

Interpreting von Neumann Model Prices as Marginal Values*

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The "prices" of goods in a generalized von Neumann model are shown to have the economic interpretation they naturally should have: the price of a good is, save mathematical niceties, the marginal rate at which the growth rate decreases as the consumption of that good increases.

1. INTRODUCTION

In his original paper on a model of a uniformly expanding economy, von Neumann [10] established the existence of an optimal growth process and a set of "equilibrium prices" by means of a fixed point theorem of topology. Later, Gale [2] generalized the model and illuminated it considerably by showing that the equilibrium price vector is nothing more than a support for a certain convex set. Rockafellar [9] generalized the setting of the Gale model still further by his notion of a "monotone process," and illustrated very elegantly the dual relationship between optimal processes and equilibrium prices. This relationship leads us to expect that the equilibrium prices must represent some sort of marginal values; but marginal values "with respect to what" has to the authors' knowledge never been precisely established.

A feature of von Neumann's original model that occasioned some criticism was the assumption that society's consumption of final goods is limited to the necessities of life. A considerable amount of effort has been spent since then in showing how to incorporate exports, imports, and final demand into the model, and this has been accomplished in a variety of different ways. (See for example [1, 5, and 8].) It is the purpose of this paper to show that if external consumption is incorporated in the "right" way, then the equilibrium prices turn out to be just the marginal rates of substitution between consumption and growth.

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We would like to state that this work originated from a seminar attended by graduate students in mathematics and economics in which various papers concerning the von Neumann model were read. In particular, Koopman's [6] lovely introductory paper made us intuitively aware of the existence of some marginal interpretation for the equilibrium prices, and so motivated the search.

2. EXTENSION OF THE GALE MODEL

We shall generalize the Gale model by including the possibility of external consumption and production. We shall then prove von Neumann's theorem for this generalized model, using essentially the same method as Gale. [2, 3].

Let $X = \{x \in \mathbb{R}^n: x \geq 0 \text{ and } \sum x_i = 1\}$. The *production space* Z is a set of vector pairs (x, y) , where $x \in X$, $y \in \mathbb{R}^n$, called *processes*, such that;

$$Z \text{ is closed, convex, and bounded above.} \quad (1)$$

$$\text{for every } i, 1 \leq i \leq n, \text{ there exists } (x, y) \in Z \text{ with } y_i > 0. \quad (2)$$

$$(x, y) \in Z \text{ implies } (x, y') \in Z \text{ whenever } y' \leq y. \quad (3)$$

Condition (1) is a customary regularity condition, where in particular "bounded above" means that there is no input x which can yield arbitrarily large outputs of any good y_i .

Condition (2) says that every good is produced by some process.

Condition (3) says that there is infinite potential for throwing away goods. (In particular, goods provided by production external to the model may be refused; this possibility is represented by process (x, y) with $y \leq 0$).

Given $b \in \mathbb{R}^n$, the largest number α such that $\alpha \geq 0$ and $y - \alpha x \geq b$ has a solution for some $(x, y) \in Z$ will, if it exists, be called the *growth rate* for b , and be denoted by $\varphi(b)$. b will be called a *requirements vector*.

Let $A = \{(x, y, b); (x, y) \in Z \text{ and } b \in \mathbb{R}^n, y \geq b\}$. Define the function $\alpha(x, y, b)$ on the domain A by

$$\alpha(x, y, b) = \min_{\substack{1 \leq i \leq n \\ x_i \neq 0}} [(y_i - b_i)/x_i].$$

$\alpha(x, y, b)$ is well-defined because $x \neq 0$. In general it is not continuous on A . For example, consider the sequence $(x^t, y^t, 0) \in A$ defined for all $t \geq 0$ by $x^t = (1 - t, t)$; $y^t = (2, t)$. Then $\lim_{t \rightarrow 0} (x^t, y^t, 0) = (x^0, y^0, 0) \in A$ but $\alpha(x^t, y^t, 0) = 1$ for all $t > 0$ while $\alpha(x^0, y^0, 0) = 2$. But we can assert

that $\alpha(x, y, b)$ is upper semicontinuous on A . Indeed, for any sequence $(x^k, y^k, b^k) \in A$ such that $\lim_{k \rightarrow \infty} (x^k, y^k, b^k) = (\bar{x}, \bar{y}, \bar{b}) \in A$, let $I = \{i: \bar{x}_i > 0\}$. Then there exists an integer K such that whenever $k \geq K$, $x_i^k > 0$, so for all $k \geq K$

$$\alpha(x^k, y^k, b^k) = \min_{\substack{1 \leq i \leq n \\ x_i^k > 0}} [(y_i^k - b_i^k)/x_i^k] \leq \min_{i \in I} [(y_i^k - b_i^k)/x_i^k].$$

Hence,

$$\begin{aligned} \limsup_{k \rightarrow \infty} \alpha(x^k, y^k, b^k) &\leq \limsup_{k \rightarrow \infty} \min_{i \in I} \frac{y_i^k - b_i^k}{x_i^k} \\ &= \min_{i \in I} \frac{\bar{y}_i - \bar{b}_i}{\bar{x}_i} = \alpha(\bar{x}, \bar{y}, \bar{b}), \end{aligned}$$

proving the assertion.

