_1^s)$, two of which are linearly stable and one of which is not.

**Lemma 2.** Let $G$ be a nondegenerate $2 \times 2$ game with three equilibria, the mixed one being $(\beta, \alpha)$. Let $s > 1$ be an odd number, $\epsilon \in (0, 1/2)$, $\delta \in [0, 1]$, and $\epsilon \delta < \min(\alpha, \beta, 1/2) \leq \max(\alpha, \beta, 1/2) < 1 - \epsilon \delta$. Then

(i) for all sufficiently large $s$, (5) has exactly three solutions, $(x_0^s, y_0^s)$, $(x_\beta^s, y_\beta^s)$, and $(x_1^s, y_1^s)$;
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(ii) \( \lim_{s \to \infty} (x_0^s, y_0^s) = (\epsilon \delta, \epsilon \delta), \lim_{s \to \infty} (x_\beta^s, y_\alpha^s) = (\beta, \alpha) \), and \( \lim_{s \to \infty} (x_i^s, y_i^s) = (1 - \epsilon \delta, 1 - \epsilon \delta) \), with \( (x_0^0, y_0^0) = (0, 0) \) and \( (x_i^1, y_i^1) = (1, 1) \) for all \( s \) in the particular situation when \( \delta = 0 \);

(iii) \( \lim_{s \to \infty} J^s(x_i^s, y_i^s) = -I \) for \( i = 0, 1, \) so the matrices \( J^s(x_i^s, y_i^s) \) are stable for all sufficiently large \( s \);

(iv) \( J^s(x_\beta^s, y_\alpha^s) \) is not a stable matrix for all large \( s \), one of its real eigenvalues converges to \( \infty \) as \( s \to \infty \), and the other converges to \( -\infty \).

Proof. Let us show that for all large enough \( s \) there are no solutions of (5) (or, equivalently, of (A1)) outside of an arbitrarily small neighborhood of \( (\epsilon \delta, \epsilon \delta), (\beta, \alpha), \) and \( (1 - \epsilon \delta, 1 - \epsilon \delta) \). To do this we shall show that for an arbitrary small fixed \( \tau > 0 \), there is a number \( s_\tau \) such that for all \( s \geq s_\tau \),

\[
\begin{align*}
T_{\epsilon, \delta}^s(z) & \leq 0 \quad \text{for} \quad z \in [0, \epsilon \delta], \quad (A2) \\
T_{\epsilon, \delta}^s(z) & \geq 1 \quad \text{for} \quad z \in [1 - \epsilon \delta, 1], \quad (A3) \\
T_{\epsilon, \delta}^s(z) & \in (\tau/2, 1 - \tau/2) \quad \text{for} \quad z \in [\epsilon \delta + \tau, 1 - \epsilon \delta - \tau], \quad (A4) \\
f_{\frac{\epsilon}{\gamma}}^s(z) & < \tau/2 \quad \text{for} \quad z \in [0, \gamma - \tau/2], \quad (A5) \\
f_{\frac{\epsilon}{\gamma}}^s(z) & > 1 - \tau/2 \quad \text{for} \quad z \in [\gamma + \tau/2, 1] \quad \gamma = \alpha, \beta.
\end{align*}
\]

Indeed, if these relations hold and if \( \tau \) is so small that \( \epsilon \delta + \frac{3}{2} \tau < \min(\alpha, \beta) \leq \max(\alpha, \beta) < 1 - \epsilon \delta - \frac{3}{2} \tau \), then there are only three possible locations for solutions (if any) of (A1) for \( s \geq s_\tau \):

\[
\begin{align*}
B_0^\tau &= [\epsilon \delta, \epsilon \delta + \tau] \times [\epsilon \delta, \epsilon \delta + \tau], \\
B^\tau &= [(f_{\beta})^{-1}(\tau/2), (f_{\beta})^{-1}(1 - \tau/2)] \times [(f_{\alpha})^{-1}(\tau/2), (f_{\alpha})^{-1}(1 - \tau/2)], \\
(B^\tau \subset [\beta - \tau/2, \beta + \tau/2] \times [\alpha - \tau/2, \alpha + \tau/2]), \\
B_1^\tau &= [1 - \epsilon \delta - \tau, 1 - \epsilon \delta] \times [1 - \epsilon \delta - \tau, 1 - \epsilon \delta].
\end{align*}
\]

As \( \tau \) goes to zero, the sets \( B_0^\tau, B^\tau, \) and \( B_1^\tau \) shrink to the limits \( (\epsilon \delta, \epsilon \delta), (\beta, \alpha), \) and \( (1 - \epsilon \delta, 1 - \epsilon \delta) \), as asserted in (ii). What remains to be proved are the existence and uniqueness of the solutions and the stability of the Jacobians.

First we establish (A2)–(A5). Let \( \Delta_1 = \frac{1}{2} - \frac{\epsilon}{1 - \epsilon} |\delta - 1/2| \) and \( \Delta_2 = \frac{1}{2} + \frac{\epsilon}{1 - \epsilon} |\delta - 1/2| \). By hypothesis \( \epsilon \delta < \min(\alpha, \beta, 1/2) \leq \max(\alpha, \beta, 1/2) < 1 - \epsilon \delta \). Hence for all sufficiently small \( \tau > 0 \),

\[
\epsilon \delta + 2 \tau < \min(\alpha, \beta, 1/2) \leq \max(\alpha, \beta, 1/2) < 1 - \epsilon \delta - 2 \tau.
\]

Clearly we can assume that \( \tau < \Delta_1 \) and \( 1 - \tau > \Delta_2 \). Since \( T_{\epsilon, \delta}(z) \leq 0 \) for \( z \in [0, \epsilon \delta] \subset [0, 1/2) \) and \( T_{\epsilon, \delta}(z) \geq 1 \) for \( z \in [1 - \epsilon \delta, 1) \subset (1/2, 1] \), inequalities (A2) and (A3) follow from statement 4 of Proposition 2.
By hypothesis $\epsilon \in (0, 1/2)$, hence we have $\frac{\tau}{1-\epsilon} > \tau$. Consequently
\[ T_{\epsilon, \delta}(z) \in (\tau, 1-\tau) \quad \text{for} \quad z \in [\epsilon \delta + \tau, 1 - \epsilon \delta - \tau]. \quad (A6) \]

By statement 1 of Proposition 2, for all sufficiently large $s$ we have
\[ T_{\epsilon, \delta}^s(\epsilon \delta + \tau) > \tau/2 \quad \text{and} \quad T_{\epsilon, \delta}^s(1 - \epsilon \delta - \tau) < 1 - \tau/2. \quad (A7) \]

Suppose that $\delta \geq 1/2$. By statement 3 of Proposition 2, $T_{\epsilon, \delta}^s(\cdot)$ is monotone increasing on $[\epsilon \delta + \tau, 1 - \epsilon \delta - \tau]$. This together with (A7) implies (A4). Suppose, on the other hand, that $\delta < 1/2$. By concavity,
\[ T_{\epsilon, \delta}^s(z) \geq \min[T_{\epsilon, \delta}^s(\epsilon \delta + \tau), T_{\epsilon, \delta}^s(1/2)] \quad \text{for} \quad \epsilon \delta + \tau \leq z \leq 1/2. \quad (A8) \]

Notice that
\[ \lim_{s \to \infty} T_{\epsilon, \delta}^s(\epsilon \delta + \tau) = \frac{\tau}{1-\epsilon} \quad \text{and} \quad T_{\epsilon, \delta}^s(1/2) = 1/2. \quad (A9) \]

Since $\tau(1-\epsilon)^{-1} < 1/2$, (A9) implies that for all sufficiently large $s$
\[ T_{\epsilon, \delta}^s(\epsilon \delta + \tau) < T_{\epsilon, \delta}^s(1/2). \quad (A10) \]

Also, $T_{\epsilon, \delta}^s(\epsilon \delta + \tau) \to T_{\epsilon, \delta}(\epsilon \delta + \tau) = \frac{\tau}{1-\epsilon} > \tau$. Hence, (A8) and (A10) entail that
\[ T_{\epsilon, \delta}^s(z) \geq T_{\epsilon, \delta}(\epsilon \delta + \tau) > \tau/2 \quad \text{for} \quad \epsilon \delta + \tau \leq z \leq 1/2. \quad (A11) \]

Similarly by convexity,
\[ T_{\epsilon, \delta}^s(z) \leq T_{\epsilon, \delta}^s(1 - \epsilon \delta - \tau) < 1 - \tau/2 \quad \text{for} \quad z \in [1/2, 1 - \epsilon \delta - \tau]. \quad (A12) \]

By (A6) and statement 4 of Proposition 2 we obtain (for all $s$)
\[ T_{\epsilon, \delta}(z) \leq T_{\epsilon, \delta}(z) < 1 - \tau \quad \text{for} \quad z \in [\epsilon \delta + \tau, 1/2] \quad (A13) \]

and
\[ T_{\epsilon, \delta}^s(z) \geq T_{\epsilon, \delta}(z) > \tau \quad \text{for} \quad z \in [1/2, 1 - \epsilon \delta - \tau]. \quad (A14) \]

Inequalities (A11)–(A14) imply (A4) for the case when $\delta < 1/2$. Finally, inequalities (A5) follow from statement 1 of Proposition 1. Thus we have proved that relations (A2)–(A5) hold.

Let $D_i = \{ z_i : T_{\epsilon, \delta}^s(z_i) \in [0, 1] \}$, $i = 1, 2$. On these domains, Eqs. (A1) can be given as follows:
\[ z_j = \Phi_i^s(z_i) = (f_i^s)^{-1}(T_{\epsilon, \delta}^s(z_i)) \quad i, j = 1, 2 \quad i \neq j \quad \gamma = \alpha, \beta. \]

We claim that there is a unique solution of (A1) belonging to $B_0^s$, and the Jacobian at this solution approaches $-I$. 

\[ z_j = \Phi_i^s(z_i) = (f_i^s)^{-1}(T_{\epsilon, \delta}^s(z_i)) \quad i, j = 1, 2 \quad i \neq j \quad \gamma = \alpha, \beta. \]

We claim that there is a unique solution of (A1) belonging to $B_0^s$, and the Jacobian at this solution approaches $-I$. 

\[ z_j = \Phi_i^s(z_i) = (f_i^s)^{-1}(T_{\epsilon, \delta}^s(z_i)) \quad i, j = 1, 2 \quad i \neq j \quad \gamma = \alpha, \beta. \]
Case 1. \( \delta = 0 \). It is easy to see that \((0, 0)\) is a solution of (A1) and, consequently (5), for all \( s \). Also \( J^s(\bar{0}) = -I \). This, in particular, implies that if \( \tau \) is small enough, this solution is unique on \( B_0^s \).

Case 2. \( \delta > 0 \). We shall prove that the curves \( z_2 = \Phi_i^s(z_1) \) and \( z_2 = (\Phi_2^s)^{-1}(z_1) \) intersect at a single point and that this point is the solution of (A1) on \( B_0^s \). First we show that the \( \Phi_i^s(\cdot) \) are defined on a subinterval of \([0, \epsilon \delta + \tau]\). Since \( \epsilon \delta + 2\tau < 1/2 \), statement 4 of Proposition 2 implies that

\[
T_{\epsilon, \delta}^s(\epsilon \delta) \leq T_{\epsilon, \delta}(\epsilon \delta) = 0.
\] (A15)

Statement 1 of Proposition 2 entails that

\[
\lim_{s \to \infty} T_{\epsilon, \delta}^s(\epsilon \delta + \tau) = T_{\epsilon, \delta}(\epsilon \delta + \tau) = \frac{\tau}{1 - \epsilon} > 0.
\] (A16)

The continuity of \( T_{\epsilon, \delta}^s(\cdot) \), together with (A15) and (A16), imply that there exists a root \( x^s \) of \( T_{\epsilon, \delta}^s(x^s) = 0 \). Statement 2 of Proposition 2 shows that the functions \( T_{\epsilon, \delta}^s(\cdot) \) are monotone on \([0, \epsilon \delta + \tau]\) for all sufficiently large \( s \). This entails that \( x^s \) is unique. Moreover, (A15) implies that \( T_{\epsilon, \delta}^s(\epsilon \delta) \leq 0 \); hence,

\[
x^s \geq \epsilon \delta.
\] (A17)

Finally, by statement 1 of Proposition 2, \( T_{\epsilon, \delta}^s(\cdot) \) converges uniformly to \( T_{\epsilon, \delta}(\cdot) \) on \([0, \epsilon \delta + \tau]\). Since \( T_{\epsilon, \delta}(\cdot) > 0 \) on \([\epsilon \delta, \epsilon \delta + \tau]\), by (A17) we conclude that

\[
\lim_{s \to \infty} x^s = \epsilon \delta.
\] (A18)

Next we check that \( T_{\epsilon, \delta}^s(\cdot) \) maps \([x^s, \epsilon \delta + \tau]\) into \([0, 1]\) for all sufficiently large \( s \). Since \( \tau(1 - \epsilon)^{-1} < 1 \) and the function \( T_{\epsilon, \delta}^s(\cdot) \) increases on \([0, \epsilon \delta + \tau]\), by (A16) we conclude that

\[
T_{\epsilon, \delta}^s(\cdot): [x^s, \epsilon \delta + \tau] \mapsto [0, 1]
\]

for all sufficiently large \( s \). It follows that the function \( \Phi_i^s(\cdot) \) is defined on \([x^s, \epsilon \delta + \tau]\). Since, \( \Phi_i^s(\cdot) \) is a composition of the increasing functions \( (f_i^s)^{-1}(\cdot) \) and \( T_{\epsilon, \delta}^s(\cdot) \), so \( \Phi_i^s(\cdot) \) is also increasing on \([x^s, \epsilon \delta + \tau]\). Consequently, \( (\Phi_i^s)^{-1}(\cdot) \) is well defined.

