PRODUCER INCENTIVES IN COST ALLOCATION

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A general problem faced by both private firms and public enterprises is how to allocate the costs of common facilities fairly among the different goods and services produced. Any such cost accounting method can create incentives among product managers within the firm for altering the production function to their advantage. It is therefore both reasonable and desirable that a method reward increased efficiency by attributing lower unit costs to products whose marginal cost of production uniformly decreases. It is shown that there is only one "symmetric" method that satisfies this "monotonicity" principle—namely, the Aumann-Shapley price mechanism based on the Aumann-Shapley value for nonatomic games. This provides a new and simple axiomatization of this method without resorting to the usual assumption of additivity.

1. INTRODUCTION

How should a firm allocate the joint costs of production among the different goods and services that it produces? This well-known problem is faced by public utilities and other regulated enterprises where such allocations determine the prices they may charge; it is also a feature of unregulated firms that wish to monitor the performance of different divisions or product lines through decentralized cost accounting.

In theory, marginal cost pricing is the only pricing mechanism that is consistent with economic efficiency in the large. Unfortunately it is typically unworkable as a cost allocation method because marginal costs need not sum to total costs, as required for an allocation. Indeed, marginal cost pricing may not even cover costs. This possibility arises in natural monopolies characterized by increasing returns to scale and declining marginal costs, such as distribution networks for transport, electric power, water, and communications services.

This paper focuses on the incentives that different cost accounting methods create for the adoption of more efficient techniques of production. In a decentralized organization, any cost accounting procedure based on production costs inevitably creates incentives for innovation and risk taking by division or product managers. A reasonable goal of good management is to adopt a system of incentives that rewards individuals for making decisions that increase the firm’s overall profits and penalizes decisions that damage profits. Yet, as Martin Shubik pointed out in his 1962 paper on incentives and cost accounting in the firm, depending on the cost accounting system employed, “it is possible that an improvement in the efficiency of a department may damage the profit statement even though it increases the overall profitability of the firm” [11, p. 329]. A minimal requirement of a cost accounting method would seem to be that innovations which result in uniformly more efficient techniques of production for some division or product line—as measured by the marginal costs of producing that

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product—should not lead to higher imputed unit costs (and lower imputed profits).

Yet, as will be shown by example, plausible and commonly advocated methods fail this "incentives" test in even the simplest cases. This includes modified forms of marginal cost pricing as well as Ramsey pricing. In fact, there is only one method that does not fail the test in a wide sense—the Aumann-Shapley price mechanism derived from the Aumann-Shapley value for nonatomic games. An analogous result characterizes the Shapley value for cooperative games [12].

2. Cost Allocation, Marginal Cost Pricing, and Ramsey Pricing

Let $i = 1, 2, \ldots, n$, designate $n$ goods or services jointly produced by a firm, and let $f(x)$ be the total cost of producing $x_i \geq 0$ units of product $i$, $i = 1, 2, \ldots, n$. A pair $(f, \bar{x})$, where $\bar{x}_i > 0$ for all $i$ and $f$ is defined for all $x$, $0 \leq x \leq \bar{x}$, defines a cost allocation problem. We assume that $f$ has continuous first partial derivatives on $D(\bar{x}) = \{x \in \mathbb{R}^n: 0 \leq x \leq \bar{x}\}$, one-sided on the boundary, and that $f(0) = 0$. A cost allocation method is a functional $\varphi$ that to every problem $(f, \bar{x})$ associates a vector of unit product costs $\varphi(f, \bar{x}) = (c_1, \ldots, c_n) \in \mathbb{R}^n$ that exactly share all costs:

$$\sum_i \bar{x}_i \varphi_i(f, \bar{x}) = f(\bar{x}).$$

(1)

In a regulated enterprise, these costs may represent product prices and condition (1) is the requirement that total revenue equal total costs (including the costs of capital).

The marginal cost pricing solution is to set $\varphi_i(f, \bar{x}) = \partial f(\bar{x})/\partial x_i$ for $1 \leq i \leq n$. As previously noted, the difficulty with this approach is that (1) may not be satisfied. A plausible way out of this dilemma is to adjust the marginal costs by a proportional factor so that (1) is satisfied. The proportionally adjusted marginal cost pricing solution is

$$\varphi_i(f, \bar{x}) = \frac{\partial f(\bar{x})/\partial x_i}{\sum_{j=1}^n \bar{x}_j \partial f(\bar{x})/\partial x_j} f(\bar{x}).$$

(2)