Now, given $b \in \mathbb{R}^n$, if the set $A_b = \{(x, y, b): (x, y) \in Z, y \geq b\}$ is nonempty, then $\max \alpha(x, y, b)$ exists (by (1), A_b is compact) and equals $\varphi(b)$:

$$\varphi(b) = \max_{\substack{(x, y) \in Z \\ y \geq b}} \alpha(x, y, b) \tag{4}$$

proving that $\varphi(b)$ is well-defined. In particular, $\varphi(b)$ is also upper semicontinuous, but not, in general, continuous, as the following example shows. Let Z be the convex set generated by the pairs $(.5, .5; 2, .5)$, $(1, 0; 2, 0)$ and $(0, 1; 2, 0)$, together with all pairs obtained from these by assumption (3) (see Fig. 1). Let $b_n = (0, 1/n)$ for each integer $n \geq 3$. Then $\varphi(b_n) = 1 - 2/n$ for all n , but $\varphi(0, 0) = \varphi(\lim_{n \rightarrow \infty} b_n) = 2$.

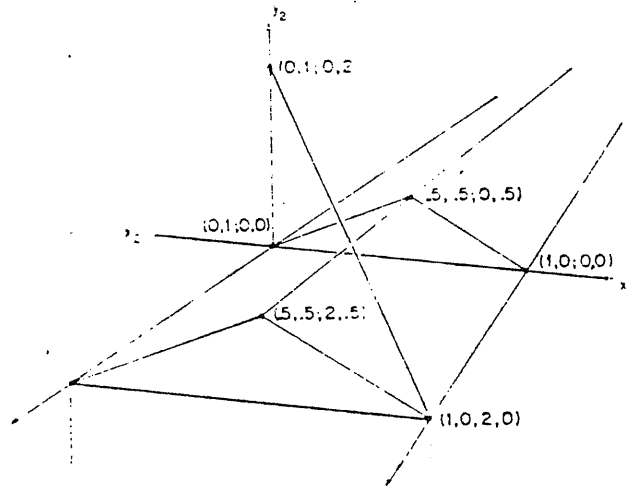


FIG. 1. A production space with discontinuous growth rate.

We may interpret b as a vector of goods either consumed or supplied externally at the end of the first production period. Thus, given input quantities x and requirements b in period 1, $y - b$ represents the quantities of goods available as inputs in period 2. $\varphi(b)$ will therefore be the maximum growth rate of production in the first period, subject to external demands and supplies b . But if $\varphi(b) = 0$, then the economy will run down after the first period. Thus we shall ordinarily consider only requirements vectors b such that $\varphi(b) > 0$; such vectors will be called *feasible*. For any feasible requirements vector b , a process $(x, y) \in Z$ such that $y - \varphi(b)x \geq b$ will be called an *optimal process* for b . If $y - \varphi(b)x = b$, then (x, y) is said to be *balanced*. By virtue of (3), to every feasible b there always corresponds a balanced optimal process.

Let $b \geq 0$, so that b represents the case of "pure consumption." If (x, y) is a balanced optimal process for b then the economy will be able to supply b amounts of goods for consumption in the first period and amounts $\varphi(b)x$ of capital inputs for use in the next period. In the second period, with $\varphi(b)x$ as input, the economy will produce $\varphi(b)b$ quantities of goods for consumption. In general, in the $(t + 1)$ st period there will be $[\varphi(b)]^t b$ available for consumption. Thus we see that $\varphi(b)$ is not just the maximum rate at which *production* can grow proportionally, given requirements b , but it can equally well be understood as the maximum rate at which *consumption* can grow in the proportions specified by b . Viewed in this light, the von Neumann model is prescription for increasing consumption as rapidly as possible, not for increasing growth at subsistence wages.

Let us also observe that, if b is "mixed," i.e., if $b_i > 0$ and $b_j < 0$ for some i and j , then those goods j for which $b_j < 0$ will of course have to be provided from external sources at a rate of increase $\varphi(b)$ in order for the economy to continue growing proportionally.

Let $B = \{b \in \mathbb{R}^n : \varphi(b) > 0\}$ denote the set of all feasible requirements vectors. For every $\alpha > 0$ define the set $B_\alpha = \{b : \varphi(b) \geq \alpha\}$. By (3), we can represent the set B_α alternatively as $\{y - \alpha x : (x, y) \in Z\}$. Since the latter set is convex, it follows that $\varphi(b)$ is a quasiconcave function on the domain B . For any $\alpha > 0$, a feasible requirements vector which maximizes the consumption of every good, and is still consistent with a growth rate of at least α will be called a *Pareto requirements vectors* for α , and the set, B_α^* , of all such vectors will be called the *Pareto boundary* of B_α . Explicitly,

$$B_\alpha^* = \{b \in B_\alpha : b' \in B_\alpha \text{ and } b' \geq b \text{ implies } b' = b\}.$$

$\varphi(b) = \alpha$ for every $b \in B_\alpha^*$, but the converse is not necessarily true.