Next we demonstrate that on the domain \( D^s = [x^s, \epsilon \delta + \tau] \), the curves \( z_2 = \Phi_1^s(z_1) \) and \( z_2 = (\Phi_2^s)^{-1}(z_1) \) intersect at a single point \( x_0^s \). Let us show first that \( \Phi_i^s(\epsilon \delta + \tau) > \epsilon \delta + \tau \) as \( s \to \infty \). Indeed, by (A4) \( T_{\epsilon, \delta}^s(\epsilon \delta + \tau) > \tau/2 \). This, together with (A5), implies that

\[
\Phi_i^s(\epsilon \delta + \tau) = (f_i^s)^{-1}(T_{\epsilon, \delta}(\epsilon \delta + \tau)) > \gamma - \tau/2
\]

\( i = 1, 2 \) \( \gamma = \alpha, \beta \).
Since we set $\epsilon \delta + 2\tau < \min(\alpha, \beta)$, the latter inequality gives

$$
\Phi_i^x(\epsilon \delta + \tau) > \gamma - \tau/2 > \epsilon \delta + \tau, \quad i = 1, 2, \quad \gamma = \alpha, \beta.
$$

Thus we have shown that

$$
\Phi_i^x(\cdot): [x^\gamma, \epsilon \delta + \tau] \mapsto [0, \Phi_i^x(\epsilon \delta + \tau)] \supset [0, \epsilon \delta + \tau], \quad i = 1, 2. \quad (A19)
$$

Now consider $(\Phi_i^x)^{-1}(\cdot)$. Since $\Phi_i^x(\cdot)$ is increasing, its inverse $(\Phi_i^x)^{-1}(\cdot)$ is also an increasing function. From (A1) we obtain

$$
\frac{d}{dz_j}(\Phi_i^x)^{-1}(z_j) = \frac{(1 - \epsilon)(d/dz_j)f_i^x(z_j)}{1 - \epsilon(1 - 2\delta)(d/dz_i)f_i^{x/2}(z_i)}, \quad i \neq j, \quad \gamma = \alpha, \beta,
$$

(A20)

where $z_i = (\Phi_i^x)^{-1}(z_j)$. The right-hand side of (A20) is defined at least on $[0, \epsilon \delta + \tau] \times [0, \epsilon \delta + \tau]$. It is positive on $(0, \epsilon \delta + \tau] \times (0, \epsilon \delta + \tau]$ for all sufficiently large $s$, because by statement 2 of Proposition 1 the denominator converges uniformly to 1, while the numerator converges to 0 from above. Further,

$$
\max_{z_j \in [0, \epsilon \delta + \tau]} \frac{d}{dz_j}(\Phi_i^x)^{-1}(z_j) \to 0 \quad \text{as} \quad s \to \infty. \quad (A21)
$$

Since $\Phi_i^x(x^\gamma) = 0$, we have $(\Phi_i^x)^{-1}(0) = x^\gamma$. For an arbitrarily small $\sigma > 0$, by (A21) the function $(\Phi_i^x)^{-1}(\cdot)$ cannot increase over $[0, \epsilon \delta + \tau]$ by more than $\sigma$, that is,

$$
(\Phi_i^x)^{-1}(\cdot): [0, \epsilon \delta + \tau] \mapsto [x^\gamma, (\Phi_i^x)^{-1}(\epsilon \delta + \tau)] \subset [x^\gamma, x^\gamma + \sigma), \quad j = 1, 2,
$$

(A22)

provided that $s$ is sufficiently large. Let $G_i^x(\cdot) = \Phi_i^x(\cdot) - (\Phi_i^x)^{-1}(\cdot)$. Since $\Phi_i^x(\cdot)$ and $(\Phi_i^x)^{-1}(\cdot)$ are increasing functions, (A19) and (A22) imply that

$$
G_i^x(x^\gamma) < -x^\gamma \quad \text{and} \quad G_i^x(\epsilon \delta + \tau) > \epsilon \delta + \tau - x^\gamma - \sigma.
$$

Since $\tau$ is positive and fixed, while $\sigma$ can be chosen arbitrarily small (in particular, less than $\tau$), (A18) entails that for all sufficiently large $s$

$$
G_i^x(x^\gamma) < 0 \quad \text{and} \quad G_i^x(\epsilon \delta + \tau) > 0.
$$

By the continuity of $G_i^x(\cdot)$ there exists $x_0^\gamma \in (x^\gamma, \epsilon \delta + \tau)$ such that $G_i^x(x_0^\gamma) = 0$, that is,

$$
\Phi_i^x(x_0^\gamma) = (\Phi_i^x)^{-1}(x_0^\gamma).
$$

(A23)
Let us show that this point is unique. Consider
\[ \frac{d}{dz_i} \Phi_i^s(z_i) = \frac{1 - \epsilon(1 - 2\delta)(d/dz_i) f_{1/2}(z_i)}{(1 - \epsilon)(d/dz_j) f_{1/2}(z_j)} \]
where \( z_j = \Phi_i^s(z_i) \quad i \neq j \quad \gamma = \alpha, \beta. \) \hfill (A24)

The right-hand side of (A24) is defined (since \( \epsilon \in (0, 1/2) \)), and the derivative in the denominator is positive on \( B_0^s \) by statement 2 of Proposition 1. By the same statement, the numerator converges uniformly to 1, while the denominator converges uniformly to 0. This implies that
\[ \min_{z_i \in [x^s_i, \epsilon \delta + \tau]} \frac{d}{dz_i} \Phi_i^s(z_i) \to \infty \quad \text{as} \quad s \to \infty. \] \hfill (A25)

From (A21) and (A25) we conclude that
\[ \min_{z_i \in [x^s_i, \epsilon \delta + \tau]} \frac{d}{dz_i} G^s_{1,2}(z_i) \to \infty \quad \text{as} \quad s \to \infty. \]

The latter relation implies that for all sufficiently large \( s \) the derivative of \( G^s_{1,2}(\cdot) \) is positive on \([x^s, \epsilon \delta + \tau]\); hence the function increases and, consequently, the point \( x_0^s \) is unique.

Now, setting \( y_0^s = \Phi_1^s(x_0^s) = (f_{1/2}^s)^{-1}(T_{1,2}^s(x_0^s)) \), we see that \( (x_0^s, y_0^s) \) satisfies the first of equations (A1). Also, by (A23), \( y_0^s = (\Phi_2^s)^{-1}(x_0^s) = ((f_{1/2}^s)^{-1}T_{1,2}^s)^{-1}(x_0^s) \), which implies that \( (f_{1/2}^s)^{-1}(T_{1,2}^s(y_0^s)) = x_0^s \) or \( f_{1/2}^s(x_0^s) = T_{1,2}^s(y_0^s) \). Consequently, \( (x_0^s, y_0^s) \) satisfies the second of equations (A1). Thus \( (x_0^s, y_0^s) \) is a solution of (A1) and, consequently of (5). As we have shown, \( x_0^s \) is unique, so \( (x_0^s, y_0^s) \) is also unique.

It remains to show that \( J^s((x_0^s, y_0^s)) \to -I \) as \( s \to \infty \). We have
\[ J^s(\tilde{z}) = \begin{pmatrix}
-1 + \epsilon(1 - 2\delta) \frac{d}{dz_1} f_{1/2}(z_1) & (1 - \epsilon) \frac{d}{dz_2} f_{1/2}(z_2) \\
(1 - \epsilon) \frac{d}{dz_1} f_{1/2}(z_1) & -1 + \epsilon(1 - 2\delta) \frac{d}{dz_2} f_{1/2}(z_2)
\end{pmatrix}. \] \hfill (A26)

By statement 2 of Proposition 1 \( J^s(\tilde{z}) \to -I \) uniformly on \([0, \epsilon \delta + \tau] \times [0, \epsilon \delta + \tau]\) as \( s \to \infty \). This implies that \( \lim_{s \to \infty} J^s((x_0^s, y_0^s)) = -I \).

Thus we have shown that for \( \delta > 0 \) there is a single solution of (4) belonging to \( B_0^s \), and this solution has the properties stated in the lemma. A similar argument applies to the domain \( B_1^s \), and the solution \( (x_1^s, y_1^s) \).

Next consider the domain \( B^s \). We shall show that there is a single solution \( (x_\beta^s, x_\alpha^s) \in B^s \) and the Jacobian at this solution has the eigenvalues converging to \( \pm \infty \) as \( s \to \infty \). Notice that by (A4) the functions \( \Phi_i^s(\cdot), \quad i = 1, 2 \), are defined on \( B^s \).
We need a clear idea of how this domain looks. By statement 4 of Proposition 1, $B^r \subset B^r_{\beta, \alpha}$, where

$$B^r_{\beta, \alpha} = \left[ \beta - \sqrt{\frac{\beta(1-\beta)}{s}} b_1^r, \beta + \sqrt{\frac{\beta(1-\beta)}{s}} b_2^r \right] \times \left[ \alpha - \sqrt{\frac{\alpha(1-\alpha)}{s}} a_1^r, \alpha + \sqrt{\frac{\alpha(1-\alpha)}{s}} a_2^r \right],$$

$$\lim_{s \to \infty} b_1^r = \lim_{s \to \infty} a_1^r = l_r = (N)^{-1}(1 - \tau/2) \quad \text{and}$$

$$\lim_{s \to \infty} b_2^r = \lim_{s \to \infty} a_2^r = L_r = -(N)^{-1}(\tau/2).$$

For $(v_1, v_2) \in L^r = [l_r, L_r] \times [l_r, L_r]$ let

$$z_1 = \beta + \sqrt{\frac{\beta(1-\beta)}{s}} v_1 \quad z_2 = \alpha + \sqrt{\frac{\alpha(1-\alpha)}{s}} v_2.$$ 

Then we can rewrite equations (A1) as follows:

$$N(-v_2) = 1 - T_{e, \delta}^r \left( \beta + \sqrt{\frac{\beta(1-\beta)}{s}} v_1 \right) + o_1^r(1), \quad (A27)$$

$$N(-v_1) = 1 - T_{e, \delta}^r \left( \alpha + \sqrt{\frac{\alpha(1-\alpha)}{s}} v_2 \right) + o_2^r(1).$$

We have several cases to consider.

