Individual contribution and just compensation

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Abstract

The "marginality principle" states that the share of joint output attributable to any single factor of production should depend only on that factor's own contribution to output. This property, together with symmetry and efficiency, uniquely determines the Shapley value. A similar result characterizes Aumann–Shapley pricing for smooth production functions with variable input levels.

1 Introduction

In a perfectly competitive market, the wage of a laborer equals his marginal product. No ethical judgment need be made as to whether marginal productivity is a "just" rule of compensation so long as competitive markets are accepted as the correct form of economic organization. Nevertheless, the idea that rewards should be in proportion to contributions has considerable ethical appeal in itself, and appears to reflect widely held views about what constitutes "just compensation" without any reference to the theory of perfect competition.

In this paper we shall ask what "compensation in accordance with contribution" means in the absence of competition. How does the marginality principle translate into a rule of distributive justice when cooperation rather than competition is the mode of economic organization?

Unfortunately, if we attempt to translate marginalism directly into a

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cooperative sharing rule, difficulties arise. For, except in very special cases, the sum of individuals' marginal contributions to output will not equal total output. If there are increasing returns from cooperation, the sum of marginal contributions will be too great; if there are decreasing returns, it will be too small.

One seemingly innocuous remedy is to compute the marginal product of all factor inputs and then adjust them by some common proportion so that total output is fully distributed. This proportional to marginal product principle is the basis of several classical allocation schemes, including the "separable costs remaining benefits" method in cost-benefit analysis [3,6]. We show by example, however, that the proportional to marginal product principle does not resolve the "adding up" problem in a satisfactory way. The reason is that the rule does not base the share of a factor solely on that factor's own contribution to output, but on all factors' contribution to output. For example, if one factor's marginal contribution to output increases while another's decreases, the share attributed to the first factor may actually decrease; that is, it may bear some of the decrease in productivity associated with the second factor.

We show that there is essentially only one cooperative sharing rule that avoids this difficulty—the Shapley value. More particularly, the Shapley value is the unique sharing rule with the following three properties:

(i) Output is fully distributed.
(ii) Factors that enter into the productive function in a symmetric way receive equal shares.
(iii) A factor's share depends only on its own contribution to output.

This result holds for all production functions with discrete factor inputs (i.e., all cooperative games). A similar result characterizes the Aumann–Shapley sharing rule on the class of smooth production functions with variable levels of input. In this sense it can be said that the Shapley value (and Aumann–Shapley prices in the continuous case) are the natural interpretations of marginalism in problems of pure cooperation.

2 Sharing as a cooperative game

Consider $n$ agents $N = \{1, 2, \ldots, n\}$ who can cooperate to produce a single joint product. The product is assumed to be perfectly divisible. The agents have different skills, so some agents may contribute more to pro-
duction than others. For each subset of agents \( S \subseteq N \), let \( v(S) \) be the total amount produced by \( S \) when the agents in \( S \) pool their skills. We assume that nothing is produced for free; that is, \( v(\emptyset) = 0 \). The set function \( v \) defines a cooperative game on the fixed set \( N \) of agents. A \textit{sharing rule} (or "solution concept") is a function \( \phi \), defined for every cooperative game on the fixed set \( N \), such that \( \phi(v) = (a_1, \ldots, a_n) \in \mathbb{R}^n \). Here \( a_i \) is interpreted to be \( i \)'s "share" of the total output \( v(N) \).

We shall be interested in sharing rules \( \phi \) that obey three properties. First, the output \( v(N) \) should be \textit{fully distributed}:

\[ \Sigma \phi_i(v) = v(N). \tag{1} \]

This property is also known as \textit{efficiency} [5].

Second, an agent's share should depend only on his structural role in the function \( v \), not on his name. We say that \( \phi \) is \textit{symmetric} if for every permutation \( \pi \) of \( N \), \( \phi_{\pi(i)}(\pi v) = \phi_i(v) \).