This solution is actually a special case of another notion, Ramsey pricing [2, 8]. Ramsey prices have the property that the corresponding market demands lead to a level of production $\bar{x}$ that maximizes consumer surplus subject to the cost-sharing requirement (1). When demands for all products are independent, a necessary condition characterizing Ramsey prices is that

$$\varphi_i(f, \bar{x}) = (\partial f(\bar{x})/\partial x_i)/(1 + k/\eta_i(\bar{x}))$$

where $k$ is constant for all $i$ and $\eta_i(\bar{x})$ is the elasticity of demand for $i$ at $\bar{x}_i$. If all elasticities are equal, Ramsey prices are just proportional to marginal costs and are given by formula (2).
3. MONOTONICITY AND AUMAN-SHAPLEY PRICES

The flaw in proportionally adjusted marginal cost pricing is that it does not create the right incentives for producers because it is not monotonic with changes in the cost function. A simple example illustrates the difficulty.

Let there be three products \( i = 1, 2, 3 \) and suppose that the cost function in period 1 is

\[
f(x) = 14x_1 + x_2 + x_3 + (x_1 + x_2 + x_3)^2.
\]

Production costs consist of a linear component in each good, plus a joint component that requires the common use of a facility exhibiting increasing marginal costs. Suppose that the production manager for good 1 devises a scheme for producing that good without benefit of the joint facility. The new cost function is

\[
g(x) = 14x_1 + x_2 + x_3 + (x_2 + x_3)^2.
\]

Total production costs are now lower at all positive levels of output, and the saving is clearly attributable to good 1. But the unit costs as calculated by proportionally adjusted marginal cost pricing (or Ramsey pricing, assuming equal and perfectly inelastic demands) are

\[
\varphi_1(f, \bar{x}) = \frac{[14 + 2(\bar{x}_1 + \bar{x}_2 + \bar{x}_3)]f(\bar{x})}{14\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + 2(\bar{x}_1 + \bar{x}_2 + \bar{x}_3)^2},
\]

\[
\varphi_1(g, \bar{x}) = \frac{[14 + 2(\bar{x}_2 + \bar{x}_3)]g(\bar{x})}{14\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + 2(\bar{x}_2 + \bar{x}_3)^2}.
\]

Evaluated at \( \bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 1 \) we obtain \( \varphi_1(f, 1) = 14.7 < 15 = \varphi_1(g, 1) \) so the improvement has operated to 1's disadvantage.

The question naturally arises: does any method avoid the above paradox? To answer it one must first be able to unambiguously tell when an improvement in production efficiency has taken place with respect to some particular product. While various formulations are possible, one can say unequivocally that \( f \) is more efficient than \( g \) with respect to product \( i \) if the marginal cost of producing \( i \) is lower in \( f \) than it is in \( g \) at all levels of production.

A cost allocation procedure is monotonic (or incentive-compatible) if it never penalizes a division for adopting a new process that is uniformly more efficient to produce; that is, if for all problems \((f, \bar{x}), (g, \bar{x})\) and all \( i \):

\[
\frac{\partial f}{\partial x_i}(x) \leq \frac{\partial g}{\partial x_i}(x) \quad \forall x \in D(\bar{x}) \quad \text{implies} \quad \varphi_i(f, \bar{x}) \leq \varphi_i(g, \bar{x}).
\]

The same logic also allows a comparison between different products (or the same product masquerading in the guise of different subscripts) in alternative cost functions. We say that \( \varphi \) is symmetrically monotonic (or symmetrically incentive-compatible) if for all problems \((f, \bar{x}), (g, \bar{x})\) and all \( i, j \),

\[
\frac{\partial f}{\partial x_i}(x) \leq \frac{\partial g}{\partial x_j}(x) \quad \forall x \in D(\bar{x}) \quad \text{implies} \quad \varphi_i(f, \bar{x}) \leq \varphi_j(g, \bar{x}).
\]
Theorem 1: There is a unique cost allocation procedure that is symmetrically monotonic, namely
\[ \varphi^A(f, \bar{x}) = \int_0^1 \left( \frac{\partial f}{\partial x_i}(t\bar{x}) \right) dt. \]

The procedure \( \varphi^A \) is the Aumann–Shapley price mechanism treated by Billera and Heath [3], Mirman and Tauman [6, 7] and others (see [3–7]), derived from the Aumann–Shapley value for nonatomic games [1].