**Case (a).** $\alpha \neq 1/2$ and $\beta \neq 1/2$. Statement 1 of Proposition 2 together with (A27) imply that

$$N(-v_2) = 1 - T_{e, \delta}^r(\beta) + o_1^r(1) \quad \text{and} \quad N(-v_1) = 1 - T_{e, \delta}^r(\alpha) + o_2^r(1). \quad (A28)$$

The equations

$$N(-v_2) = 1 - T_{e, \delta}(\beta), \quad N(-v_1) = 1 - T_{e, \delta}(\alpha),$$

have a unique solution, since by (A6) the right-hand sides belong to $(0, 1)$. Denote this solution by

$$v_1^0 = -(N)^{-1}(1 - T_{e, \delta}(\beta)), \quad v_2^0 = -(N)^{-1}(1 - T_{e, \delta}(\alpha)).$$

For all sufficiently large $s$, a solution $(v_1^0 + o_1^r(1), v_2^0 + o_2^r(1))$ of (A28) exists and, consequently $(x_\beta^x, y_\alpha^x)$ is a solution of (A1), where $x_\beta^x = \beta + \sqrt{\frac{\beta(1-\beta)}{s}} [v_1^0 + o_1^r(1)], y_\alpha^x = \alpha + \sqrt{\frac{\alpha(1-\alpha)}{s}} [v_2^0 + o_2^r(1)]$. For $\gamma = \alpha, \beta$, (A6) implies that $1 - T_{e, \delta}(\gamma) \in (\tau/2, 1 - \tau/2)$, hence $(x_\beta^x, y_\alpha^x)$ belongs to $B^r$. Thus, we have shown that a solution of (5) exists.
Next we prove that the solution is unique. Assume to the contrary that there are at least two solutions of (A1) belonging to $\mathcal{B}^r$: $(z_1^i, z_2^i)$, $i = 1, 2$. To simplify notation, we suppress the dependence on $s$. Hence for some $\tilde{z}_j$ between $z_1^j$ and $z_2^j$

$$z_i^2 - z_i^1 = \frac{d}{dz_j} \Phi_j^i(\tilde{z}_j)(z_j^2 - z_j^1), \quad i \neq j,$$

which implies that

$$\frac{d}{dz_1} \Phi_1^i(\tilde{z}_1) \frac{d}{dz_2} \Phi_2^i(\tilde{z}_2) = 1,$$

or, by (A24),

$$\frac{1 - \epsilon(1 - 2\delta)(d/dz_1)f_{1/2}^s(\tilde{z}_1)}{(1 - \epsilon)(d/dz_2)f_{\alpha}^s(\tilde{z}_1)} = \frac{(1 - \epsilon)(d/dz_1)f_{\beta}^s(\tilde{z}_2)}{1 - \epsilon(1 - 2\delta)(d/dz_2)f_{1/2}^s(\tilde{z}_2)}. \quad (A29)$$

If $\tau$ is so small that $1/2 \notin [\gamma - \tau/2, \gamma + \tau/2]$ for $\gamma = \alpha$ or $\gamma = \beta$, then by statement 2 of Proposition 1, the numerators in (A24) converge uniformly to 1 on $\mathcal{B}^r$. This has two implications. First, (A29) is equivalent to

$$(1 - \epsilon)^2 \frac{d}{dz_2} f_{\alpha}^s(\Phi_1^i(\tilde{z}_1)) \frac{d}{dz_1} f_{\beta}^s(\Phi_2^i(\tilde{z}_2)) = 1 + o_s(1). \quad (A30)$$

Second, since the denominators in (A24) are positive, $(d/dz_i)\Phi_i^i(\cdot) > 0$ and thus $\Phi_i^i(\cdot)$ is increasing on $\mathcal{B}^r$. The solution $(v_1^0, v_2^0)$ is unique, which entails that

$$\lim_{s \to \infty} \sqrt{s}|z_1^i - z_2^i| = 0, \quad i = 1, 2. \quad (A31)$$

Since $\Phi_j^i(z_j^k) = z_j^k$, $i \neq j$, and the functions are continuous and monotone, we conclude that $\Phi_j^i(\tilde{z}_j)$ lies between $z_j^k$, $k = 1, 2$. Hence, by (A31), we obtain

$$\lim_{s \to \infty} \sqrt{s}|z_j^i - \Phi_j(\tilde{z}_j)| = 0, \quad i \neq j. \quad (A32)$$

By statement 5 of Proposition 1, we conclude from (A31) and (A32) that the left-hand side of (A30) unboundedly increases (as $s[1 + o_s(1)](1 - \epsilon)^2 \exp\{-\{(v_1^0)^2 + (v_2^0)^2)/2\}/2\pi\sqrt{\alpha(1 - \alpha)}\beta(1 - \beta)}$ as $s \to \infty$, which contradicts this equality. Thus, the solution $(x_\beta, y_\alpha)$ must be unique.

Let us show that $J^s((x_\beta, y_\alpha))$ has eigenvalues converging to $\pm \infty$ as $s \to \infty$. By (A26) we see that the eigenvalues $\lambda_{1,2}$ of $J^s((x_\beta, y_\alpha))$ are

$$\lambda_{1,2} = -1 + o_s(1) \pm (1 - \epsilon) \sqrt{d/dz_1 f_{\beta}^s(x_\beta) d/dz_2 f_{\alpha}^s(y_\alpha)[1 + o_s^2(1)]}.$$
We took into account here that, by statement 2 of Proposition 1, 
\( (d/dz_i)f_{1/2}(z_i) \to 0 \) as \( s \to \infty \) uniformly on \( B^x \). By statement 5 of Proposition 1, we obtain

\[
\lambda_{1,2} = \pm \sqrt{s}[1 + o^2_s(1)](1 - \epsilon) \exp\left[-\frac{[(\nu_0^1)^2 + (\nu_0^2)^2)/4]}{\sqrt{2\pi \sqrt{4\alpha(1 - \alpha)}\beta(1 - \beta)}} \right].
\]

Hence, one of the eigenvalues converges (at the rate \( \sqrt{s} \)) to \( \infty \), while the other converges (at the same rate) to \( -\infty \). Thus, we have proved the statement of the lemma for the case when \( \alpha \neq 1/2 \) and \( \beta \neq 1/2 \).

**Case (b).** \( \alpha \neq 1/2 \) and \( \beta = 1/2 \), or \( \alpha = 1/2 \) and \( \beta \neq 1/2 \). It suffices to consider the situation \( \alpha \neq 1/2 \) and \( \beta = 1/2 \). Let \( \tau \) be so small that \( 1/2 \not\in [\alpha - \tau/2, \alpha + \tau/2] \). By statements 1 and 6 of Proposition 2 we obtain from (A27) that

\[
N(-\nu_2) = 1 - \frac{1 - 2\epsilon(1 - \delta)}{2(1 - \epsilon)} - \frac{\epsilon}{1 - \epsilon}(1 - 2\delta)N(-\nu_1) + o^1_s(1),
\]

\[
N(-\nu_1) = 1 - T_{\epsilon, \delta}(\alpha) + o^2_s(1).
\]

Consider the equations

\[
N(-\nu_2) = 1 - \frac{1 - 2\epsilon(1 - \delta)}{2(1 - \epsilon)} - \frac{\epsilon}{1 - \epsilon}(1 - 2\delta)N(-\nu_1),
\]

\[
N(-\nu_1) = 1 - T_{\epsilon, \delta}(\alpha),
\]

or, equivalently,

\[
N(-\nu_1) = 1 - T_{\epsilon, \delta}(\alpha),
\]

\[
N(-\nu_2) = \frac{1}{2} + \frac{\epsilon}{1 - \epsilon}(1 - 2\delta)\left[T_{\epsilon, \delta}(\alpha) - \frac{1}{2}\right].
\]

(A34)

Let us show that

\[
\frac{1}{2} + \frac{\epsilon}{1 - \epsilon}(1 - 2\delta)\left[T_{\epsilon, \delta}(\alpha) - \frac{1}{2}\right] \in (\tau, 1 - \tau),
\]

(A35)

which would imply that (A34) has a unique solution in \( L^x \), namely,

\[
\nu_1^0 = -(N)^{-1}(1 - T_{\epsilon, \delta}(\alpha)),
\]

\[
\nu_2^0 = -(N)^{-1}\left(\frac{1}{2} + \frac{\epsilon}{1 - \epsilon}(1 - 2\delta)\left[T_{\epsilon, \delta}(\alpha) - \frac{1}{2}\right]\right).
\]

Indeed, by (A6), \( \tau - 1/2 < T_{\epsilon, \delta}(\alpha) - 1/2 < 1/2 - \tau \), where \( \tau - 1/2 < 0 \) by choice of \( \tau \). Furthermore, \( |1 - 2\delta| \leq 1 \) and \( 0 < \epsilon(1 - \epsilon)^{-1} < 1 \), the latter since \( \epsilon \in (0, 1/2) \). Consequently,

\[
\tau - 1/2 < \frac{\epsilon}{1 - \epsilon}(1 - 2\delta)\left[T_{\epsilon, \delta}(\alpha) - \frac{1}{2}\right] < 1/2 - \tau,
\]

which implies (A35).
As above, \((v^0_1, v^0_2)\) determines a solution \((x^s_{1/2}, y^s_\alpha)\) of (A1), which we claim is unique. Assuming to the contrary that there are at least two solutions of (A1), we obtain (A29). Since the solution \((v^0_1, v^0_2)\) is unique, (A31) holds. We claim that (A32) also holds. Now \(\Phi^s_1(\cdot)\) is exactly as in the previous case, which implies that (A32) holds for \(j = 1\). On the other hand, for \(j = 2\), Proposition 1 (statement 5) implies that as \(s \to \infty\) the numerator of (A24) increases without bound if \(\delta > 1/2\), decreases without bound if \(\delta < 1/2\), and is equal to 1 if \(\delta = 1/2\). Since the denominator is always positive, in all of these situations the derivative of \(\Phi^s_2(\cdot)\) preserves its sign on \(B^*\) for all sufficiently large \(s\). Hence the function \(\Phi^s_2(\cdot)\) is monotone. By continuity, (A32) follows from (A31) for \(j = 2\). From (A32) and statement 5 of Proposition 1 we conclude that the left-hand side of (A29) converges to 0 from above. For the right-hand side there are three possibilities: it converges to a negative limit, namely, \(-\frac{1-\epsilon}{\epsilon(2\delta-1)} \exp\{\frac{1}{2}[(v^0_2)^2 - (v^0_1)^2]\}\), if \(\delta < 1/2\); it converges to infinity if \(\delta = 1/2\); it converges to a finite positive limit, namely, \(\frac{1-\epsilon}{\epsilon(2\delta-1)} \exp\{\frac{1}{2}[(v^0_2)^2 - (v^0_1)^2]\}\), if \(\delta > 1/2\). Thus, the limits of the right-hand side and the left-hand side do not coincide, which is impossible. This contradiction proves that the solution \((x^s_{1/2}, y^s_\alpha)\) is unique.

Next let us consider the eigenvalues of the Jacobian. By statement 2 of Proposition 1 we see that \((d/dz_1)f^s_{1/2}(x^s_{1/2}) = o_s(1)\). Hence, by statement 5 of this proposition and (A26) we obtain

\[
\lambda_{1,2} = -1 + \frac{1}{2} \epsilon(1-2\delta) \frac{d}{dz_1} f^s_{1/2}(x^s_{1/2}) + o_s^2(1) \\
\pm \frac{1}{2} \left\{ \epsilon(1-2\delta)^2 \left[ \left( \frac{d}{dz_1} f^s_{1/2}(x^s_{1/2}) \right)^2 + 1 + o_s^2(1) \right] \right. \\
+ 4(1-\epsilon)^2 \frac{d}{dz_1} f^s_{1/2}(x^s_{1/2}) \frac{d}{dz_2} f^s_{2/2}(y^s_\alpha) \right\}^{1/2},
\]

that is,

\[
\lambda_{1,2} = \sqrt{s} [1 + o_s(1)] \frac{\exp[-1/2(v^0_1)^2]}{\sqrt{2\pi}} \\
\times \left\{ \epsilon(1-2\delta) \pm \sqrt{\epsilon(1-2\delta)^2 + \frac{2(1-\epsilon)^2 \exp[(v^0_2)^2 - (v^0_1)^2]/2)}{\alpha(1-\alpha)} \right\}.
\]

Hence, for all sufficiently large \(s\) there is one positive eigenvalue converging to \(\infty\) at the rate \(\sqrt{s}\) as \(s \to \infty\) and a negative one converging to \(-\infty\) at the same rate. Thus we have proved the statement of the lemma for \(\alpha \neq 1/2\) and \(\beta = 1/2\). A similar proof holds if \(\alpha = 1/2\) and \(\beta \neq 1/2\).

Case (c). \(\alpha = \beta = 1/2\). Now Eqs. (A1) are identical and \(\Phi^s_1(\cdot) = \Phi^s_2(\cdot)\), which we shall henceforth designate by \(\Phi^s(\cdot)\). If there is a unique solution
\((z_1, z_2)\) of (A1), then \(z_1 = z_2\). Moreover \((0, 0)\) is the unique solution of

\[
N(-v_2) = 1 - \frac{1 - 2\epsilon(1 - \delta)}{2(1 - \epsilon)} - \frac{\epsilon}{1 - \epsilon}(1 - 2\delta)N(-v_1),
\]

\[
N(-v_1) = 1 - \frac{1 - 2\epsilon(1 - \delta)}{2(1 - \epsilon)} - \frac{\epsilon}{1 - \epsilon}(1 - 2\delta)N(-v_2).
\]

As above, this solution generates a solution \((x_{1/2}^f, y_{1/2}^f)\) of (A1). What remains to be shown is that the latter is unique and the statement concerning the eigenvalues.

Assume by way of contradiction, that there are at least two solutions of (A1). Then we obtain (A29). Since \((0, 0)\) is the unique solution of the above equations, (A31) holds. Arguing as we did in the preceding case, we see that \(\Phi^s(\cdot)\) is a monotone function on \(B^r\) for all sufficiently large \(s\). By the continuity of this function, (A31) entails (A32). Statement 5 of Proposition 1 implies that

\[
\lim_{s \to \infty} \frac{1 - \epsilon(1 - 2\delta)(d/dz_i)f_{1/2}^s(\bar{z}_i)}{(1 - \epsilon)(d/dz_j)f_{1/2}^s(\Phi^s(\bar{z}_i))} = \begin{cases} 
0 & \text{if } \delta = 1/2 \\
\frac{\epsilon(1 - 2\delta)}{1 - \epsilon} & \text{otherwise.}
\end{cases}
\]

(A36)

Since \(\epsilon \in (0, 1/2)\), we have \(|\frac{\epsilon(1 - 2\delta)}{1 - \epsilon}| < 1\). By (A36) this implies that (A29) is impossible. Consequently, the solution of (A1) must be unique.