A third important property is that each agent's share should depend only on his own contribution to output. The problem immediately arises of how to define "own contribution" unambiguously. Consider the following example. Two agents, 1, 2, each working by himself can produce two units per period; working together they can produce ten units per period. Thus \( v(1) = v(2) = 2, v(1,2) = 10 \) (and \( v(\emptyset) = 0 \)). What is \( i \)'s contribution? Relative to the state of full cooperation (i.e., the set \( \{1,2\} \)), each agent contributes eight units at the margin. But relative to the state of noncooperation, each agent contributes only two units at the margin. The meaning of "marginal contribution" is ambiguous because one cannot say a priori which of the coalitions will actually form.

One way around this impasse is the following idea, due to Shapley [5]. Imagine that the agents arrive on the scene of cooperation in some random order. If, say, agent 1 arrives first, then 1 is credited with a marginal contribution of \( v(1) - v(\emptyset) = 2 \) units. Agent 2 arrives next and is credited with a marginal contribution of \( v(1,2) - v(1) = 8 \) units. In this case, there is a premium on arriving last. To place all of the agents on the same footing, we can average the agents' expected marginal contributions over all \( n! \) orderings. The result is the Shapley value

\[ \phi^*(v) = \sum_{S \subseteq N-i} \frac{|S|!(|N-S|-1)!}{|N|!} [v(S+i) - v(S)]. \tag{2} \]

Instead of determining contributions according to a random ordering of the players, let us postulate merely that each agent's share should depend
in some fashion on his own contributions. A full description of agent i’s marginal contributions, at all possible levels of output, is contained in the partial derivative of $v$ with respect to $i$, namely, the function $v_i$ defined as follows:

$$
v_i(S) = v(S + i) - v(S) \quad \text{if } i \notin S,\]
$$

$$
= v(S) - v(S - i) \quad \text{if } i \in S.
$$

(3)

The sharing rule $\phi$ satisfies the marginality principle if $\phi_i(v)$ is solely a function of $v_i$: in other words, if for every $i \in N$ and every two games $v$, $w$ on $N$,

$$
v_i = w_i \Rightarrow \phi_i(v) = \phi_i(w).
$$

(4)

If a sharing rule does not satisfy the marginality principle, it is subject to serious distortions. For, if one agent’s share depends on another’s contributions, the first agent can be rewarded (or punished) for actions undertaken by the second. Such rules force dependencies among agents that are not at all necessary for cooperation, and may penalize individual initiative.

Consider the proportional to marginal product rule:

$$
\phi_i(v) = \left[ \frac{v_i(N)}{\sum_{j \in N} v_j(N)} \right] v(N).
$$

(5)

This rule does not satisfy the marginality principle because $\phi_i(v)$ depends on $v(N)/\sum_{j \in N} v_j(N)$, which involves other agents’ marginal contributions. This dependence can lead to unfortunate results, as shown by the following example. Suppose that two agents can pool their labor and resources to produce grain. Agent A alone can produce 20 bushels per year by his own labor on his own land (net of his own subsistence requirements). Agent B alone can grow 60 bushels per year (net of subsistence). Working together they can produce 100 bushels per year (net of subsistence). The marginal contributions to the grand coalition are 40 for A and 80 for B. Hence, the proportional to marginal product rule implies that A’s share of net output is $33\frac{1}{3}$ and B’s is $66\frac{2}{3}$. Now suppose that A works more efficiently than before: for example, A tightens his belt and eats 1 bushel per year less. Thus A adds 1 bushel a year to net output whether working alone or with B. Simultaneously, suppose that B uses up 21 bushels per year more, either through waste or self-indulgence. Combining both effects, the new output function will be
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\[ \phi: 0 \]
A: 21
B: 39
A, B: 80

A’s marginal contribution to every coalition of which it is a member has increased by 1, and B’s has decreased by 21. Yet the proportional to marginal product rule now gives A 32.8 bushels and B 47.2 bushels. Thus, even though A becomes more efficient, A is penalized for B’s becoming less efficient.

The following result [7] shows that this sort of injustice can be avoided in essentially only one way. (This generalizes a result of Loehman and Winston [4], who showed it for a special class of cooperative games.)

