A QUICK PROOF OF WAGNER'S EQUIVALENCE THEOREM

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Wagner's Equivalence Theorem is the statement that the Four Colour Theorem for planar graphs holds if, and only if, every graph requiring five colours for a vertex colouration contracts to a graph that contains the complete 5-graph. This equivalence was first proved by K. Wagner [**1**]; somewhat simpler proofs were given later by R. Halin [**2, 3**] and O. Ore [**4**]. All of these proofs are rather complicated, and involve a characterization of a certain class of maximal graphs that do not contract to a complete 5-graph. The purpose of this paper is to present a relatively short proof from first principles, using only Kuratowski's Theorem characterizing planar graphs. Proposition 1 proved below is also derived in [**4**] as Theorem 10.5.1; however, the proof given here is more direct. Also, a portion of our proof of Wagner's Equivalence Theorem follows an argument presented in [**4**; pp. 167–169] in a different context.

In general, the notation and terminology used here will follow Ore [**4, 5**]. However, for convenience, we shall introduce certain terms explicitly.

Let G and H be graphs, which we always assume are finite and have neither loops nor multiple edges. $V(G)$ shall denote the vertex set of G. If $S \subseteq V(G)$, then $G(S)$ denotes the graph whose vertex set is S, and whose edges are precisely those edges of G having both ends in S. A *contraction* of G onto H is a function f from $V(G)$ onto $V(H)$ such that for all distinct $x, y \in V(H)$: (1) $G(f^{-1}(x))$ is connected; (2) x is adjacent to y in H if and only if there is an edge of G from the set $f^{-1}(x)$ to the set $f^{-1}(y)$. A contraction may be thought of as a sequence of *elementary contractions*, in which a pair of adjacent vertices are identified and all other adjacencies between vertices are preserved (multiple edges arising from the identification being replaced by single edges). H is a *subcontraction* of G, $G > H$, if H is a subgraph of the image of a contraction of G. We remark that the relation " $>$ " is transitive. G is *conformal* to H if G is isomorphic to a graph obtained from H by inserting and deleting vertices along some of the edges of H. The complete graph on n vertices shall be denoted by K_n.

PROPOSITION 1: *If G is 4-connected and non-planar then $G > K_5$.*

Proof: By Kuratowski's Theorem we are reduced to proving that if G contains a subgraph H conformal to the Kuratowski graph $K_{3,3}$, then $G > K_5$. Let the principal vertices of H form the two classes $\{A, B, C\}$ and $\{D, E, F\}$. Each pair of principal vertices in opposite classes is joined by an *H line*. The H-lines are disjoint except possibly at the ends; they may also be subdivided by some vertices called *midvertices*. We first contract the midvertices (if any) of each H-line to a single midvertex. Since the principal vertices are not involved in the contraction, no three of them separate the contracted graph (which we still call G).

Let P be a path connecting A and B in $G-\{D, E, F\}$. Traversing P from A, let w be the last vertex on an H-line from A, and let x be the first vertex after w on an

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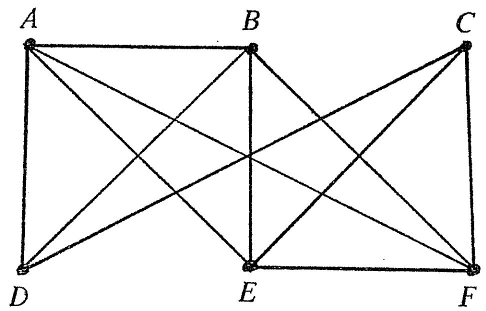
H-line from B or C, say from B. Let P_{wx} be the section of P from w to x, inclusive. Then P_{wx} contains no vertex of H in its interior and $w \neq x$. There is some vertex among $\{D, E, F\}$, say F, such that neither w nor x is a midvertex of any H-line emanating from it. Let Q be a path in $G-\{A, B, C\}$ from F to $\{D, E\}$. Traversing Q from F, let y be the last vertex of Q on an H-line from F. Let z be the next vertex that is in $P_{wx} \cup H$. If $z \notin P_{wx}$ then z lies on an H-line from D or E. Contract the pairs (z, D or E), (y, F), (x, B), (w, A) along their respective H-lines. The image of $H \cup P_{wx} \cup Q_{yz}$ under this contraction is conformal to L_1, which contracts to K_5 by contracting C and D. Hence also $G > K_5$.