Arguing as above, we find that the eigenvalues satisfy

\[
\lambda_{1,2} = \sqrt{\frac{s}{2\pi}} \left[1 + o_s(1)\right] \left\{\epsilon(1 - 2\delta) \pm \sqrt{[\epsilon(1 - 2\delta)]^2 + 4(1 - \epsilon)^2}\right\},
\]

so they converge to \(\pm \infty\) as \(s \to \infty\). Thus, we have proved the statement of the lemma for the case when \(\alpha = \beta = 1/2\). This completes the consideration for the domain \(B^r\), or, equivalently, for the root \((x_{1/2}^f, y_{1/2}^f)\). Since we have already considered the domains \(B_0^r\) and \(B_1^r\), the lemma is proved.

**Lemma 4.** Let \(G\) be a nondegenerate \(2 \times 2\) game with three equilibria. Let \(s > 1\) be an odd number, \(\epsilon \in (0, 1/2)\) and \(\delta \in [0, 1]\). Every solution \(\bar{z}^s(t)\) of (2) belonging to \(\mathcal{K}\) for \(0 \leq t < \infty\), converges to a limit as \(t \to \infty\), and the limit must be a singular point. If \(\bar{z}^s(t)\) belongs to \(\mathcal{K}\) for \(-\infty < t < \infty\), then either \(\lim_{t \to -\infty} \bar{z}^s(t) = \text{exists and } \lim_{t \to -\infty} \bar{z}^s(t) \neq \lim_{t \to \infty} \bar{z}^s(t), \) or else \(\bar{z}^s(\cdot)\) is constant.

**Proof.** Let us show first that \(\bar{z}^s(t)\) converges to a singular point as \(t \to \infty\). If, there is a time instant \(t^*\) such that \(\bar{z}^s(t)\) belongs to one of the sets \(\mathcal{K}_{++}, \mathcal{K}_{+-}, \mathcal{K}_{-+}\) and \(\mathcal{K}_{--}\) for \(t \geq t^*\), then both coordinates \(\bar{z}_1^s(\cdot)\) and \(\bar{z}_2^s(\cdot)\) are monotone functions on \([t^*, \infty)\). Hence, they converge as \(t \to \infty\), and
this limit must coincide with a singular point. To establish convergence of the positive half-trajectories, it therefore suffices to show that starting from some time instant \( \tilde{z}^s(\cdot) \) belongs to one of these sets.

Assume that \( \tilde{z}^s(0) \in \mathcal{K}_{++} \). (The same argument applies when \( \tilde{z}^s(0) \in \mathcal{K}_{+-} \)). Then either \( \tilde{z}^s(t) \) belongs to \( \mathcal{K}_{+-} \) for all \( t \geq 0 \) and there is nothing to prove, or \( \tilde{z}^s(\cdot) \) escapes from \( \mathcal{K}_{+-} \) at some finite time \( t^* \). Let \( \tilde{z}^* = \tilde{z}^s(t^*) \).

Then either \( g_{1}^s(\tilde{z}^*) > 0 \) and \( g_{2}^s(\tilde{z}^*) = 0 \), or \( g_{1}^s(\tilde{z}^*) = 0 \) and \( g_{2}^s(\tilde{z}^*) < 0 \). In the first case \( \tilde{z}^s(\cdot) \) enters \( \mathcal{K}_{++} \), and in the second it enters \( \mathcal{K}_{--} \). Notice that we cannot have \( g_{1}^s(\tilde{z}^*) > 0 \) and \( g_{2}^s(\tilde{z}^*) < 0 \) or \( g_{1}^s(\tilde{z}^*) = 0 \) and \( g_{2}^s(\tilde{z}^*) = 0 \).

Indeed, if both coordinates of \( \tilde{g}^s(\cdot) \) are nonzero at \( \tilde{z}^* \), then it is an interior point of \( \mathcal{K}_{+-} \). But \( \tilde{z}^s(\cdot) \) cannot exit from this set at an interior point. On the other hand, if both coordinates equal zero, \( \tilde{z}^s(\cdot) \) cannot move from \( \tilde{z}^* \), which means that it remains at this point forever.

Thus, we have shown that if \( \tilde{z}^s(\cdot) \) leaves \( \mathcal{K}_{+-} \), then it enters either \( \mathcal{K}_{++} \) or \( \mathcal{K}_{--} \). Let us prove that \( \tilde{z}^s(\cdot) \) does not escape from \( \mathcal{K}_{++} \) once it enters \( \mathcal{K}_{++} \). (A similar argument applies to \( \mathcal{K}_{--} \).) In other words, we shall show that \( \mathcal{K}_{++} \) is a positive invariant set for (2).

It is sufficient to prove that \( \tilde{z}^s(t) \in \mathcal{K}_{++} \) for all \( t > 0 \) provided \( \tilde{z}^s(0) \in \mathcal{K}_{++} \). To prove this it is enough to show (see Aubin, 1991, Theorems 1.2.1 and 1.2.3) that for every \( \tilde{z} \in \mathcal{K}_{++} \)

\[
\tilde{g}^s(\tilde{z}) \in T_{++}(\tilde{z}).
\]

Recall that \( T_{++}(\tilde{z}) \) stands for the contingent cone to \( \mathcal{K}_{++} \) at \( \tilde{z} \) defined by (6). Let us prove that (A37) holds for every \( \tilde{z} \in \mathcal{K}_{++} \). If \( g_{1}^s(\tilde{z}) > 0 \) and \( g_{2}^s(\tilde{z}) > 0 \), then \( \tilde{z} \) is an interior point of \( \mathcal{K}_{++} \). Hence \( \tilde{z} + \Delta \tilde{v} \in \mathcal{K}_{++} \) for every \( \tilde{v} \in R^2 \) if \( \Delta \) is sufficiently small. Consequently, \( T_{++}(\tilde{z}) = R^2 \), which implies (A37). If \( g_{1}^s(\tilde{z}) = 0 \), then (A37) holds because \( \tilde{0} \in T_{++}(\tilde{z}) \). Now let \( g_{1}^s(\tilde{z}) > 0 \) and \( g_{2}^s(\tilde{z}) = 0 \). (If \( g_{1}^s(\tilde{z}) = 0 \) and \( g_{2}^s(\tilde{z}) > 0 \) the same argument applies.) Fix a sequence \( \tau_k, k \geq 1 \), of positive numbers converging to 0 as \( k \to \infty \). Letting \( \tilde{v} = \tilde{g}^s(\tilde{z}) \) and \( \Delta_k = \tau_k / g_{1}^s(\tilde{z}) \), we obtain

\[
\tilde{z} + \Delta_k \tilde{v} = \tilde{z} + \Delta_k (g_{1}^s(\tilde{z}), 0) = \tilde{z} + \Delta_k g_{1}^s(\tilde{z}) \tilde{e} = \tilde{z} + \tau_k \tilde{e} \quad \text{where} \quad \tilde{e} = (1, 0).
\]

Since \( f^s_{\beta}(\cdot) \) is increasing (statement 2 of Proposition 1), for all sufficiently large \( k \) we have

\[
g_{2}^s(\tilde{z} + \Delta_k \tilde{v}) = g_{2}^s(\tilde{z} + \tau_k \tilde{e}) > g_{2}^s(\tilde{z}) = 0.
\]

By continuity, \( g_{1}^s(\tilde{z}) > 0 \) implies that \( g_{1}^s(\tilde{z} + \Delta_k \tilde{v}) = g_{1}^s(\tilde{z} + \tau_k \tilde{e}) > 0 \) for all sufficiently large \( k \). Thus, \( \tilde{z} + \Delta_k \tilde{v} \) is an interior point of \( \mathcal{K}_{++} \) for all sufficiently large \( k \). This implies that \( d(\tilde{z} + \Delta_k \tilde{v}, \mathcal{K}_{++}) = 0 \). By (6) it follows that \( \tilde{v} = \tilde{g}^s(\tilde{z}) \in T_{++}(\tilde{z}) \). This establishes (A37).
To prove the second statement of the lemma, assume that \( \bar{z}^s(\cdot) \) is a solution of (2) such that \( \bar{z}^s(t) \in \mathcal{H} \) for \(-\infty < t < \infty\) and that \( \bar{z}^s(\cdot) \) is not constant. We shall show that

\[
\lim_{t \to -\infty} \bar{z}^s(t) \neq \lim_{t \to \infty} \bar{z}^s(t). \tag{A38}
\]

We have already proved that there is a time instant \( t^* \) such that \( \bar{z}^s(\cdot) \) does not escape one of the sets \( \mathcal{H}_{++}, \mathcal{H}_{+}, \mathcal{H}_{-} \) and \( \mathcal{H}_{--} \) for \( t \geq t^* \). It is suffices to consider only the case when \( \bar{z}^s(t) \in \mathcal{H}_{++} \) for \( t \geq t^* \), since the other possibilities can be treated similarly. If \( \bar{z}^s(t) \in \mathcal{H}_{++} \) for all \( t < t^* \), then both \( z_1^s(\cdot) \) and \( z_2^s(\cdot) \) are monotone functions. Consequently, \( \lim_{t \to -\infty} \bar{z}^s(t) \) exists. Since, by assumption, \( \bar{z}^s(\cdot) \) is not constant, the monotonicity of \( z_i^s(\cdot) \) implies that (A38) holds for the case when \( \bar{z}^s(\cdot) \) does not escape from \( \mathcal{H}_{++} \). Otherwise, there must be a point \( t' < t^* \) such that \( \bar{z}^s(t') \notin \mathcal{H}_{++} \). We claim that in this case one of the following inclusions holds for all \( t < t^* \)

\[
\bar{z}^s(t) \in \mathcal{H}_{--} \setminus \mathcal{H}_{++}, \tag{A39}
\]

\[
\bar{z}^s(t) \in \mathcal{H}_{-} \setminus \mathcal{H}_{++}. \tag{A40}
\]

Without loss of generality we can assume that

\[
t^* = \min\{t: \bar{z}^s(t) \in \mathcal{H}_{++}\}. \tag{A41}
\]

Note that we can express \( \mathcal{H} \) as follows:

\[
\mathcal{H} = \mathcal{H}_{++} \cup (\mathcal{H}_{++} \setminus \mathcal{H}_{++}) \cup (\mathcal{H}_{-} \setminus \mathcal{H}_{++}) \cup (\mathcal{H}_{--} \setminus \mathcal{H}_{++}).
\]

If \( \bar{z}^s(t') \notin \mathcal{H}_{++} \) for some \( t' < t^* \), then there are only three possibilities:

\( \bar{z}^s(t') \in \mathcal{H}_{--} \setminus \mathcal{H}_{++}, \bar{z}^s(t') \in \mathcal{H}_{-} \setminus \mathcal{H}_{++}, \) and \( \bar{z}^s(t') \in \mathcal{H}_{--} \setminus \mathcal{H}_{++}. \)

Consequently, to prove (A39) and (A40), it is enough to show that, first, \( \bar{z}^s(\cdot) \) cannot escape from \( \mathcal{H}_{--} \setminus \mathcal{H}_{++} \) or \( \mathcal{H}_{-} \setminus \mathcal{H}_{++} \) before \( t^* \) and that, second, it never enters \( \mathcal{H}_{--} \setminus \mathcal{H}_{++} \) before \( t^* \).

Let us show that \( \bar{z}^s(\cdot) \) cannot escape from \( \mathcal{H}_{--} \setminus \mathcal{H}_{++} \) before \( t^* \). (For \( \mathcal{H}_{-} \setminus \mathcal{H}_{++} \) the argument is the same.) If \( \bar{z}^s(\cdot) \) escapes from \( \mathcal{H}_{--} \setminus \mathcal{H}_{++} \) at \( \bar{t} < t^* \), either

\[
g_1^s(\bar{z}^s(\bar{t})) = 0, \tag{A42}
\]

or

\[
g_2^s(\bar{z}^s(\bar{t})) = 0. \tag{A43}
\]

We claim that neither is possible. Consider (A42) first. There are two possibilities:

\[
g_2^s(\bar{z}^s(\bar{t})) \geq 0, \tag{A44}
\]

or

\[
g_2^s(\bar{z}^s(\bar{t})) < 0. \tag{A45}
\]
Since $\tilde{z}^s(\tilde{t}) \notin \mathcal{H}_{++}$, (A42) and (A44) cannot both hold. Suppose on the other hand that (A42) and (A45) hold, that is, $\tilde{z}^s(\tilde{t}) \in \mathcal{H}_-$ for all $t \geq \tilde{t}$, and, in particular, $\tilde{z}^s(t^*) \in \mathcal{H}_-$. But $\tilde{z}^s(t^*) \in \mathcal{H}_{++}$, by assumption, so $\tilde{z}^s(t^*) \in \mathcal{H}_{++} \cap \mathcal{H}_-$ which implies that $\bar{g}^s(\tilde{z}^s(t^*)) = 0$. This means that $\tilde{z}^s(\cdot)$ is a constant, which is ruled out by hypothesis. We therefore conclude that (A42) and (A45) cannot both hold, and neither can (A42) and (A44). Consequently, (A42) always leads to a contradiction. A similar contradiction arises if (A43) holds. Consequently, neither (A42) nor (A43) is possible, which implies that $\tilde{z}^s(\cdot)$ does not escape $\mathcal{H}_{++} \setminus \mathcal{H}_{++}$ before $t^*$.