**Theorem 1.** The Shapley value is the unique sharing rule that is symmetric, fully distributes all gains, and satisfies the marginality principle.

**Proof:** Fix a set of agents \( N = \{1, 2, \ldots, n\} \). For every game \( \nu \) defined on \( N \), it is clear that the Shapley value \( \Phi \) satisfies the properties of symmetry (S), full distribution (D), and marginality (M).

Conversely, let \( \phi \) be a sharing rule defined for all games \( \nu \) on \( N \) such that S, D, and M hold. First we shall show that an agent whose marginal contribution to every coalition is zero (i.e., a dummy) gets nothing. Consider the game \( w \) that is identically zero for all \( S \subseteq N \). This game is symmetric in all agents, so all agents receive equal shares. Because \( \sum_{i \in N} \phi_i(w) = 0 \), it follows that \( \phi_i(w) = 0 \) for all \( i \in N \). Now suppose that \( i \) is a dummy in an arbitrary game \( \nu \) on \( N \). Then \( \nu_i \) is identically zero. Hence \( \nu_i = w_i \), so marginality implies that \( \phi_i(\nu) = \phi_i(w) = 0 \). Thus dummies get nothing under \( \phi \).

Next we show that \( \phi(\nu) \) must be the Shapley value for every game \( \nu \) on \( N \). Consider first the case where \( \nu \) is a unanimity game: For some non-empty subset \( R \subseteq N \), \( \nu = \nu_R \), where

\[
\nu_R(S) = \begin{cases} 
1 & \text{if } R \subseteq S, \\
0 & \text{if } R \nsubseteq S.
\end{cases}
\]

It is easy to see that every \( i \notin R \) is a dummy in \( \nu_R \). Hence, the preceding implies that \( \phi_i(\nu_R) = 0 \) for all \( i \notin R \). Because \( \nu_R \) is symmetric with respect to all agents \( i \in R \), symmetry implies that \( \phi_i(\nu_R) = \phi_j(\nu_R) \) for all \( i, j \in R \). This, together with full distribution, implies that \( \phi_i(\nu_R) = 1/|R| \) for all
i \in R. This argument shows that \( \phi(v) \) is the Shapley value whenever \( v_R \) is a unanimity game. A similar argument shows that \( \phi(cv) \) is the Shapley value whenever \( c \) is a constant and \( v \) is a unanimity game.

For a general game \( v \) on \( N \), we may write

\[
v = \sum_{R \subseteq N} c_R v_R,
\]

where each \( v_R \) is a unanimity game and the coefficients \( c_R \) are real numbers. Let \( I(v) \) be the least number of nonzero terms in such an expression for \( v \). (If \( v \) is identically zero, \( I(v) = 0 \).) The proof that \( \phi(v) \) is the Shapley value for \( v \) is by induction on \( I(v) \). We have already shown that this is the case when \( I(v) = 0 \) or \( I(v) = 1 \).

Assume now that \( \phi(v) \) is the Shapley value whenever the index of \( v \) is at most \( I \), and let \( v \) have index \( I + 1 \) with expression

\[
v = \sum_{k=1}^{I+1} c_{R_k} v_{R_k}, \quad \text{all } c_{R_k} \neq 0.
\]

Let \( R = \cap_{k=1}^{I+1} R_k \) and choose \( i \notin R \). If \( i \) is a dummy, then \( \phi_i(v) = 0 \), which is also the Shapley value of \( i \). If \( i \) is not a dummy, consider the game

\[
w = \sum_{k: i \in R_k} c_{R_k} v_{R_k}.
\]

Since \( i \notin R \), the index of \( w \) is at most \( I \). Furthermore, \( w_i(S) = v_i(S) \) for all \( S \subseteq N \), so marginality implies that \( \phi_i(v) = \phi_i(w) \). By the induction hypothesis,

\[
\phi_i(v) = \phi_i(w) = \sum_{k: i \in R_k} \frac{c_{R_k}}{|R_k|},
\]

which is the Shapley value of \( i \).