Let $P = \{p \in \mathbb{R}^n : p \geq 0 \text{ and } p \cdot x = 1\}$. Any such p is called a *price vector*. For any $(x, y, b) \in A$, and any $p \in P$, either $p \cdot (y - b)/p \cdot x$ is undefined, or

$$\frac{p \cdot (y - b)}{p \cdot x} \geq \min_{\substack{1 \leq i \leq n \\ x_i \neq 0}} \frac{e_i \cdot (y - b)}{e_i \cdot x} = \alpha(x, y, b),$$

where e_i is the unit vector in \mathbb{R}^n with "1" in the i th position. Since $e_i \in P$, it follows that

$$\min_{p \in P} \frac{p \cdot (y - b)}{p \cdot x} = \alpha(x, y, b).$$

Further, since $\min_{p \in P} p \cdot (y - b)/p \cdot x < 0$ (perhaps $= -\infty$) when $y \geq b$, we may therefore represent $\varphi(b)$, for any $b \in B$, in the compact form

$$\varphi(b) = \max_{(x, y) \in Z} \min_{p \in P} \frac{p \cdot (y - b)}{p \cdot x}, \quad b \in B, \tag{5}$$

where we adopt the convention that

$$\begin{aligned} & +\infty && \text{if } a > 0 \\ a/0 & = -\infty && \text{if } a < 0 \\ & \text{undefined} && \text{if } a = 0. \end{aligned} \tag{6}$$

Given $b \in \mathbb{R}^n$, the *dual problem* to the max min program of (5) is the following min max program, or, more precisely, inf sup program:

$$\psi(b) = \inf_{p \in P} \sup_{(x, y) \in Z} \frac{p \cdot (y - b)}{p \cdot x}. \tag{7}$$

Let $\beta(p, b) = \sup_{(x, y) \in Z} p \cdot (y - b)/p \cdot x$. We claim that $\beta(p, b)$ is defined for every $p \in P$ and $b \in \mathbb{R}^n$, though it may be $\pm \infty$. It suffices merely to observe that for given p , not both $p \cdot x = 0$ and $p \cdot (y - b) = 0$ for all $(x, y) \in Z$, by virtue of (3). (The supremum may not be attained, however, even if it is finite.)

But the infimum is attained for every b , and is finite or $-\infty$. First, $\psi(b) < \infty$ because Z is bounded above. Next, let $\lim_{n \rightarrow \infty} \beta(p^n, b) = \psi(b)$. Without loss of generality we may assume that $p^n \rightarrow \bar{p}$ for some limit point $\bar{p} \in P$. If $\beta(\bar{p}, b) > \psi(b)$, we can find $(\bar{x}, \bar{y}) \in Z$ such that $\bar{p} \cdot (\bar{y} - b)/\bar{p} \cdot \bar{x}$ is defined and $\beta(\bar{p}, b) \geq \bar{p} \cdot (\bar{y} - b)/\bar{p} \cdot \bar{x} > \psi(b)$. But then for all sufficiently large n , $p^n \cdot (\bar{y} - b)/p^n \cdot \bar{x}$ is defined, so $\psi(b) = \lim_{n \rightarrow \infty} p^n \cdot (\bar{y} - b)/p^n \cdot \bar{x} = \bar{p} \cdot (\bar{y} - b)/\bar{p} \cdot \bar{x} > \psi(b)$, a contradiction.

If $\psi(b) > 0$, it may be interpreted as the smallest *interest factor* β for which there exists a system of prices \bar{p} such that no process is profitable at prices \bar{p} after requirements have been met, i.e., such that

$$\frac{\bar{p} \cdot (y - b)}{\bar{p} \cdot x} \text{ is undefined or } \leq \beta \quad \text{for all } (x, y) \in Z, \quad (8)$$

or equivalently,

$$\bar{p} \cdot (y - b - \beta x) \leq 0 \quad \text{for all } (x, y) \in Z. \quad (9)$$

Any $\bar{p} \in P$ satisfying (9) with $\beta = \psi(b)$ will be called an *equilibrium price vector* for b . An equilibrium price vector is therefore just a non-negative support vector at the point b for the convex set $B_{\psi(b)}$.