Next let us show that $\tilde{z}^s(\cdot)$ cannot enter $\mathcal{H}_- \setminus \mathcal{H}_{++}$ before $t^*$. Assume to the contrary that there is $t' < t^*$ such that $\tilde{z}^s(t') \in \mathcal{H}_- \setminus \mathcal{H}_{++}$. Since $\mathcal{H}_-$ is a positive invariant set for (2), the latter entails that $\tilde{z}^s(t) \in \mathcal{H}_-$ for all $t \geq t'$, and, in particular, $\tilde{z}^s(t^*) \in \mathcal{H}_-$. But, by assumption, $\tilde{z}^s(t^*) \in \mathcal{H}_{++}$, so $\tilde{z}^s(t^*) \in \mathcal{H}_{++} \cap \mathcal{H}_-$, which implies that $\bar{g}^s(\tilde{z}^s(t^*)) = 0$. This means that $\tilde{z}^s(\cdot)$ is a constant, which is ruled out by hypothesis. Consequently, $\tilde{z}^s(\cdot)$ cannot enter $\mathcal{H}_- \setminus \mathcal{H}_{++}$ before $t^*$.

In sum, we have showed that if $\tilde{z}^s(\cdot)$ leaves $\mathcal{H}_{++}$ before $t^*$, then either (A39) or (A40) holds. If (A39) takes place, then $\tilde{z}^s_1(\cdot)$ is increasing before entering $\mathcal{H}_{++}$, and is nondecreasing in $\mathcal{H}_{++}$. Consequently, (A38) holds. A similar argument shows that (A38) follows from (A40). This completes the proof of Lemma 4.

Next we shall establish a counterpart of Lemma 4 for games with a unique equilibrium. We shall restrict attention to $\delta < 1/2$, because this is the only situation not covered by Lemma 1. The geometry of the situation depends on the location of $\beta$ and $\alpha$ with respect to $1/2$. For simplicity we shall consider only the case $\beta < 1/2$ and $\alpha < 1/2$, the other cases can be studied similarly.

The argument needed to prove Lemma 6 involves several steps. We shall give them subsequently. Throughout this chain of arguments we shall be assuming that $0 < \epsilon < 1/2$, $0 < \delta < 1/2$, $\epsilon \delta < \min(\beta, \alpha) \leq \max(\beta, \alpha) < 1/2$. Remember that system (2) cannot have phase-1-gones in this case, hence, the only type of closed orbits to be excluded are cycles.

In Hahn (1967, p. 66) it is shown that if $L^s \subset \mathcal{H}$ is a cycle of (2) and $L^s$ is the finite closed domain bounded by $L^s$, then

$$\int_{L^s} \text{div} \bar{g}^s(z)d\bar{z} = 0. \tag{A46}$$

Given $\sigma \in [0, 1/2)$, let $\Pi_1^s = [1/2 - \sigma, 1/2 + \sigma] \times [0, 1]$ and $\Pi_2^s = [0, 1] \times [1/2 - \sigma, 1/2 + \sigma]$. By statement 2 of Proposition 1, for every $\sigma > 0$ there exists $s(\sigma)$ such that

$$0 < \frac{d}{dz_i} f_{i/2}^s(z_i) < \sigma \quad \text{for} \quad \bar{z} \in \mathcal{H} \setminus \Pi_i^s, \quad i = 1, 2, \quad \text{and} \quad s \geq s(\sigma).$$
Hence,

\[ \text{div } \mathbf{g}^s(\vec{z}) < 2\epsilon\sigma - 2 < -1 \quad \text{for} \quad \vec{z} \in \mathcal{K} \setminus (\Pi_1^s \cup \Pi_2^s) \quad \text{and} \quad s \geq s(\sigma). \]

\[ (A47) \]

**Proposition 3.** Let \( \sigma \in (0, 1/2) \) and \( \mathcal{L}^s \subset \mathcal{K} \) be a cycle of system (2) for some \( s \geq s(\sigma) \). Then \( L^s \cap (\Pi_1^s \cup \Pi_2^s) \neq \emptyset \), where \( L^s \) denotes the finite closed domain bounded by \( \mathcal{L}^s \).

**Proof.** Assume to the contrary that \( \mathcal{L}^s \) is a cycle of system (2) for some \( s \geq s(\sigma) \), but \( L^s \cap (\Pi_1^s \cup \Pi_2^s) = \emptyset \). Then for all \( \vec{z} \in L^s \) inequality (A47) holds; hence (A46) cannot hold. Thus \( \mathcal{L}^s \) is not a cycle.

This immediately yields the following.

**Proposition 4.** Let \( s_1, s_2, \ldots \) be a sequence of odd numbers converging to infinity. If \( \mathcal{L}^{s_i} \) is a cycle of system (2) with \( s = s_i \), then there exists a sequence of positive numbers \( \sigma_1, \sigma_2, \ldots \) converging to zero and such that

\[ L^{s_i} \cap (\Pi_1^{\sigma_i} \cup \Pi_2^{\sigma_i}) \neq \emptyset, \]

where \( L^{s_i} \) designates the finite closed domain bounded by \( \mathcal{L}^{s_i} \).

The plan of the proof runs as follows. First we shall consider the limit behavior of (2) as \( s \to \infty \). Then we shall identify some sets that are always "almost" inside any cycle of (2) provided that \( s \) is large enough, and show that the Lebesgue measure of these sets is bounded from below as \( s \to \infty \) by a positive value \( S \) which is a function of \( \beta, \alpha, \epsilon \) and \( \delta \). The divergence involved in (A46) equals \(-2\) plus a vanishing term everywhere inside a cycle, except for two narrow strips \( \Pi_1^{\sigma_i} \) and \( \Pi_2^{\sigma_i} \) where it unboundedly increases. Thus, the impact of the negative part in (A46) is bounded from above by \(-2S\) plus a vanishing term, while, as we shall show, the impact of the positive part never exceeds \( 2\epsilon(1 - 2\delta) \) plus a vanishing term. Consequently, if \( S > \epsilon(1 - 2\delta) \), the integral involved in (A46) is negative for all sufficiently large \( s \). By (A46) this excludes cycles for system (2). This line of argument is summarized in Lemma 5.

Having outlined the general plan of argument, we now implement it. Consider the limit system (3). We have shown in Section 3 that \( \mathcal{K} \) splits into nine domains \( R^{ij} \) inside of which the right-hand side of (3) is a linear function (see Tables I and II). This vector field is shown schematically in Fig. 7.

Since the right-hand side of (3) is a discontinuous function, the standard notion of a solution does not apply in this situation, and we have to understand solutions of this system of ordinary differential equations in the
sense of Filippov (1960, p. 101). Namely, an absolutely continuous vector function \( \vec{z}(\cdot) \) is a Filippov solution of (3) on a finite interval \([t_1, t_2]\) of time if, for almost all \( t \in [t_1, t_2] \),

\[
\frac{d}{dt} \vec{z}(t) \in \bar{G}(\vec{z}(t)) - \vec{z}(t),
\]

where

\[
\bar{G}(\vec{z}) = \begin{cases} 
\{ \bar{g}(\vec{z}) \} & \text{for } \vec{z} \in \bigcup_{i,j=1}^{3} R^{ij}, \\
\bigcap_{\mu > 0} \text{conv} \{ \bar{g}(\vec{x}) : ||\vec{x} - \vec{z}|| < \mu, \ \vec{x} \in \mathcal{K} \} & \text{for } \vec{z} \in \mathcal{K} \setminus \bigcup_{i,j=1}^{3} R^{ij}.
\end{cases}
\]

Here \( \text{conv} \ B \) designates the convex hull of \( B \subset R^2 \). The existence of solutions of system (3) on \([0, \infty)\) and their continuity with respect to the initial state follow from Theorems 4 and 5 in Filippov (1960, pp. 111–112). We can show that solutions of (3) which start in \( \mathcal{K} \) never leave this set, that is, \( \mathcal{K} \) is a positive invariant set of (3). Figure 7 gives a geometrical intuition for this statement, showing that close to the boundary of \( \mathcal{K} \) the trajectories of (3) are pushed away from the boundary. If \( C^{ij} \) is a closed subset of
$R^{ij}$, then, by statement 1 of Proposition 1, the vector field $\tilde{g}^s(\cdot)$ converges to $\tilde{g}(\cdot)$ uniformly on $C^{ij}$ as $s \to \infty$. Hence, using Theorem 3 from Filippov (1960, p. 109), we obtain the following statement.

**Proposition 5.** Let $\tilde{z}^{s_i}(\cdot)$ be a solution of system (2) with $s = s_i$ for some sequence of odd numbers $s_i$, $i \geq 1$, converging to infinity. Further, let $\tilde{z}^{s_i}(t) \in \mathcal{K}$ for all $t$ in some finite time interval $[t_1, t_2]$. Then there exists an infinite subsequence $\{s_i\} \subseteq \{s_i\}$ such that $\tilde{z}^{s_i}(\cdot)$ converges uniformly on $[t_1, t_2]$ to a solution of (3).

The expression for the set-valued mapping $\tilde{G}(\cdot)$ shows that the inclusion $\bar{0} \in \tilde{G}(\tilde{z}) - \tilde{z}$ holds only for $\tilde{z} = (\beta, \alpha)$. Hence, $(\beta, \alpha)$ is the unique singular point of (3) in $\mathcal{K}$. The following statement concerning the uniqueness of the solutions of (3) can be obtained from Theorem 14 by Filippov (1960, p. 125).

**Proposition 6.** Let $\tilde{z}(\cdot)$ be a solution of (3) on $[0, T]$, $0 < T < \infty$, starting at $t = 0$ from $\tilde{z}^0 \in \mathcal{K}$ and such that $\tilde{z}(t) \neq (\beta, \alpha)$ for $t \in [0, T]$. Then such a solution is unique.

Figure 7 gives an intuition for the following result.

**Proposition 7.** Let $\tilde{z}(\cdot)$ be a solution of (3) on $[0, T]$, $0 < T < \infty$, starting at $t = 0$ from a point $\tilde{z}^0 \in \mathcal{K} \setminus \{ (\beta, \alpha) \}$. Then $\tilde{z}(t) \neq (\beta, \alpha)$ for all $t \in [0, T]$.

The simple structure of the right-hand side of (3) suggests that we can characterize the behavior of its solutions. Indeed, by Fig. 7 we conclude that any such solution must pass through the sets $R^{ij}$ in the order given in Fig. 8. If $\tilde{z}^0 \in R^{ij}$ then, integrating Eq. (3), we obtain

$$\tilde{z}(t) = \tilde{z}^0 + [1 - \exp(-t)](\tilde{g}^{ij} - \tilde{z}^0) \quad \text{for} \quad 0 \leq t < t^{ij},$$

where $t^{ij}$ designates the time instant (possibly infinite) when $\tilde{z}(\cdot)$ exits from $R^{ij}$ for the first time. By Table II we conclude that each such trajectory spends a finite time in each $R^{ij}$.

![FIG. 8. Possible paths of the limit system through the sets $R^{ij}$.](image)
We claim that this time is bounded above by a constant specific to this set. To see this, consider the situation \( \bar{z}^0 \in R^{11} \). Since \( \delta < 1/2, \epsilon \delta < \alpha \), and \( \alpha < 1/2 \) by hypothesis, we have \( 1 - \epsilon(1 - \delta) > \alpha \). But \( g^{11}_2 = 1 - \epsilon(1 - \delta) \), which implies that \( z_2(t) \) increases to \( 1 - \epsilon(1 - \delta) \) as \( t \to \infty \). Hence \( \bar{z}^0(\cdot) \) can only spend a finite time in \( R^{11} \). (In fact, it cannot spend more than \( \bar{t}^{11} = \lim \frac{1 - \epsilon(1 - \delta)}{1 - \epsilon(1 - \delta) - \alpha} \) in \( R^{11} \).) Recall, that a trajectory of (3) can get stuck only at the point \((\beta, \alpha)\). By Table II we conclude that each solution of (3) which does not pass through \((\beta, \alpha)\) must visit \( R^{11}, R^{12} \) and \( R^{21} \) infinitely often.