It remains to show that \( \phi_i(v) \) is the Shapley value when \( i \in R = \cap_{k=1}^{I+1} R_k \). By symmetry, \( \phi_i(v) \) is a constant \( c \) for all members of \( R \); likewise the Shapley value is some constant \( c' \) for all members of \( R \). Because both allocations sum to \( v(N) \) and are equal for all \( i \notin R \), it follows that \( c = c' \).

Q.E.D.

If it is desired to stay entirely within the class of superadditive games, the foregoing proof may be modified as follows. Every superadditive game \( v \) may be written in the form \( v = u - \sum c_R v_R \), where all \( c_R > 0 \), \( u \) is superadditive, and \( u \) is symmetric in the sense that \( u(S) \) depends only on the cardinality of \( S \). (Every unanimity game \( v_R \) is also superadditive.) Proceed
by induction on the minimum number of nonzero terms in such an expression, observing that the deletion of any term $c_i v_i$ leaves a superadditive game. The result holds in the base case $v = u$, because symmetry and full distribution imply that $\phi_i(u) = u(N)/|N|$, which accords with the Shapley value.

3 Aumann–Shapley pricing

In this section we show that a similar set of axioms characterizes the Aumann–Shapley pricing mechanism. Consider a firm that produces a single homogeneous product as a function of several resource inputs. Let $y = f(x_1, x_2, \ldots, x_n)$ be the maximum quantity of output that can be produced with $x_i$ units of each resource $i (i = 1, 2, \ldots, n)$. Assume that $f(x)$ is defined for all $x$ on some bounded domain of the form $D = D(\bar{x}) = \{x \in \mathbb{R}^n : 0 \leq x \leq \bar{x}\}$, where $\bar{x} > 0$. The target level of production is $f(\bar{x})$. Assume that $f$ has continuous first partial derivatives on $D$ (the one-sided derivative applies on the boundary of $D$). Assume further that there are no fixed costs—that is, $f(0) = 0$. A pair $(f, \bar{x})$ satisfying these conditions will be called a variable-input production problem.

How much product should be attributed to each of the $n$ inputs? This question arises in a firm that wants to allocate profit to different inputs for purposes of internal accounting and control. Assume for simplicity that output is measured in terms of revenues, and inputs are measured in terms of costs. The firm wants to allocate total revenue $\bar{y} = f(\bar{x})$ among the various inputs $i = 1, 2, \ldots, n$ so that a net profit can be imputed to each input (or groups of inputs that constitute "profit centers"). In other words, the firm wishes to find a vector of unit prices $(p_1, \ldots, p_n)$ such that $\sum_{i \in N} p_i \bar{x}_i = f(\bar{x})$.

A pricing rule is a function $\psi$ defined for all production problems $(f, \bar{x})$ on some fixed set of inputs $N = (1, 2, \ldots, n)$ such that $\psi_i(f, \bar{x}) = p_i$ is the unit price associated with $i$. Full distribution requires that

$$\sum_{i \in N} \bar{x}_i \psi_i(f, \bar{x}) = f(\bar{x}).$$

(7)

If $f(\bar{x})$ is the total revenue from production and $\bar{x}_i$ is the total cost of input $i$, then $\psi_i(f, \bar{x})$ may be interpreted as the imputed revenue per unit cost of $i$, and $\psi_i(f, \bar{x}) - 1$ is $i$'s imputed rate of profit. A pricing rule might be used as part of a compensation or bonus scheme to reward different profit centers according to their imputed profitability. Alternatively, management might treat the rule as an internal accounting method for
monitoring the performance of profit centers or divisions over time. In either case it is reasonable to require that the unit price imputed to $i$ depends only on $i$'s own contribution to revenue or product. That is, $\psi_i(f, \bar{x})$ should be a function only of the partial derivative $f_i(x) = \partial f(x)/\partial x_i$, $0 \leq x \leq \bar{x}$ (and possibly of $i$ itself).