Notice that the goods b produced for consumption by a process are not included in calculating the profitability of that process; only capital inputs to production are valued in this model. Thus the equilibrium prices do not, necessarily, represent money payments from consumers to producers, or the marginal value of goods to consumers. Indeed, as there is no money involved in the model, and no trading of goods, the terms "equilibrium price" and "interest factor" must be given a rather special interpretation. Now the only criterion of value in this model is the growth rate, i.e., the common growth rate of both consumption and production. As consumption requirements (the vector b) go up, the growth rate will typically go down (or stay the same). We are therefore led to expect that the equilibrium price of a good j must, roughly speaking, represent the marginal rate at which the growth rate decreases as the consumption of j increases. In particular, where p is an equilibrium price vector for b , $p \cdot b$ would represent the "payment" (in decreased growth) that society would have to make for a marginal increase of consumption in the proportions b .

We shall show precisely in what sense the above statements are true in Theorem 2 below. First, however, we give a revised form of Gale's generalization of von Neumann's theorem [2, Theor. 1]. For this purpose, we need to consider the problem of "singularities" in the model.

For any $b \in B$, let $P(b)$ denote the set of all equilibrium price vectors for b , and let $X(b)$ denote the set of optimal process inputs for b , $X(b) = \{x: (x, \varphi(b)x) \in Z\}$. A *singularity at b* is a pair (p, x) such that $p \in P(b)$, $x \in X(b)$, and $p \cdot x = 0$. From an economic point of view, a singularity is "unnatural" in that it implies the capital inputs to a fastest growing process have zero value in some system of equilibrium prices. From a mathematical point of view, singularities are undesirable because they cause certain expressions to be undefined, and more importantly

because von Neumann's theorem (and our theorem on marginal values) may fail in the presence of singularities.

The following theorem corresponds to Gale's Theorem 1.

THEOREM 1. *For every feasible b , $P(b) \neq \emptyset$ and $\psi(b) \leq \varphi(b)$. If Z has no singularities at b , then $\psi(b) = \varphi(b)$.*

Proof. Let $\alpha = \varphi(b) > 0$, and let

$$C = B_\alpha - b = \{y - b - \alpha x : (x, y) \in Z\}.$$

By definition of $\varphi(b)$, $C \cap \mathbb{R}_+^n = \emptyset$, where

$$\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i > 0, 1 \leq i \leq n\}.$$

There exists a balanced optimal process for b , so $0 \in C$, and by the preceding, 0 is on the boundary of C . Hence there is a support vector $p \in \mathbb{R}^n$, $p \neq 0$, such that $p \cdot c \leq 0$ for all $c \in C$. Since $0 \in C$, (3) implies $-e_i \in C$ for every unit vector $e_i \in \mathbb{R}^n$, hence $p_i \geq 0$ for all i . Thus $\bar{p} = p/\|p\| \in P(b)$. Further, $\bar{p} \cdot (y - b)/\bar{p} \cdot x$ is either undefined or less than or equal to α for all $(x, y) \in Z$; hence,

$$\beta(\bar{p}, b) = \sup_{(x, y) \in Z} \frac{\bar{p} \cdot (y - b)}{\bar{p} \cdot x},$$

which is defined, satisfies

$$\psi(b) \leq \beta(\bar{p}, b) \leq \alpha = \varphi(b). \quad (10)$$

proving the first statement of the theorem.

Suppose now that Z has no singularities and let $\beta(\hat{p}, b) = \psi(b)$. For any optimal process $(\hat{x}, \hat{y}) \in Z$, $\hat{p} \cdot \hat{x} > 0$, and so

$$\psi(b) = \beta(\hat{p}, b) = \sup_{(x, y) \in Z} (\hat{p} \cdot (y - b)/\hat{p} \cdot x) \geq \hat{p} \cdot (\hat{y} - b)/\hat{p} \cdot \hat{x} \geq \alpha. \quad \blacksquare$$

For the case $b = 0$ Gale calls Z *regular* if $x > 0$ for all optimal inputs x . More generally, for any $b \in B$ we say Z is *regular at b* if $x > 0$ for all $x \in X(b)$. If Z is regular at b , then clearly there can be no singularities at b , so Theorem 1 holds. However, regularity is a considerably stronger condition than we need, and its economic basis seems questionable—it means in an optimal process every good is used as an input, and hence in particular there must be a surplus of every good after consumption requirements are met, for use in the following period. This is unrealistic because it excludes the possibility of producing some goods for con-

sumption only. By contrast, the exclusion of singularities is much less restrictive.

We noted in the example above that $\varphi(b)$ may be discontinuous. This situation always implies the existence of singularities, as we shall now show.