In what follows we shall consider only such solutions.

**Proposition 8.** Let \( s_1, s_2, \ldots \) be a sequence of odd numbers converging to infinity, and let \( \mathcal{L}^{s_i} \subset \mathcal{R} \) be a cycle formed by a solution of system (2) with \( s = s_i \). Then the limit system (3) has a cycle \( \mathcal{L} \) and there exists a subsequence \( \{s_{i'}\} \subseteq \{s_i\} \) such that \( \mathcal{L}^{s_{i'}} \) converges to \( \mathcal{L} \).

**Proof.** If we knew that the periods of the cycles \( \mathcal{L}^{s_i} \) were uniformly bounded above, then Proposition 8 would be a trivial consequence of Proposition 5. Unfortunately we do not know this.

Fix a finite time instant \( T > 2 \sum_{i,j=1} g^{12} \), where \( g^{12} \) designates the upper bound for the time which a trajectory of (3) can spend in \( R^{12} \). Let \( \bar{z}^{s_i}(\cdot) \) be a solution that forms the cycle \( \mathcal{L}^{s_i} \), that is,

\[
\mathcal{L}^{s_i} = \{ \bar{z} \in R^2 : \bar{z} = \bar{z}^{s_i}(t) \text{ for some } t \in [0, T_{s_i}) \},
\]

where \( T_{s_i} \) designates the period of \( \bar{z}^{s_i}(\cdot) \), that is, the smallest \( T > 0 \) such that \( \bar{z}^{s_i}(t + T') = \bar{z}^{s_i}(t) \) for every \( t \geq 0 \). By Proposition 5 there exists a subsequence \( s_{i'}, i' \geq 1 \), and a solution \( \bar{z}^0(\cdot) \) of (3) such that

\[
\lim_{i' \to \infty} \sup_{t \in [0, T]} ||\bar{z}^{s_{i'}}(t) - \bar{w}^0(t)|| = 0.
\]

To simplify notation, let us assume that the whole sequence \( s_i, i \geq 1 \), has this property; that is,

\[
\lim_{i \to \infty} \sup_{t \in [0, T]} ||\bar{z}^{s_i}(t) - \bar{w}^0(t)|| = 0. \tag{A48}
\]

By Proposition 4 \( L^{s_i} \cap (\Pi_1^{s_i} \cup \Pi_2^{s_i}) \neq \emptyset \), where \( \sigma_i \to 0 \). We may assume that \( \bar{z}^0(0) \in \Pi_1^{s_i} \cup \Pi_2^{s_i} \). Then (A48) entails that \( \bar{z}^0(0) \in \Pi_1^0 \cup \Pi_2^0 \). This set does not contain \((\beta, \alpha)\). Consequently by Proposition 7 \( \bar{z}^0(\cdot) \) does not pass through \((\beta, \alpha)\).

By construction, \( T \) is greater than twice the sum of the times which any solution of (3) can spend in \( R^{ij}, i, j = 1, 2, 3 \). Hence, during the time interval \([0, T]\), the solution \( \bar{z}^0(\cdot) \) must enter and leave \( R^{21} \) at least twice. Consequently, it crosses the left border of \( R^{22} \cup R^{23} \) (that is, \( \Gamma \)) at least twice. Let \( t' \) be the first time instant such that \( t' > 0 \) and \( \bar{z}^0(t') \in \Gamma \). Also, let \( t'' < T \) be the first time after \( t' \) such that \( \bar{z}^0(\cdot) \) re-enters \( \Gamma \).
Since \( \tilde{z}^0(\cdot) \) does not pass through \((\beta, \alpha)\), we conclude that 
\[ \min\{z^0_2(t'), z^0_2(t'')\} > \alpha. \]
Hence, by statement 1 of Proposition 1, in a small neighborhood of the set \( \gamma = \{\beta\} \times [\min(z^0_2(t'), z^0_2(t''))] \), the functions \( g^0_1(\cdot) \) converge uniformly to \( 1 - \epsilon + \epsilon \delta - \epsilon \zeta \) as \( i \to \infty \). Consequently, for all sufficiently large \( i \), these functions are uniformly bounded away from zero provided that the neighborhood is small enough. If \( z^0_1(t') \neq \beta \) or/and \( z^0_1(t'') \neq \beta \), this implies the existence of uniquely defined small shifts of time \( \tau'_i \) and \( \tau''_i \) such that 
\[ z^0_1(t' + \tau'_i) = \beta \] or/and 
\[ z^0_1(t'' + \tau''_i) = \beta \] and 
\[ \lim_{i \to \infty} \sup_{i \geq j} \max\{|\tau'_i|, |\tau''_i|\} = 0. \] (A49)

If \( z^0_2(t' + \tau'_i) = z^0_2(t'' + \tau''_i) \), then \( \bar{z}^0(t' + \tau'_i) = \bar{z}^0(t'' + \tau''_i) \) and by the uniqueness of the solutions of system (2) we conclude that \( T_{s_i} = t'' - t' + \tau''_i - \tau'_i \) for all \( i \geq 1 \). By (A49), the \( T_{s_i} \) are uniformly bounded by \( T \) for all sufficiently large \( i \). Moreover, 
\[ \bar{z}^0(t' - \bar{z}^0(t' + \tau'_i)) = [\bar{z}^0(t' - \bar{z}^0(t' + \tau'_i))] \to 0 \] as \( i \to \infty \). Indeed, the first term goes to zero by the continuity of \( \bar{z}^0(\cdot) \) and (A49). The second term vanishes by (A48) because by (A49) \( [t' + \tau'_i, t'' + \tau''_i] \subset [0, T] \) for all sufficiently large \( i \). Similarly, \( \bar{z}^0(t'') - \bar{z}^0(t' + \tau'_i) \to 0 \) as \( i \to \infty \). Since \( \bar{z}^0(t' + \tau'_i) = \bar{z}^0(t'' + \tau''_i) \), we conclude that \( \bar{z}^0(t) = \bar{z}^0(t'') \). Thus, 
\[ \mathcal{L}^0 = \{ \bar{z} \in \mathbb{R}^2: \bar{z} = \bar{z}^0(t) \text{ for some } t \in [0, T_0) \} \]
is a cycle of (3) of the period \( T_0 = t'' - t' \).

To complete the proof of Proposition 8, it suffices to show that \( z^0_2(t' + \tau'_i) \) coincides with \( z^0_2(t'' + \tau''_i) \) for all \( i \). (Actually, since we are interested in what happens on an infinite sequence, everything that depends upon a finite number of terms of this sequence does not change our conclusion.) Assume to the contrary that \( z^0_2(t' + \tau'_i) \neq z^0_2(t'' + \tau''_i) \) for \( i \geq 1 \). Figure 9 illustrates the case when \( z^0_2(t' + \tau'_i) > z^0_2(t'' + \tau''_i) \). Since \( \mathcal{L}^0 = \{ \bar{z}^0(t' + \tau'_i) \} \) is a cycle, \( \bar{z}^0(t' + \tau'_i + T_{s_i}) = \bar{z}^0(t' + \tau'_i) \) and \( T_{s_i} = t'' + \tau''_i - t' - \tau'_i \). The latter inequality holds because \( \bar{z}^0(t' + \tau'_i) \neq \bar{z}^0(t'' + \tau''_i) \) and the fact that \( t'' + \tau''_i \) is the first time after \( t'' + \tau'_i \) when \( \bar{z}^0(\cdot) \) hits \( \Gamma \), while \( t' + \tau'_i + T_{s_i} \) is a moment after \( t' + \tau'_i \) when \( \bar{z}^0(\cdot) \) reaches \( \Gamma \).

Since \([t', t''] \subset (0, T)\), (A48) and (A49) imply that \( \bar{z}^0(t' + \tau'_i) \) and \( \bar{z}^0(t' + \tau''_i) \) belong to an arbitrarily small neighborhood of \( \gamma \) for all sufficiently large \( i \). The function \( g^0_1(\cdot) \) is bounded below by a positive constant in a small neighborhood of \( \gamma \). Hence, for all sufficiently large \( i \) there are time instants \( \tau_i^+ \in (t'' + \tau''_i, T_{s_i}) \) such that \( z^0_1(t_i^+) > z^0_1(t'' + \tau''_i) = \beta \). Let \( \tau_i^- \) be first time after \( \tau_i^+ \) when \( \bar{z}^0(\cdot) \) hits 
\[ \gamma^i = \{\beta\} \times [\min\{z^0_2(t' + \tau'_i), z^0_2(t'' + \tau''_i)\}, \max\{z^0_2(t' + \tau'_i), z^0_2(t'' + \tau''_i)\}]. \]
FIG. 9. The existence of a periodic solution of the limit system.

Clearly such a time instant exists and \( \tau_i^- \leq t' + \tau_i' + T_{\gamma_i} \). Consider the closed curve \( \mathcal{S}^i \) formed by \( \mathcal{S}^i \) and \( \gamma^i \), where

\[
\mathcal{S}^i = \{ \tilde{z} \in \mathbb{R}^2 \colon \tilde{z} = \tilde{z}^{\gamma_i}(t) \text{ for some } t \in [t' + \tau_i', t'' + \tau_i''] \}.
\]

Due to the uniqueness of the solutions of (2), the trajectory \( \tilde{z}^{\gamma_i}(\cdot) \) does not intersect \( \mathcal{S}^i \) in the time interval \( (\tau_i^-, t' + \tau_i' + T_{\gamma_i}) \). Hence, \( \tilde{z}^{\gamma_i}(t) \) approaches \( \tilde{z}^{\gamma_i}(\tau_i^-) \) from the right as \( t \to \tau_i^- - 0 \), which implies that \( \frac{d}{dt} z_2^{\gamma_i}(\tau_i^-) \leq 0 \). This is not possible because \( \tilde{z}^{\gamma_i}(\tau_i^-) \in \gamma^i \) and \( \gamma^i \) belongs for all sufficiently large \( i \) to a neighborhood of \( \gamma \) on which \( g_1^{\gamma_i}(\cdot) \) is bounded away from zero by a positive constant. Thus, the assumption that \( z_2^{\gamma_i}(t' + \tau_i') \neq z_2^{\gamma_i}(t'' + \tau_i'') \) leads to a contradiction. Consequently, we must have that \( \tilde{z}^{\gamma_i}(t' + \tau_i') = \tilde{z}^{\gamma_i}(t'' + \tau_i'') \) for all sufficiently large \( i \). This completes the proof of Proposition 8.

An immediate consequence is the following.

**Proposition 9.** Let there be a sequence \( s_1, s_2, \ldots \) of odd numbers converging to infinity. For each \( i \geq 1 \), let \( \mathcal{S}^{s_i} \subset \mathcal{N} \) be a cycle of system (2) with \( s = s_i \). Then there is a subsequence \( \{s_{r_i}\} \subseteq \{s_i\} \) such that \( \lim_{r_i \to \infty} \mu(L^{s_{r_i}}) = \mu(L) \). Here \( \mu(B) \) designates the area of \( B \), \( L^{s_i} \) stands for the finite closed set
bounded by $\mathcal{L}_y$, and $L$ is the finite closed domain bounded by $\mathcal{L}$, the cycle of $(3)$ to which $\mathcal{L}_y$ converges.