The function $\psi$ satisfies the *marginality principle* if and only if for every $i \in N$ and for every two production functions $f$ and $g$ defined on the same domain $D = \{x \in R^n: 0 \leq x \leq \bar{x}\}$,

$$f_i(x) = g_i(x) \text{ for all } x \in D \text{ implies } \psi_i(f, \bar{x}) = \psi_i(g, \bar{x}). \quad (8)$$

A method that does *not* have this property is the proportional to marginal product pricing rule:

$$\psi_i(f, \bar{x}) = \lambda f_i(\bar{x}) \quad \text{where } \lambda = \frac{f(\bar{x})}{\sum_{j \in N} \bar{x}_j f_j(\bar{x})}.$$ 

Like its relative in the finite case, this method behaves in an unsatisfactory way when the production function shifts. In particular, it may impute a lower unit revenue to $i$ even though $i$'s marginal revenue product increases over all possible levels of output. The reason is that $i$'s imputed revenue may be dragged down through its dependence on the other factors' marginal revenue products.

We claim that there is only one plausible pricing mechanism that is fully distributive and satisfies (8), namely, Aumann–Shapley pricing:

$$\psi_i^{AS}(f, \bar{x}) = \int_0^1 \frac{df(t \bar{x})}{dx_i} \, dt. \quad (9)$$

To establish this fact, we need continuity plus a condition analogous to symmetry. One natural formulation of symmetry is the following: If $f$ is symmetric in the two factors $i$ and $j$, that is, if

$$f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = f(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n) \text{ for all } x, 0 \leq x \leq \bar{x},$$

then $\psi_i(f, x) = \psi_j(f, x)$. In fact, we need a somewhat stronger condition that identifies inputs which are essentially the same. Consider the case where two inputs represent the same item expressed in different units. For example, let $x_1$ be the number of quarts of gasoline and $x_2$ the number of pints of gasoline used in production. The number of gallons of gasoline is therefore $y = x_1/4 + x_2/8$. If the imputed price per gallon is $p$, then it is natural to assign a unit price of $p/4$ to $x_1$ and $p/8$ to $x_2$. 
In general, define a linear aggregate of the input factors \(x_1, \ldots, x_n\) to be an input of the form \(y_i = \sum_{j=1}^{m} a_{ij}x_j\), where all \(a_{ij} \geq 0\). Let \(g(y)\) be a production function of \(m\) aggregate inputs \(y_1, \ldots, y_m\), each of which is a linear combination of inputs \(x_1, \ldots, x_n\). Thus \(y = Ax\), where \(A\) is a nonnegative matrix. Consider the production function \(f\) defined directly on \(x\) as follows: \(f(x) = g(Ax)\). The pricing rule \(\psi\) is aggregation invariant if whenever \(A \geq 0\) and \(A\bar{x} = \bar{y} > 0\), \(f = g \circ A\) implies \(\psi(f, \bar{x}) = [\psi(g, A\bar{x})]A\). In other words, the prices to the \(y\) inputs are imputed to the \(x\) inputs via the linear system \(A\). (A similar property was defined by Billera and Heath [2]). Notice that symmetry follows from aggregation invariance by taking \(A\) to be a transposition matrix.

For every \(\bar{x} \in \mathbb{R}^N, \bar{x} > 0\), let \(C^1(\bar{x})\) be the Banach space of all continuously differentiable, real-valued functions \(f\) defined on \(D = D(\bar{x})\) such that \(f(0) = 0\), with norm

\[
\|f\| = \sup_{x \in D}|f(x)| + \sum_{i=1}^{n} \sup_{\bar{x} \in D} \left| \frac{\partial f(x)}{\partial x_i} \right|.
\]

The pricing rule \(\psi\) is continuous if for every fixed \(\bar{x} > 0\) \(\psi(f, \bar{x})\) is continuous in \(f\) in the topology of \(C^1(\bar{x})\).

**Theorem 2.** Aumann–Shapley pricing is the unique pricing rule that is continuous, aggregation invariant, fully distributive, and satisfies the marginality principle.

**Proof:** The reader may verify that the Aumann–Shapley pricing rule has the required properties. (Aggregation invariance follows from Corollary 4 in [2].)