LEMMA 1. *If $\varphi(b)$ is discontinuous at $\bar{b} \in B$, then for every $x \in X(\bar{b})$ there is a singularity (p, x) for some $p \in P(\bar{b})$.*

Proof. Since $\varphi(b)$ is in any case upper semicontinuous, $\varphi(b)$ discontinuous at \bar{b} implies there is a sequence $\{b^n\}$, $b^n \in B$, such that $b^n \rightarrow \bar{b}$ and $\limsup_{n \rightarrow \infty} \varphi(b^n) < \varphi(\bar{b})$. Let $\varphi(\bar{b}) = \bar{\alpha}$. Given $\bar{x} \in X(\bar{b})$, let $\bar{y} = \bar{\alpha}\bar{x} + \bar{b}$, $(\bar{x}, \bar{y}) \in Z$, and let $I = \{i: \bar{x}_i = 0, 1 \leq i \leq n\}$. Suppose first that there exists $(x^*, y^*) \in Z$ such that $y^* - \bar{b} - \bar{\alpha}x^* = c$, where $c_i > 0$ for all $i \in I$. For every λ , $0 \leq \lambda \leq 1$, we have

$$(\lambda x^* + (1 - \lambda)\bar{x}, \lambda y^* + (1 - \lambda)\bar{y}) \in Z,$$

and

$$[\lambda y^* + (1 - \lambda)\bar{y}] - \bar{\alpha}[\lambda x^* + (1 - \lambda)\bar{x}] = \bar{b} + \lambda c.$$

For every small $\epsilon > 0$,

$$\begin{aligned} & [\lambda y^* + (1 - \lambda)\bar{y}] - (\bar{\alpha} - \epsilon)[\lambda x^* + (1 - \lambda)\bar{x}] \\ &= \bar{b} + \epsilon\bar{x} + \lambda[c + \epsilon(x^* - \bar{x})]. \end{aligned}$$

Choose λ_0 , $0 < \lambda_0 < 1$, such that

$$\lambda_0 \leq \min_{i \in I} \frac{\epsilon \bar{x}_i}{2 | c_i + \epsilon(x_i^* - \bar{x}_i) |}.$$

(There exist $i \in I$ since $\bar{x} \neq 0$). Choose $N(\epsilon)$ so large that, for all $n \geq N(\epsilon)$,

$$\begin{aligned} b_i^n - \bar{b}_i &\leq \epsilon \bar{x}_i / 2 && \text{for all } i \notin I, \\ b_i^n - \bar{b}_i &\leq \lambda_0 c_i && \text{for all } i \in I. \end{aligned}$$

Then

$$[\lambda_0 y^* + (1 - \lambda_0)\bar{y}] - (\bar{\alpha} - \epsilon)[\lambda_0 x^* + (1 - \lambda_0)\bar{x}] \geq b^n$$

and

$$\varphi(b^n) \geq \bar{\alpha} - \epsilon \quad \text{for all } n \geq N(\epsilon).$$

Hence, since ϵ was arbitrary,

$$\limsup_{n \rightarrow \infty} \varphi(b^n) \geq \bar{\alpha},$$

contradicting our hypothesis. Hence if we let

$$C = \{y_I - b - \bar{\alpha}x_I : (x, y) \in Z\},$$

where x_I and y_I represent the projections of x and y into the coordinates $i \in I$, then the above argument shows that $C \subset \mathbb{R}^I$ is a convex set not meeting the strictly positive orthant of \mathbb{R}^I . Since $0 \in C$ and $-e_i \in C$ for all $i \in I$ we conclude that there is a support vector $q \in \mathbb{R}^I$ such that $q \geq 0$, $\sum_{i \in I} q_i = 1$ and $q \cdot c \leq 0$ for all $c \in C$. If we define $p_i = 0, i \in I, p_i = q_i, i \in I$, then $p \in P(b)$ and $p \cdot \bar{x} = 0$. ■

In the example of Fig. 1, the vector $\bar{p} = (0, 1)$ is orthogonal to the unique optimal solution $(1, 0; 2, 0)$.

Now we are prepared to prove our theorem relating equilibrium prices to marginal values. A perturbation vector $d \in \mathbb{R}^n, d \neq 0$, is *feasible* for $b \in B$ if $b + td \in B$ for all sufficiently small $t > 0$. The *right derivative* of φ in the *direction* d , evaluated at b , is given by

$$\varphi_a^+(b) = \lim_{t \rightarrow 0^+} \frac{\varphi(b + td) - \varphi(b)}{t}. \tag{11}$$

THEOREM 2. *For any $b \in B$, if Z has no singularities at b and d is feasible for b then $\varphi_a^+(b)$ exists and*

$$\varphi_a^+(b) = - \min_{x \in X(b)} \max_{p \in P(b)} \frac{p \cdot d}{p \cdot x} = - \max_{p \in P(b)} \min_{x \in X(b)} \frac{p \cdot d}{p \cdot x}. \tag{12}$$

Proof. By Lemma 1, φ is continuous at b . Since d is feasible, $P(b + td)$ and $X(b + td)$ are defined and nonempty for all sufficiently small $t > 0$. We claim that $\lim_{t \rightarrow 0^+} X(b + td) = X(b)$ in the sense that whenever $x^n \in X(b + t_n d), x^n \rightarrow \bar{x}$, and $t_n \rightarrow 0^+$, then $\bar{x} \in X(b)$. Indeed, $(x^n, \varphi(b + t_n d) x^n) \in Z$ for all sufficiently small t_n , so $(\bar{x}, \varphi(b) \bar{x}) \in Z$, i.e., $\bar{x} \in X(b)$, by the continuity of φ and the closure of Z . Likewise, we claim $\lim_{t \rightarrow 0^+} P(b + td) = P(b)$. Let $p^n \in P(b + t_n d)$ such that $p^n \rightarrow \bar{p}$ and $t_n \rightarrow 0^+$; then $p^n \cdot (y - b - t_n d - \varphi(b + t_n d) x) \leq 0$ for all $(x, y) \in Z$ implies $\bar{p} \cdot (y - b - \varphi(b) x) \leq 0$ for all $(x, y) \in Z$, by the continuity of φ .