**Proposition 10.** Let there be a sequence $s_1, s_2, \ldots$ of odd numbers converging to infinity. Let $\mathcal{L}_{s_i} \subset \mathcal{L}$ be a cycle of system $(2)$ with $s = s_i$ and $L^{s_i}$ the closed finite domain bounded by $\mathcal{L}_{s_i}$. Then

$$\limsup_{s \to \infty} \left[ \int_{L^{s_i}} \operatorname{div} \bar{g}^{s_i}(\bar{z}) \, d\bar{z} + 2\mu(L^{s_i}) \right] \leq 2\epsilon(1 - 2\delta).$$

**Proof.** Let $\sigma_i, i \geq 1$, be the sequence of positive numbers guaranteed by Proposition 4. We have

$$\int_{L^{s_i}} \operatorname{div} \bar{g}^{s_i}(\bar{z}) \, d\bar{z} = \sum_{j=1}^{4} I_j, \quad \text{(A50)}$$

where

$$I_1 = \int_{L^{s_i} \setminus (\Pi_1^{s_i} \cup \Pi_2^{s_i})} \operatorname{div} \bar{g}^{s_i}(\bar{z}) \, d\bar{z}, \quad I_2 = \int_{L^{s_i} \cap \Pi_i^{s_i} \setminus \Pi_2^{s_i}} \operatorname{div} \bar{g}^{s_i}(\bar{z}) \, d\bar{z},$$

$$I_3 = \int_{L^{s_i} \cap \Pi_1^{s_i} \setminus \Pi_2^{s_i}} \operatorname{div} \bar{g}^{s_i}(\bar{z}) \, d\bar{z}, \quad I_4 = \int_{L^{s_i} \cap \Pi_1^{s_i} \cap \Pi_2^{s_i}} \operatorname{div} \bar{g}^{s_i}(\bar{z}) \, d\bar{z}.$$

By statement 2 of Proposition 1 we may assume that

$$\frac{d}{dz_j} f^{s_i}_{1/2}(z_j) \in [0, \sigma_i] \quad \text{for } \bar{z} \notin \Pi_j^{s_i}, \quad j = 1, 2. \quad \text{(A51)}$$

We have

$$\mu(L^{s_i} \setminus (\Pi_1^{s_i} \cup \Pi_2^{s_i})) \leq \mu(L^{s_i})$$

and

$$\mu(L^{s_i} \setminus (\Pi_1^{s_i} \cup \Pi_2^{s_i})) \geq \mu(L^{s_i}) - 4\sigma_i.$$ 

By (4) and (A51) this implies that

$$-2[\mu(L^{s_i}) - 4\sigma_i] \leq I_1 \leq -2\mu(L^{s_i}) + 2\epsilon(1 - 2\delta)\sigma_i\mu(L^{s_i}).$$

Hence,

$$\lim_{i \to \infty} I_1 + 2\mu(L^{s_i}) = 0. \quad \text{(A52)}$$
By (4) and (A51) we also have

\[ I_2 \leq \int_{L^u \cap (\Pi^i_1 \setminus \Pi^i_2)} \left\{ -2 + \epsilon(1 - 2\delta) \left[ \frac{d}{dz_1} f^{s_i}_{1/2}(z_1) + \sigma_i \right] \right\} d\tilde{z} \]

\[ \leq \epsilon(1 - 2\delta) \int_{L^u \cap (\Pi^i_1 \setminus \Pi^i_2)} \left[ \frac{d}{dz_1} f^{s_i}_{1/2}(z_1) + \sigma_i \right] d\tilde{z} \]

\[ \leq \epsilon(1 - 2\delta) \int_{\Pi^i_1 \setminus \Pi^i_2} \left[ \frac{d}{dz_1} f^{s_i}_{1/2}(z_1) + \sigma_i \right] d\tilde{z} \]

\[ = \epsilon(1 - 2\delta) \int_{1/2 - \sigma_i}^{1/2 + \sigma_i} \left\{ \int_{[0,1/2 - \sigma_i] \cup [1/2 + \sigma_i, 1]} \left[ \frac{d}{dz_1} f^{s_i}_{1/2}(z_1) + \sigma_i \right] dz_2 \right\} dz_1 \]

\[ = \epsilon(1 - 2\delta)(1 - 2\sigma_i) \int_{1/2 - \sigma_i}^{1/2 + \sigma_i} \left[ \frac{d}{dz_1} f^{s_i}_{1/2}(z_1) + \sigma_i \right] dz_1 \]

\[ = \epsilon(1 - 2\delta)(1 - 2\sigma_i) \left[ f^{s_i}_{1/2}(1/2 + \sigma_i) - f^{s_i}_{1/2}(1/2 - \sigma_i) + 2\sigma_i^2 \right] \]

\[ < \epsilon(1 - 2\delta)(1 + 2\sigma_i^2). \quad (A53) \]

Similarly,

\[ I_2 < \epsilon(1 - 2\delta)(1 + 2\sigma_i^2). \quad (A54) \]

Finally, by (4),

\[ I_4 = \int_{L^u \cap (\Pi^i_1 \cap \Pi^i_2)} \left\{ -2 + \epsilon(1 - 2\delta) \left[ \frac{d}{dz_1} f^{s_i}_{1/2}(z_1) + \frac{d}{dz_2} f^{s_i}_{1/2}(z_2) \right] \right\} d\tilde{z} \]

\[ \leq \epsilon(1 - 2\delta) \int_{L^u \cap (\Pi^i_1 \cap \Pi^i_2)} \left[ \frac{d}{dz_1} f^{s_i}_{1/2}(z_1) + \frac{d}{dz_2} f^{s_i}_{1/2}(z_2) \right] d\tilde{z} \]

\[ \leq \epsilon(1 - 2\delta) \int_{\Pi^i_1 \cap \Pi^i_2} \left[ \frac{d}{dz_1} f^{s_i}_{1/2}(z_1) + \frac{d}{dz_2} f^{s_i}_{1/2}(z_2) \right] d\tilde{z} \]

\[ = \epsilon(1 - 2\delta) \int_{1/2 - \sigma_i}^{1/2 + \sigma_i} \int_{1/2 - \sigma_i}^{1/2 + \sigma_i} \left[ \frac{d}{dz_1} f^{s_i}_{1/2}(z_1) + \frac{d}{dz_2} f^{s_i}_{1/2}(z_2) \right] dz_1 dz_2 \]

\[ = 4\epsilon(1 - 2\delta)\sigma_i \left[ f^{s_i}_{1/2}(1/2 + \sigma_i) - f^{s_i}_{1/2}(1/2 - \sigma_i) \right] \]

\[ < 4\epsilon(1 - 2\delta)\sigma_i. \quad (A55) \]

From (A50)–(A55) we obtain the statement of the proposition.

To exclude cycles of system (2) for all sufficiently large $s$ it suffices to show that

\[ \mu(L^s) > \epsilon(1 - 2\delta) \quad (A56) \]

for all large enough odd numbers $s$. (This follows directly from (A46).) By Proposition 9 the limit points of $\mu(L^s)$, $i \geq 1$, are the areas of the domains.
bounded by cycles of the limit system (3). This suggests that to establish (A56) one has to prove that \( \mu(L) > \epsilon(1 - 2\delta) \) for every cycle of (3) which is a limit of some subsequence \( s_{i'}, i' \geq 1 \) of odd numbers converging to infinity. This would exclude cycles for system (2) with all sufficiently large \( s \). From now on let us consider only those cycles of (3) which are limits of cycles of (2) with \( s = s_i \) for some converging to infinity sequence \( s_i, i \geq 1 \), of odd numbers. We shall refer to them simply as “cycles.”

By Propositions 4 and 8 we conclude that every cycle of (3) must meet \( \Pi_1^0 \cup \Pi_2^0 \). A simple geometrical argument based on Fig. 7 shows that each cycle of (3) meeting \( \Pi_1^0 \) must enclose the set \( S_1 \). Shown in Fig. 10. Similarly, cycles of (3) meet \( \Pi_2^0 \) if and only if

\[
\tan \psi = \frac{1 - \beta - \epsilon(1 - \delta)}{1/2 - \epsilon(1 - \delta)} \leq \tan \psi_0 = \frac{\beta}{1/2 - \alpha}.
\] (A57)

Also, the cycles of (3) which intersect with \( \Pi_2^0 \) must enclose the set \( S_2 \) given in Fig. 10. Since \( \epsilon\delta < \beta \) by hypothesis,

\[
\tan \phi = \frac{\alpha - \epsilon\delta}{1/2 - \epsilon\delta} < \tan \phi_0 = \frac{\alpha}{1/2 - \beta}.
\]

This implies that \( S_1 \) is a triangle. Its area is given by the expression

\[
\mu(S_1) = (1/2 - \beta)^2 \frac{\alpha - \epsilon\delta}{1 - 2\epsilon\delta}.
\]

FIG. 10. The triangulars that are encircled by cycles of the limit system.
Under restriction (A57) the set $S_2$ is also a triangle. Its area reads as follows:

$$\mu(S_2) = (1/2 - \alpha)^2 \frac{1 - \beta - \epsilon(1 - \delta)}{1 - 2\epsilon(1 - \delta)}. $$

Due to the restrictions on the parameter values, we know that $\alpha - \epsilon(1 - \delta) > 0$, $1 - 2\epsilon\delta > 1 - 2\alpha > 0$, $1 - \epsilon(1 - \delta) > 1 - 2\epsilon > 0$ and $1 - \beta - \epsilon(1 - \delta) > 1/2 - \epsilon > 0$. It follows that $\mu(S_1)$ and $\mu(S_2)$ are positive.

As a consequence of Propositions 4 and 9 we obtain the following statement.

**Proposition 11.** Let there be a sequence of odd numbers $s_i$, $i \geq 1$, converging to infinity. Also, let $L^{s_i} \subset \mathcal{H}$ be a cycle of system (2) with $s = s_i$ such that $L^{s_i} \cap \Pi_j^{s_i} \neq \emptyset$ for $j = 1$ or $j = 2$ and some sequence $\sigma_i$, $i \geq 1$, of positive numbers converging to zero. Then

$$\liminf_{i \to \infty} \mu(L^{s_i}) \geq \mu(S_j).$$

In view of inequality (A56), to exclude cycles of system (2) for all sufficiently large $s$ it suffices that

$$\epsilon(1 - 2\delta) < \min\left[\mu(S_1), \mu(S_2)\right].$$

Thus we have established the following.

**Lemma 6.** Let $G$ be a nondegenerate $2 \times 2$ game with the unique equilibrium $(\beta, \alpha)$, which is fully mixed. Let $\epsilon \in (0, 1/2)$, $\delta \in [0, 1/2)$, $\max(\beta, \alpha) < 1/2$, and $\epsilon\delta < \min(\alpha, \beta)$. If

$$\epsilon(1 - 2\delta) < \min\left[(1/2 - \beta)^2 \frac{\alpha - \epsilon\delta}{1 - 2\epsilon\delta}, (1/2 - \alpha)^2 \frac{1 - \beta - \epsilon(1 - \delta)}{1 - 2\epsilon\delta}\right],$$

then for all sufficiently large $s$ solutions of (2) belonging to $\mathcal{H}$ do not form cycles.

It is easy to see that the bound on $\epsilon$ given by Lemma 6 vanishes as $\min(\alpha, \beta, 1/2 - \alpha, 1/2 - \beta) \to 0$. By Fig. 7 we can conclude that the area of any set encircled by a trajectory of (3) passing through $\Pi_1^0 \cup \Pi_2^0$ vanishes as $\min(\alpha, \beta, 1 - \alpha, 1 - \beta, |1/2 - \alpha|, |1/2 - \beta|) \to 0$.

Prolonging beyond $R_2^{21}$ the trajectory of (3) that forms the border of $S_1$, we can identify more sets which lie inside any cycle of (3). Taking into account their areas, we can improve the bound given by Lemma 6.

**Theorem 2.** Let $G$ be a nondegenerate $2 \times 2$ game with the unique equilibrium $(\beta, \alpha)$, which is fully mixed. Let $s > 1$ be an odd number, $\epsilon \in (0, 1/2)$, $\delta \in [0, 1]$, and $\epsilon\delta < \min(\alpha, \beta, 1/2) \leq \max(\alpha, \beta, 1/2) < 1 - \epsilon\delta$. When $\delta \geq 1/2$ and $s$ is sufficiently large, the adaptive process (1) converges with
probability one to a deterministic limit as $t \to \infty$. Moreover, the limit $(\tilde{x}_\beta^s, \tilde{y}_\alpha^s)$ is arbitrarily close to $(\beta, \alpha)$ when $s$ is sufficiently large. When $\delta < 1/2$, for all sufficiently large $s$ we have the following possibilities:

(a) if $\beta \neq 1/2$ and $\alpha \neq 1/2$, $(X^t, Y^t)$ converges with positive probability to $(\tilde{x}_\beta^s, \tilde{y}_\alpha^s)$ from every initial state $(X^1, Y^1)$, provided that the initial state is rich; moreover, if $\epsilon$ is sufficiently small, $(X^t, Y^t)$ converges to $(\tilde{x}_\beta^s, \tilde{y}_\alpha^s)$ with probability one;

(b) if $\beta = 1/2$ or $\alpha = 1/2$ or $\beta = \alpha = 1/2$, then $(X^t, Y^t)$ converges to $(\tilde{x}_\beta^s, \tilde{y}_\alpha^s)$ with probability zero for every initial state.

Proof: Given Lemmas 1 and 3, the statement for the case $\delta \geq 1/2$ holds by an argument similar to the one given in the proof of Theorem 2 by Kaniowski and Young (1995). This argument also allows us to derive the part concerning convergence with probability one in statement (a) from Lemmas 3 and 6 in the case when $\delta < 1/2$ and $\epsilon$ is sufficiently small. Statement (b) follows from results on the unattainability of linearly unstable points in stochastic approximation (see Arthur et al., 1988, Lemma 2 or Pemantle, 1990, Theorem 1). In Kaniowski and Young (1995, p. 352) one can see how this was done for a learning process similar to the one here but with $\epsilon = 0$. The present situation is sufficiently similar that we need not go into the details here. Thus it remains only to prove the part of statement (a) concerning local convergence.