Conversely, let \(\psi\) be a pricing rule with these properties. Consider first the case where the production function \(f\) is a polynomial in \(x_1, \ldots, x_n\) and \(f(0) = 0\). Aumann and Shapley [1] showed that in this instance \(f(x)\) can be written in the form

\[
f(x) = \sum_{k=1}^{l} c_k P_k(x), \quad \text{where} \quad P_k(x) = \left( \sum_{j=1}^{n} b_{kj}x_j \right)^{r_k},
\]

\(r_k\) is a positive integer for all \(k\), and all \(b_{kj} \geq 0\).

Let the index \(I\) of \(f\) be the least number of nonzero terms in any expression for \(f(x)\) of form (10). (If \(f\) is identically zero, let \(I = 0\).) Assume that the theorem is false for polynomial \(f\) and we shall derive a contradiction. Let \(I^*\) be the least index \((I^* \geq 0)\) for which there exists a problem \((f, \bar{x})\), such that \(\bar{x} > 0\), \(f\) is polynomial with index \(I^*\), and \(\psi(f, \bar{x}) \neq \psi^{\text{AS}}(f, \bar{x})\).
If \( I^* = 0 \), then \( f \) is identically zero. Let \( y = \sum_{i=1}^{I^*} x_i \) and \( g(y) = f(x) \). Then \( g \) is identically zero, and, by full distribution, \( y \phi(g,y) = g(y) = 0 \). Since \( y > 0 \), it follows that \( \psi_i(g,y) = 0 \). By aggregation invariance, it follows that for all \( i \), \( \psi_i(f,\bar{x}) = 0 = \psi_i^{AS}(f,\bar{x}) \), a contradiction.

Assume then that \( I^* > 0 \). Consider an expression of form (10) for \( f \). Let \( J^+ \) be the subset of indices \( i \) for which \( x_i \) has a positive coefficient \( b_{ki} > 0 \) in every term \( c_k P_k(x) \), and let \( J^0 \) be the set of all other indices. If \( i \in J^0 \), then by deleting the terms \( c_k P_k(x) \) in which \( x_i \) has a zero coefficient, we obtain a polynomial function \( g(x) \) with index less than \( I^* \) such that, for this particular \( i \),

\[
g_i(x) = f_i(x) \quad \text{for all } x \in D(\bar{x}).
\]

Thus by the induction assumption \( \psi_i(g,\bar{x}) = \psi_i^{AS}(g,\bar{x}) \). From the marginality principle it follows that

\[
\psi_i(f,\bar{x}) = \psi_i(g,\bar{x}) = \psi_i^{AS}(g,\bar{x}) = \psi_i^{AS}(f,\bar{x}).
\]

If \( J^+ \) is empty, it follows that \( \psi(f,\bar{x}) = \psi^{AS}(f,\bar{x}) \), contrary to assumption. If \( J^+ \) consists of a single index, say \( i \), then the fact that \( \psi \) and \( \psi^{AS} \) must both satisfy (7) implies (because \( \bar{x}_i > 0 \)) that \( \psi_i(f,\bar{x}) = \psi_i^{AS}(f,\bar{x}) \).

The case \( J^+ \geq 2 \) remains. Without loss of generality let \( 1 \in J^+ \), so \( b_{k1} > 0 \) for all \( k \), \( 1 \leq k \leq I^* \). Define an aggregate product \( y_1 \) as follows:

\[
y_1 = x_1 + \sum_{j=2}^{n} a_{1j} x_j, \quad \text{where } a_{1j} = \min_{1=k \leq I^*} \left\{ \frac{b_{kj}}{b_{k1}} \right\} \geq 0.
\]

Let \( y_j = x_j \) for \( 2 \leq j \leq n \), and let \( A \) be the nonnegative, \( n \times n \) matrix such that \( y = Ax \). Let

\[
g(y) = \sum_{k=1}^{I^*} c_k Q_k(y),
\]

where

\[
Q_k(y) = \left[ b_{k1} y_1 + \sum_{j=2}^{n} (b_{kj} - b_{k1} a_{1j}) y_j \right]_{\ast} \quad \text{for } 1 \leq k \leq I^*.
\]