Proof: It is clear that the Aumann–Shapley price mechanism is symmetrically monotonic. Conversely, let \( \varphi \) be symmetrically monotonic. Consider the case where \( f \) has form \( f(x) = ax_1^{q_1}x_2^{q_2} \cdots x_n^{q_n} \), where the \( q_i \) are nonnegative integers, not all zero. We will show that for all \( a \) and all \( \bar{x} > 0 \),

(4) \[ \varphi_i(f, \bar{x}) = (aq_i\bar{x}_1^{q_1} \cdots \bar{x}_i^{q_i-1} \cdots \bar{x}_n^{q_n}) \Big/ \sum_n q_j \quad \text{for all } i. \]

If \( a = 0 \) then \( \partial f(x)/\partial x_i = 0 \) for all \( x \), so by symmetric monotonicity \( \varphi_i(f, \bar{x}) = \varphi_j(f, \bar{x}) \) for all \( i \neq j \). Since \( \sum_i x_i \varphi_i(f, \bar{x}) = 0 \) and \( \bar{x} > 0 \), it follows that \( \varphi_i(f, \bar{x}) = 0 \) for \( 1 \leq i \leq n \). By symmetric monotonicity conclude that for any problem \( (f, \bar{x}) \),

(5) \[ \frac{\partial f}{\partial x_i}(x) = 0 \quad \forall x \in D(\bar{x}) \quad \text{implies } \varphi_i(f, \bar{x}) = 0. \]

Now consider arbitrary \( a \). The proof proceeds by lexicographic induction on \( q = (q_1, \ldots, q_n) \) where \( q' <_{lex} q \) means that the first component \( k \) where \( q' \) differs from \( q \) has \( q'_k < q_k \).

Let \( Q \) be the set of all \( q \) for which (4) fails for some \( a \) and some \( \bar{x} > 0 \). If \( Q \) is nonempty then it has a least element \( q^* \). Choose \( i < n \). If \( q^*_i = 0 \) then by (5) \( \varphi_i(f, \bar{x}) = 0 \) so (4) holds. Thus \( q^*_i > 0 \). Define

\[ h(x) = \frac{aq_i^{q_i^*}}{q_i^{q_i^*} + 1} x_1^{q_1^*} x_2^{q_2^*-1} \cdots x_i^{q_i^*-1} \cdots x_n^{q_n^*}. \]

Since \( q^{**} = (q_1^*, \ldots, q_i^* - 1, \ldots, q_n^* + 1) <_{lex} q^* \), \( q^{**} \notin Q \), and \( h(x) \) satisfies the induction hypothesis. Since \( \partial h(x)/\partial x_n = \partial f(x)/\partial x_i \) for \( 0 \leq x \leq \bar{x} \), symmetric monotonicity implies that

\[ \varphi_i(f, \bar{x}) = \varphi_n(h, \bar{x}) = aq_i^* \bar{x}_1^{q_1^*} \cdots \bar{x}_i^{q_i^*-1} \cdots \bar{x}_n^{q_n^*} / \sum q_j^* \]

so \( f \) satisfies (4) for \( 1 \leq i \leq n - 1 \). By (1) it follows that (4) also holds when \( i = n \). This contradiction completes the induction for all single-term polynomials.

For a general polynomial \( P(x) \) let \( \bar{q} \) be the maximum power of \( x_i \) in \( P(x) \), \( 1 \leq i \leq n \), and apply a similar argument using lexicographic induction on \( (\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_n) \).

It remains only to show that \( \varphi \) is continuous in \( f \). Fix \( \bar{x} > 0 \) and let \( C^1(\bar{x}) \) be the Banach space of all continuously differentiable real-valued functions \( f \) defined
on $D = D(\bar{x})$ such that $f(0) = 0$, with norm

$$\|f\| = \sup_{x \in D} |f(x)| + \sum_{i=1}^{n} \sup_{x \in D} |\partial f(x)/\partial x_i|.$$ 

For any given $f$ and $\varepsilon > 0$ there is a polynomial $P_\varepsilon(x)$ such that $\|f - P_\varepsilon(x)\| < \varepsilon$. Therefore the polynomials

$$Q^-_\varepsilon(x) = P_\varepsilon(x) - \varepsilon \sum_{i=1}^{n} x_i \quad \text{and} \quad Q^+_\varepsilon(x) = P_\varepsilon(x) + \varepsilon \sum_{i=1}^{n} x_i$$

have the property that, for all $i$,

$$\frac{\partial Q^-_\varepsilon(x)}{\partial x_i} = \frac{\partial P_\varepsilon(x)}{\partial x_i} - \varepsilon \leq \frac{\partial f(x)}{\partial x_i} \leq \frac{\partial P_\varepsilon(x)}{\partial x_i} + \varepsilon = \frac{\partial Q^+_\varepsilon(x)}{x_i} \quad \forall x \in D.$$

By symmetric monotonicity

$$\varphi_i(Q^-_\varepsilon, \bar{x}) \leq \varphi_i(f, \bar{x}) \leq \varphi_i(Q^+_\varepsilon, \bar{x}) \quad \text{for all} \ i.$$