We shall use an idea originally suggested by Hill et al. (1980, Theorem 4.1) for one-dimensional processes and later extended to higher dimensions by Arthur et al. (1988, Theorem 2). Assume, contrary to statement (a), that $P_{(X^1, Y^1)}\{(X^t, Y^t) \to (\tilde{x}_\beta^s, \tilde{y}_\alpha^s)\} = 0$ for some initial state $(X^1, Y^1) \in \mathcal{Z}$. Then for every positive $\sigma$ and every time instant $t_0 > 0$

$$P_1 = P_{(X^1, Y^1)}\{(X^t, Y^t) \to (\tilde{x}_\beta^s, \tilde{y}_\alpha^s), (X^n, Y^n) \in U_\sigma, n \geq t_0\} = 0,$$

where $U_\sigma = \{\tilde{z} \in \mathcal{Z}: ||\tilde{z} - (\tilde{x}_\beta^s, \tilde{y}_\alpha^s)|| \leq \sigma\}$. The random vector $(X^t, Y^t)$ assumes a finite number of values, namely,

$$V_{(X^1, Y^1)} = \left\{\tilde{z} \in \mathcal{Z}: z_1 = \frac{a_1 + i}{a_1 + t - 1}, z_2 = \frac{b_1 + j}{b_1 + t - 1}, 0 \leq i \leq t - 1, 0 \leq j \leq t - 1\right\}.$$

Hence,

$$P_1 = \sum_{\tilde{z} \in V_{(X^1, Y^1)} \cap U_\sigma} P\{(X^t, Y^t) \to (\tilde{x}_\beta^s, \tilde{y}_\alpha^s), (X^n, Y^n) \in U_\sigma, n \geq t_0\} = \tilde{z} \mid \tilde{z} \in V_{(X^1, Y^1)} \cap U_\sigma\} P_{(X^1, Y^1)}\{(X^t, Y^t) = \tilde{z}\}.$$  (A58)
Since we assumed that the initial state is rich, each strategy is played by both sides with positive probability at any time instant. Hence, for every \( \bar{z} \in V_{(X^1, Y^1)}, t \geq 1, \)

\[
P_{(X^1, Y^1)}\{(X^t, Y^t) = \bar{z}\} > 0.
\]

Each of these probabilities is a finite product of positive numbers. Since \( P_1 = 0, \) by (A58) we conclude that for every \( \bar{z} \in V_{(X^1, Y^1)} \cap U_\sigma \)

\[
P\{(X^t, Y^t) \to (\bar{x}^s_\beta, \bar{y}^s_\alpha), (X^n, Y^n) \in U_\sigma, n > t_0 | (X^n, Y^n) = \bar{z}\} = 0.
\]

(A59)

We claim that this is impossible. Namely, there exist \( \sigma > 0, \) a finite time \( t_0, \) and a point \( \bar{z}^0 \in V_{(X^1, Y^1)} \cap U_\sigma \) such that the value of the left-hand side of (A59) is positive. Thus we must have

\[
P_{(X^1, Y^1)}\{(X^t, Y^t) \to (\bar{x}^s_\beta, \bar{y}^s_\alpha)\} > 0
\]

for every initial state.

For \( \delta < 1/2, \) by Lemma 3, the Jacobian \( J^s(\bar{x}^s_\beta, \bar{y}^s_\alpha) \) is a stable matrix for all sufficiently large \( s \) provided that \( \beta \neq 1/2 \) and \( \alpha \neq 1/2. \) From now on we shall be considering only such \( s. \) To simplify notation, let us designate \( J^s(\bar{x}^s_\beta, \bar{y}^s_\alpha) \) by \( J^s. \) By Lyapunov's theorem (see Marcus and Minc, 1964, p. 160), for every \( \lambda \in (0, -\min(\text{Re}\lambda_1^s, \text{Re}\lambda_2^s)) \) there exists a symmetric, positive definite \( 2 \times 2 \) matrix \( C_\lambda \) such that \( C_\lambda(J^s + \lambda I) + [(J^s)^T + \lambda I]C_\lambda \) is positive definite. Hence, \( \langle C_\lambda(J^s \bar{z}, \bar{z}) \rangle \leq -\lambda \langle C_\lambda \bar{z}, \bar{z} \rangle \) for every \( \bar{z} \in R^2. \) Here \( \lambda_1^s \) and \( \lambda_2^s \) designate the eigenvalues of \( J^s. \) Recall that \( I \) stands for the identity matrix in \( R^2. \) Also, \( \langle \cdot, \cdot \rangle \) designates the Euclidean scalar product in \( R^2. \) Finally, the sign \( ^T \) stands for transposition. By Taylor's expansion we conclude that there exists a small \( \sigma > 0 \) such that

\[
\langle C_\lambda \bar{z}^s(\bar{z}), \bar{z} - (\bar{x}^s_\beta, \bar{y}^s_\alpha) \rangle \leq \frac{\lambda}{2} \langle C_\lambda [\bar{z} - (\bar{x}^s_\beta, \bar{y}^s_\alpha)], \bar{z} - (\bar{x}^s_\beta, \bar{y}^s_\alpha) \rangle
\]

provided that \( \bar{z} \in U_\sigma. \)

For each \( t \geq 1 \) and \( \bar{z} \in \mathcal{H} \) let

\[
\tilde{\xi}^t(\bar{z}) = \begin{cases} 
\xi^t(\bar{z}) & \text{if } \bar{z} \in U_\sigma; \\
\xi^t((\bar{x}^s_\beta, \bar{y}^s_\alpha)) & \text{otherwise};
\end{cases}
\]

\[
\tilde{\psi}^t(\bar{z}) = \begin{cases} 
\psi^t(\bar{z}) & \text{if } \bar{z} \in U_\sigma; \\
\psi^t((\bar{x}^s_\beta, \bar{y}^s_\alpha)) & \text{otherwise}.
\end{cases}
\]

Then \( \tilde{\xi}^t(\cdot), t \geq 1, \) and \( \tilde{\psi}^t(\cdot), t \geq 1, \) are mutually independent sequences of independent random variables. Define

\[
\tilde{X}^{t+1} = \tilde{X}^t + (1/a^{t+1})[\tilde{\xi}^t(\tilde{X}^t, \tilde{Y}^t)) - \tilde{X}^t], \quad t \geq 1, \quad \tilde{X}^1 = X^1
\]

\[
\tilde{Y}^{t+1} = \tilde{Y}^t + (1/b^{t+1})[\tilde{\psi}^t(\tilde{X}^t, \tilde{Y}^t) - \tilde{Y}^t], \quad t \geq 1, \quad \tilde{Y}^1 = Y^1.
\]
Taking into account \( \tilde{g}^s((\tilde{x}^s, \tilde{y}^s)) = \tilde{0} \), we obtain
\[
E\tilde{\xi}^t(\tilde{z}) = \begin{cases} 
(1 - \epsilon)f^{s}_\alpha(z_2) + \epsilon[(1 - 2\delta)f^{s}_{1/2}(z_1) + \delta] & \text{if } \tilde{z} \in U_\sigma \\
\tilde{x}^s_\beta, & \text{otherwise};
\end{cases}
\]
\[
E\tilde{\psi}^t(\tilde{z}) = \begin{cases} 
(1 - \epsilon)f^{s}_{1-\beta}(1 - z_1) + \epsilon[(1 - 2\delta)f^{s}_{1/2}(z_2) + \delta] & \text{if } \tilde{z} \in U_\sigma \\
\tilde{y}^s_\alpha & \text{otherwise}.
\end{cases}
\]

Now \((\tilde{x}^s_\beta, \tilde{y}^s_\alpha)\) is a single solution on \(\mathcal{H}\) of the following nonlinear equations
\[
\tilde{g}^s_1(\tilde{z}) = 0 \quad \tilde{g}^s_2(\tilde{z}) = 0,
\]
where
\[
\tilde{g}^s_i(\tilde{z}) = \begin{cases} 
g^s_i(\tilde{z}) & \text{if } \tilde{z} \in U_\sigma, \\
\tilde{x}^s_\beta - z_1 & \text{otherwise},
\end{cases} \quad \tilde{g}^s_2(\tilde{z}) = \begin{cases} 
g^s_2(\tilde{z}) & \text{if } \tilde{z} \in U_\sigma, \\
\tilde{y}^s_\alpha - z_2 & \text{otherwise}.
\end{cases}
\]

For \(\tilde{z} \in \mathcal{H}\)
\[
(C_\lambda(\tilde{g}^s_1(\tilde{z}), \tilde{g}^s_2(\tilde{z})), [\tilde{z} - (\tilde{x}^s_\beta, \tilde{y}^s_\alpha)])
\]
\[
= \begin{cases} 
(C_\lambda(\tilde{g}^s(\tilde{z})), [\tilde{z} - (\tilde{x}^s_\beta, \tilde{y}^s_\alpha)]) & \text{if } \tilde{z} \in U_\sigma, \\
(C_\lambda(\tilde{x}^s_\beta, \tilde{y}^s_\alpha), [\tilde{z} - (\tilde{x}^s_\beta, \tilde{y}^s_\alpha)]) & \text{if } \tilde{z} \not\in U_\sigma.
\end{cases}
\]

This inequality shows that \(V(\tilde{z}) = (C_\lambda[\tilde{z} - (\tilde{x}^s_\beta, \tilde{y}^s_\alpha)], \tilde{z} - (\tilde{x}^s_\beta, \tilde{y}^s_\alpha))\) is a Lyapunov function for the process \((\tilde{X}^t, \tilde{Y}^t), t \geq 1\). Since \((\tilde{X}^t, \tilde{Y}^t), t \geq 1\), is a stochastic approximation process, Theorem 7.3 of Nevelson and Hasminskii (1976) allows us to conclude that with probability one
\[
(\tilde{X}^t, \tilde{Y}^t) \to (\tilde{x}^s_\beta, \tilde{y}^s_\alpha) \quad \text{as } t \to \infty.
\]

Hence, for a fixed \(\nu \in (0, 1)\) there is \(t_0\) depending on \(\sigma\) and \(\nu\) such that
\[
\tilde{P}_0 = P((\tilde{X}^t, \tilde{Y}^t) \to (\tilde{x}^s_\beta, \tilde{y}^s_\alpha), (\tilde{X}^n, \tilde{Y}^n) \in U_\sigma, n \geq t_0) > 1 - \nu > 0.
\]

Since the random vector \((\tilde{X}^t, \tilde{Y}^t), t \geq 1\), takes only values from \(V^{t_0}_{(X^t, Y^t)}\), we get that
\[
\tilde{P}_1 = \sum_{\tilde{z} \in V^{t_0}_{(X^t, Y^t)} \cap U_\sigma} P((\tilde{X}^t, \tilde{Y}^t) \to (\tilde{x}^s_\beta, \tilde{y}^s_\alpha), (\tilde{X}^n, \tilde{Y}^n) \in U_\sigma, n \geq t_0 | (\tilde{X}^{t_0}, \tilde{Y}^{t_0}) = \tilde{z}) \cdot P((\tilde{X}^{t_0}, \tilde{Y}^{t_0}) = \tilde{z}).
\]

This sum is positive; hence, there exists \(\tilde{z}^0 \in V^{t_0}_{(X^t, Y^t)} \cap U_\sigma\) such that
\[
P((\tilde{X}^t, \tilde{Y}^t) \to (\tilde{x}^s_\beta, \tilde{y}^s_\alpha), (\tilde{X}^n, \tilde{Y}^n) \in U_\sigma, n \geq t_0 | (\tilde{X}^{t_0}, \tilde{Y}^{t_0}) = \tilde{z}^0) > 0.
\]
Since inside $U_\sigma$, the random processes $(X^t, Y^t)$, $t \geq 1$, and $(\tilde{X}^t, \tilde{Y}^t)$, $t \geq 1$, are determined by the same expression, we conclude that

$$P\{(\tilde{X}^t, \tilde{Y}^t) \to (\tilde{x}_\beta^t, \tilde{y}_\alpha^t), \ (\tilde{X}^n, \tilde{Y}^n) \in U_\sigma, \ n \geq t_0| (\tilde{X}^{t_0}, \tilde{Y}^{t_0}) = \tilde{z}^0\}$$

$$= P\{(X^t, Y^t) \to (\tilde{x}_\beta^t, \tilde{y}_\alpha^t), \ (X^n, Y^n) \in U_\sigma, \ n \geq t_0| (X^{t_0}, Y^{t_0}) = z^0\}.$$

Consequently, we have proved that there exist $\sigma > 0$, a finite time $t_0$, and a point $z^0 \in V^{t_0}_{(x^1, y^1)} \cap U_\sigma$ such that

$$P\{(X^t, Y^t) \to (\tilde{x}_\beta^t, \tilde{y}_\alpha^t), \ (X^n, Y^n) \in U_\sigma, \ n \geq t_0| (X^{t_0}, Y^{t_0}) = z^0\} > 0.$$

This concludes the proof of Theorem 2.

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