Then \( g(Ay) = f(x) \). By choice of the coefficients \( a_{1j} \), \( y_1 \) is the only variable that has positive coefficients in every term of (11). By the preceding argument, we can conclude that for all \( \bar{y} > 0 \) and all \( i \),

\[
\psi_i(g,\bar{y}) = \psi_i^{AS}(g,\bar{y}).
\]
For all \( \overline{x} > 0 \) let \( \overline{y} = A\overline{x} > 0 \) and conclude by aggregation invariance for \( \psi \) and \( \psi^{AS} \),

\[
\psi(f, \overline{x}) = \psi(g, \overline{y})A = \psi^{AS}(g, \overline{y})A = \psi^{AS}(f, \overline{x}),
\]
a contradiction. This establishes that \( \psi \) is identical with Aumann–Shapley pricing for all problems \( (f, \overline{x}) \) in which \( f \) is a polynomial in \( C^1(\overline{x}) \). Since the polynomials are dense in \( C^1(\overline{x}) \) and both \( \psi(f, \overline{x}) \) and \( \psi^{AS}(f, \overline{x}) \) are continuous in \( f \), it follows that \( \psi(f, \overline{x}) = \psi^{AS}(f, \overline{x}) \) for all \( f \) in \( C^1(\overline{x}) \). This concludes the proof of Theorem 2.

Several variants of Theorem 2 can be obtained by strengthening the marginality principle. One natural variation is to require that the unit price of each product \( i \) be monotone nondecreasing with respect to \( i \)'s marginal contributions. We say that a pricing rule \( \psi \) is monotonic if for every \( i \in N \), every positive \( \overline{x} \in \mathbb{R}^N \), and every \( f, g \in C^1(\overline{x}) \),

\[
\frac{\partial f(x)}{\partial x_i} \geq \frac{\partial g(x)}{\partial x_i} \quad \text{for all } x \in D(\overline{x}) \text{ implies } \psi_i(f, \overline{x}) \geq \psi_i(g, \overline{x}). \tag{12}
\]

If we assume monotonicity instead of marginality, then the continuity assumption may be dropped in Theorem 2 (see [8] Theorem 2). The reason is that every \( f \) in \( C^1(\overline{x}) \) may be sandwiched between two sequences of polynomials in \( C^1(\overline{x}) \) that converge to \( f \) from above and below in the topology of \( C^1(\overline{x}) \).

The marginality principle may be strengthened in another way to incorporate symmetry. We say that \( \psi \) satisfies the symmetric marginality principle if every two inputs with equal partial derivatives are priced equally. That is, for every \( i, j \in N \), every positive \( \overline{x} \in \mathbb{R}^N \), and every \( f, g \in C^1(\overline{x}) \),

\[
\frac{\partial f(x)}{\partial x_i} = \frac{\partial g(x)}{\partial x_j} \quad \text{for all } x \in D(\overline{x}) \text{ implies } \psi_i(f, \overline{x}) = \psi_j(g, \overline{x}). \tag{13}
\]

The symmetric marginality principle, together with continuity and full distribution, uniquely characterizes Aumann–Shapley pricing. The method of proof is different than that used to derive Theorem 2 (see [8], Theorem 1).

Finally, we may combine (12) and (13) into the following single condition: for every \( i, j \in N \), every positive \( \overline{x} \in \mathbb{R}^N \), and every \( f, g \in C^1(\overline{x}) \),

\[
\frac{\partial f(x)}{\partial x_i} \geq \frac{\partial g(x)}{\partial x_j} \quad \text{for all } x \in D(\overline{x}) \text{ implies } \psi_i(f, \overline{x}) \geq \psi_j(g, \overline{x}). \tag{14}
\]
This allows us to drop both aggregation invariance and continuity. That is, (14) and full distribution uniquely characterize Aumann–Shapley pricing on $C^1(\bar{x})$ (see [8], Theorem 1).

REFERENCES