Since $\bar{p} \cdot \bar{x} > 0$ for all $\bar{p} \in P(b), \bar{x} \in X(b)$ by hypothesis, the above argument implies that we can choose $t_0 > 0$ such that whenever $0 \leq t \leq t_0$, then $b + td \in B$ and for every $p \in P(b + td)$ and $x \in X(b + td), p \cdot x > 0$. (This uses the fact that the pairs (p, x) lie in a compact set). Fix such a t_0 .

For any $t, 0 \leq t \leq t_0$, let $(x^t, y^t) \in Z$ such that

$$y^t - \varphi(b + td) x^t \geq b + td. \tag{13}$$

Then for any $p \in P(b)$,

$$p \cdot [y^t - b - td - \varphi(b + td)x^t] \geq 0 \quad (14)$$

while

$$p \cdot [y^t - b - \varphi(b)x^t] \leq 0. \quad (15)$$

Subtracting (15) from (14) we obtain

$$-tp \cdot d + [\varphi(b) - \varphi(b + td)]p \cdot x^t \geq 0$$

and

$$[\varphi(b) - \varphi(b + td)]/t \geq p \cdot d/p \cdot x^t \quad \text{for all } p \in P(b). \quad (16)$$

Likewise, choose $p^t \in P(b - td)$, $0 \leq t \leq t_0$. For any $x \in X(b)$ choose $(x, y) \in Z$ such that $y - \varphi(b)x \geq b$. Then

$$p^t \cdot [y - b - td - \varphi(b + td)x] \leq 0,$$

and

$$p^t \cdot [y - b - \varphi(b)x] \geq 0.$$

whence

$$[\varphi(b) - \varphi(b + td)]/t \leq p^t \cdot d/p^t \cdot x \quad \text{for all } x \in X(b). \quad (17)$$

From (16) and (17) it follows that

$$\min_{x \in X(b)} \frac{p^t \cdot d}{p^t \cdot x} \geq \frac{\varphi(b) - \varphi(b + td)}{t} \geq \max_{p \in P(b)} \frac{p \cdot d}{p \cdot x^t}$$

and

$$\begin{aligned} \limsup_{t \rightarrow 0^-} \min_{x \in X(b)} \frac{p^t \cdot d}{p^t \cdot x} &\geq \limsup_{t \rightarrow 0^+} \frac{\varphi(b) - \varphi(b + td)}{t} \\ &\geq \liminf_{t \rightarrow 0^+} \frac{\varphi(b) - \varphi(b - td)}{t} \\ &\geq \liminf_{t \rightarrow 0^-} \max_{p \in P(b)} \frac{p \cdot d}{p \cdot x^t}. \end{aligned}$$

But $\min_{x \in X(b)} p^t \cdot d/p^t \cdot x$ continuous in p^t and P compact implies

$$\limsup_{t \rightarrow 0^-} \min_{x \in X(b)} \frac{p^t \cdot d}{p^t \cdot x} = \min_{x \in X(b)} \frac{\bar{p} \cdot d}{\bar{p} \cdot x}$$

for some limit point \bar{p} of $\{p^t\}$, $\bar{p} \in P(b)$. Likewise,

$$\liminf_{t \rightarrow 0^+} \max_{p \in P(b)} \frac{p \cdot d}{p \cdot x^t} = \max_{p \in P(b)} \frac{p \cdot d}{p \cdot \bar{x}}$$

for some $\bar{x} \in X(b)$. Hence,

$$\begin{aligned} \max_{p \in P(b)} \min_{x \in X(b)} \frac{p \cdot d}{p \cdot x} &\geq \limsup_{t \rightarrow 0^+} \frac{\varphi(b) - \varphi(b + td)}{t} \\ &\geq \liminf_{t \rightarrow 0^+} \frac{\varphi(b) - \varphi(b + td)}{t} \geq \min_{x \in X(b)} \max_{p \in P(b)} \frac{p \cdot d}{p \cdot x}. \end{aligned}$$