L_1:

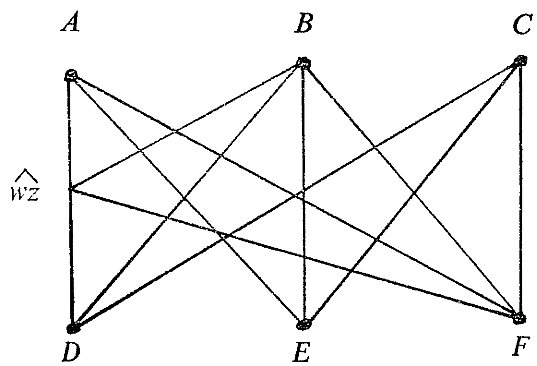


If $z \in P_{wx}$ we consider two cases:

Case 1: Not both $w = A$ and $x = B$.

Suppose $w \neq A$. If $z \neq x$, we can contract z into w along P_{wx} and the pairs (x, B), (y, F) along H-lines to obtain a subcontraction of G conformal to L_2. If $z = x$, then since $Q \cap \{A, B, C\} = \varnothing$, $x \neq B$ and we can again find a subcontraction of G conformal to L_2. By contracting (E, A) and (C, D) in L_2, we obtain K_5.

L_2:



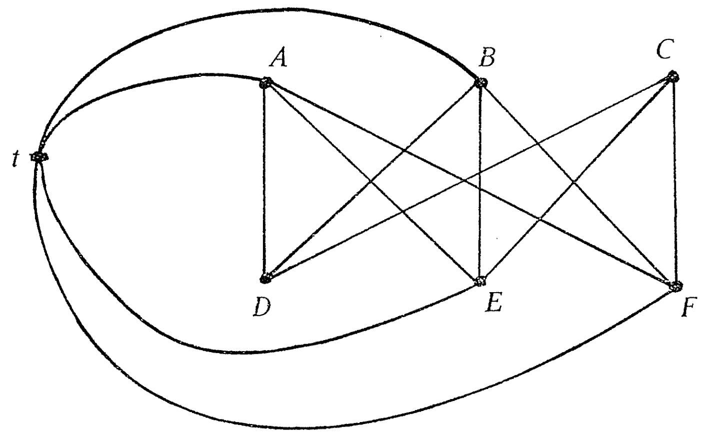
A similar argument can be given when $x \neq B$.

Case 2: Both $w = A$ and $x = B$.

Still traversing Q from F, let v be the next vertex after z that is in H. v then lies on an H-line from D or E (not F), say E. Contract the interiors of P_{wx} and Q_{yv} to a single vertex t with edges to A, B, v, y. Contracting the pairs (v, E), (y, F) along

H-lines, we obtain a subcontraction of G conformal to L_3. By contracting (C, E) and (A, D) in L_3, we obtain K_5.

L_3:



Thus in all cases, $G > K_5$.

THEOREM (*Wagner*).
(*H*) : "*G is 5-chromatic implies* $G > K_5$" *holds if, and only if,*
(4CC): "*Every planar graph can be vertex coloured in four colours.*"

Proof. (*H*) implies (4*CC*) is immediate, by Kutatowski's Theorem.

Conversely, let (4*CC*) hold, and let G be a 5-chromatic graph that does not admit K_5 as a subcontraction. We may assume in addition (by applying preliminary contractions and deletions of edges to G if necessary) that every proper subcontraction of G is 4-colourable. We shall show that G is 4-connected.

Suppose to the contrary that there is a set S of vertices of G such that $|S| \leqslant 3$ and $G-S$ separates into the components $C_1, C_2, \ldots, C_n$ $(n \geqslant 2)$. We may assume that S is the smallest such set, and hence that every vertex of S is joined to each C_i by an edge. Let $C_i' = G(VC_i) \cup S)$, $1 \leqslant i \leqslant n$. Let T be a maximal independent subset of S in G. Each C_i' can be coloured in four colours such that all vertices of T are given the same colour α, and the colours used for $S-T$ are all different from α. Indeed, choose $j \neq i$ and contract C_j to a vertex t; then contract t and T to a vertex t'. Because T was maximal independent in S, t' is adjacent to every vertex of $S-T$. The contracted graph can be coloured in four colours. Hence, restoring the vertices of T with the colour of t', which is different from the colours of $S-T$, we obtain a 4-colouring of C_i' as claimed.

Therefore, if T is a maximal independent subset of S such that $G(S-T)$ is empty or can be coloured in essentially only one way, then the 4-colourings of the various C_i' can be combined to give a 4-colouring of G, which is impossible. However, when $|S| \leqslant 3$, an easy analysis of cases shows that S always has such a maximal independent subset T. Hence the supposition that G is not 4-connected has led to a contradiction.

Since $G \not> K_5$, it follows from Proposition 1 that G must be planar. But then G is 4-colourable by (4*CC*), contrary to the choice of G. Hence (*H*) holds for all graphs G.

References

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