Since

$$\begin{aligned} \max_{p \in P(b)} \min_{x \in X(b)} \frac{p \cdot d}{p \cdot x} &\leq \min_{x \in X(b)} \max_{p \in P(b)} \frac{p \cdot d}{p \cdot x}, \\ \varphi_d^+(b) &= - \lim_{t \rightarrow 0^+} \frac{\varphi(b) - \varphi(b + td)}{t} \end{aligned}$$

exists and

$$\varphi_d^+(b) = - \max_{p \in P(b)} \min_{x \in X(b)} \frac{p \cdot d}{p \cdot x} = - \min_{x \in X(b)} \max_{p \in P(b)} \frac{p \cdot d}{p \cdot x}. \quad \blacksquare$$

COROLLARY. *If $b \in \text{int } B$ and there is a unique equilibrium price vector $\bar{p} \in P(b)$ and a unique optimal $\bar{x} \in X(b)$, and if $\bar{p} \cdot \bar{x} > 0$, then $\nabla \varphi(b)$ exists and*

$$\nabla \varphi(b) = - \bar{p} / \bar{p} \cdot \bar{x}. \quad (18)$$

Proof. Under the hypotheses e_i and $-e_i$ are feasible for b for every i , $1 \leq i \leq n$. Hence,

$$\partial \varphi / \partial b_i = \varphi_{e_i}^-(b) = -\varphi_{-e_i}^+(b) = -\bar{p}_i \bar{p} \cdot \bar{x}$$

for every i .

COROLLARY. *Let the hypotheses of Theorem 2 hold. If, in addition, $b \in \text{int } B$, $\bar{p} \in P(b)$ is the unique equilibrium price vector for b , and $\bar{p}_i = 0$, then the marginal rate of substitution of good i for good j is \bar{p}_j / \bar{p}_i .*

Proof. The marginal rate of substitution of i for j is that μ such that

$$\varphi_{\mu e_i - e_j}^-(b) = 0. \quad (19)$$

Since $b \in \text{int } B$, $\mu e_i - e_j$ is feasible for all μ and $\varphi_{\mu e_i - e_j}^-(b) = 0$ if and only if

$$\min_{x \in X(b)} [(\mu \bar{p}_i - \bar{p}_j) / \bar{p} \cdot x] = 0,$$

hence if and only if $\mu = \bar{p}_j / \bar{p}_i$. ($\bar{p} \cdot x > 0$ for all $x \in X(b)$ by hypothesis).

Theorem 2 is very similar to a theorem first conceived by Harlan Mills [7], and proved by A. C. Williams [11] on the marginal value of

linear programs. Mills's result has been generalized to quasiconvex-quasiconcave programs by Gol'stein [4]. The von Neumann model can be recast as a quasiconvex-quasiconcave program by noticing that the minimax problem of (7);

$$\inf_{p \in P} \sup_{(x, y) \in Z} (p \cdot (y - b) / p \cdot x)$$

can be rewritten as

$$\inf_{p \in P} \sup_{x \in X} f(x, p, b)$$

where

$$f(x, p, b) = \sup_{(x, y) \in Z} (p \cdot (y - b) / p \cdot x).$$

It may then be verified that $f(x, p, b)$ is quasiconcave in x and quasiconvex in p . Unfortunately, we cannot readily apply Goldstein's theorem to obtain Theorem 2 because $f(x, p, b)$ is not continuous on $X \times P$ as that theorem requires. The authors are indebted to R. T. Rockafellar for bringing Gol'stein's work to their attention.

We close with an example that illustrates some of the foregoing theory.

Let Z be the set of all vector pairs $(x, 1 - x; y_1, y_2)$ such that $y_1^2 + y_2^2 = 4x(1 - x)$, $0 \leq x \leq 1$, together with all pairs implied by the disposal assumption (3). Z is depicted in Fig. 2.

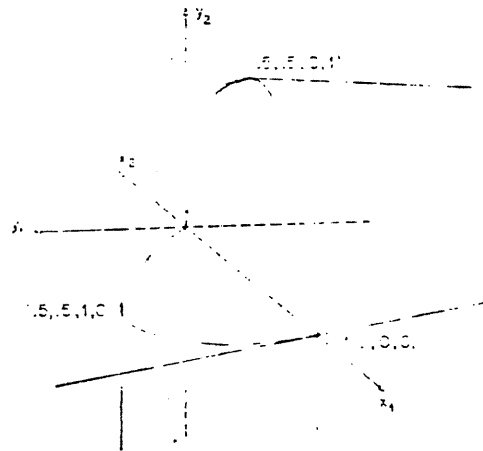


FIG. 2. The production space Z .

We propose to study the feasible set B . In particular, given $\bar{\alpha} > 0$ let us find the Pareto boundary $B_{\bar{\alpha}}^*$. Given $b = (b_1, b_2) \in B_{\bar{\alpha}}^*$, an optimal process (x, y) for b will in general have the property that

$$(y_1 - b_1)/x = (y_2 - b_2)/(1 - x) = \bar{\alpha} \quad (20)$$

and in fact $\bar{\alpha}$ will be the largest α for which (20) has a solution in Z : equivalently, the largest α such that for some x , $0 \leq x \leq 1$,

$$[\alpha x + b_1]^2 + [\alpha(1 - x) + b_2]^2 = 4x(1 - x). \quad (21)$$

Regarding α as an implicit function of x , we set $d\alpha/dx = 0$ to solve for $\bar{\alpha}$ and a corresponding $\bar{x} \in X(b)$. We thus obtain

$$(\bar{\alpha}\bar{x} + b_1) \bar{\alpha} - (\bar{\alpha}(1 - \bar{x}) + b_2) \bar{\alpha} = 2 - 4\bar{x} \quad (22)$$

and

$$\bar{x} = [\bar{\alpha}^2 + (b_2 - b_1) \bar{\alpha} + 2]/(2\bar{\alpha}^2 + 4). \quad (23)$$

Substituting (23) in (21) we obtain the following equation for $\bar{\alpha}$,

$$\bar{\alpha}^4 + 2(b_1 + b_2) \bar{\alpha}^3 + (b_1 + b_2)^2 \bar{\alpha}^2 + 4(b_1 + b_2) \bar{\alpha} + 4(b_1^2 + b_2^2 - 1) = 0. \quad (24)$$

The transformation

$$(b_1, b_2) \rightarrow ((b_1 + b_2)/\sqrt{2}, (b_2 - b_1)/\sqrt{2}) = (\omega_1, \omega_2)$$

is a rotation of $\pi/4$ in the (b_1, b_2) -plane. In the transformed coordinates we obtain from (24) the following defining equation for $\bar{\alpha} = \varphi(b)$.

$$[\omega_1 + (\varphi(b)/\sqrt{2})]^2 + (2\omega_2^2/(\varphi^2(b) + 2)) = 1. \quad (25)$$

For each $\alpha > 0$, let

$$E_\alpha = \{(\omega_1, \omega_2): [\omega_1 + \alpha/\sqrt{2}]^2 + (2\omega_2^2/(\alpha^2 + 2)) = 1\}.$$

E_α is an ellipse with center at $(\omega_1, \omega_2) = (-\alpha/\sqrt{2}, 0)$. The Pareto boundary, B_α^* , is just the arc of E_α bounded by the two points which maximize b_1 or b_2 , namely, the points

$$(b_1, b_2) = \frac{-x + (\alpha^2 + 4)^{1/2}}{2}, -\frac{\alpha^2}{2(\alpha^2 + 4)^{1/2}} - \frac{\alpha}{2} \quad (26)$$

and

$$(b_1, b_2) = -\frac{\alpha^2}{2(\alpha^2 + 4)^{1/2}} - \frac{\alpha}{2}, \frac{-x + (\alpha^2 + 4)^{1/2}}{2}. \quad (27)$$

Notice that the ellipse corresponding to the inadmissible growth rate $\alpha = 0$ is just the unit circle. The feasible set B is given by

$$B = \{b \in \mathbb{R}^2: \exists b' \text{ such that } b \leq b' \text{ and } \|b'\| < 1\}.$$

Figure 3 shows the set B and several of the sets B_α^* . The curves Γ_1 and Γ_2 are given by (26) and (27) as parametric functions of α ($\alpha > 0$). Any

point b in the interior of the region B^* bounded by the positive arc of the unit circle and the curves Γ_1 and Γ_2 lies on a Pareto boundary B_α^* for some $\alpha > 0$. For any such b , the marginal rate of substitution between b_1 and b_2 is the slope of the unique tangent to the ellipse $E_{\sigma(b)}$ at the point b . For any point $b \in B - B^*$, the marginal rate of substitution of b_1 for b_2 is zero if $b_1 > 0$ and ∞ if $b_1 < 0$.

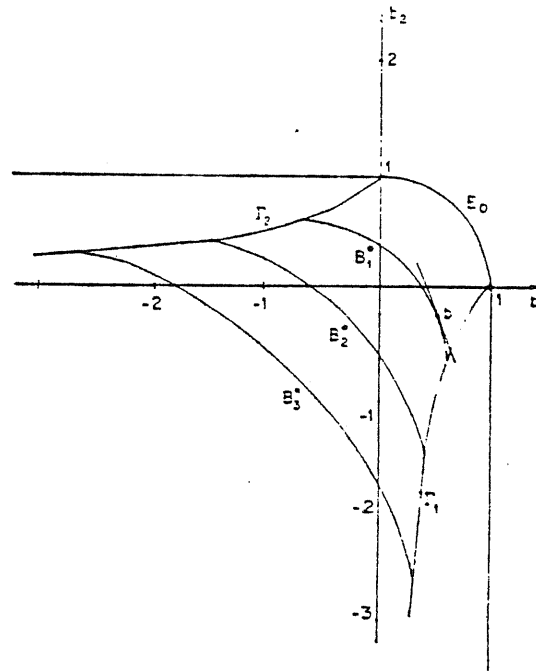


FIG. 3. The feasible set B with Pareto boundaries.

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