On polynomials $\varphi$ is equal to $\varphi^{AS}$ and is therefore additive, so

$$\varphi^{AS}_i(P_\varepsilon, \bar{x}) - \varepsilon \leq \varphi_i(f, \bar{x}) \leq \varphi^{AS}_i(P_\varepsilon, \bar{x}) + \varepsilon.$$

But $\varphi^{AS}(f, \bar{x})$ is continuous in its first argument, hence

$$\lim_{\varepsilon \to 0} \varphi^{AS}_i(P_\varepsilon, \bar{x}) = \varphi_i(f, \bar{x}) = \varphi^{AS}_i(f, \bar{x}).$$

Q.E.D.

4. OTHER CHARACTERIZATIONS OF AUAMANN-SHAPLEY PRICES

The usual axiomatization of the Aumann–Shapley price mechanism and its parent, the Shapley value [10] relies heavily on additivity. In the cost allocation context [3–7] the additivity principle says that if the cost function $f(x)$ can be decomposed into separate and independent components $f'(x) + f''(x)$ for all $x \in D$—such as investment costs plus operating costs, or capital costs plus labor costs—then allocating these costs separately should come to the same thing as allocating them jointly:

$$\varphi(f' + f'', \bar{x}) = \varphi(f', \bar{x}) + \varphi(f'', \bar{x}).$$

It is evident on its face that the Aumann–Shapley value is additive. The argument often given for additivity is that it guarantees a result independent of the particular way in which accounts categorize costs [3, 5]. This argument is perhaps more compelling for accountants than for those who must bear the costs.

Moreover, in addition to additivity several other crucial axioms are needed in the usual characterization of Aumann-Shapley prices. (See, however, Samet and Tauman [9], who replace additivity by two other conditions.) One is a version of monotonicity (called “positivity” in [7]) which says that

$$\frac{\partial f}{\partial x_i}(x) \geq 0 \quad \forall i \text{ and } \forall x \in D(\bar{x}) \quad \text{implies} \quad \varphi(f, \bar{x}) \geq 0.$$
That is, if \( f \) is monotone increasing in \( x \), then assessed costs should be nonnegative. This condition is closely related to, and has the same rationale as monotonicity, since it says that incremental increases in the costs of production should be associated with increases in assessed costs.

Instead of treating the zero cost function as the status quo, the above can be generalized to compare an arbitrary \( f \) and \( g \) as follows:

\[
\frac{\partial f(x)}{\partial x_i} \leq \frac{\partial g(x)}{\partial x_i} \quad \forall i \text{ and } \forall x \in D(\bar{x}) \quad \text{implies } \varphi(f, x) \leq \varphi(q, x).
\]

The spirit of this axiom and monotonicity (3) is the same. However, (3) is stronger in that it applies to a uniform change in the cost of producing a particular good, whereas (7) applies only when the cost of producing all goods changes in the same direction. In effect, (3) allows one to compare situations in which \( i \)'s technology improves and simultaneously \( j \)'s deteriorates; whereas (7) only allows comparisons in which all products improve or deteriorate together. An axiom such as (7) is not enough to characterize Aumann–Shapley prices since it is satisfied by such schemes as \( \varphi_i = f(\bar{x})/n\bar{x}_i \) i.e., divide costs equally among the product lines.

To characterize Aumann–Shapley prices using (3) one needs a condition on the allocation of costs when different products are aggregated.

Suppose for example that \( g(y_1, \ldots, y_m) \) is the joint cost of producing \( m \) types of gasoline, where the quantity \( y_i \geq 0 \) of each type is a blend of \( n \) refinery grades \( x_1, \ldots, x_n \), \( y_i = \sum_{j=1}^{n} a_{ij} x_j \), all \( a_{ij} \geq 0 \). Costs could be written just as well in terms of \( x_1, \ldots, x_n \geq 0 \) as follows: \( f(x_1, \ldots, x_n) = g(Ax) \). The procedure \( \varphi \) is aggregation invariant if costs are aggregated in the same manner as product quantities, that is, if \( \varphi(f, \bar{x}) = (\varphi(g, A\bar{x}))A \) whenever \( A \) is a nonnegative matrix such that \( A\bar{x} = \bar{y} > 0 \) and

\[
f(x) = g(Ax) \quad \text{for all } 0 \leq x \leq \bar{x} \text{ and } 0 \leq y \leq \bar{y}.
\]

This is a somewhat stronger version of aggregation invariance as defined in [3].

**Theorem 2:** The Aumann–Shapley price mechanism is the unique allocation procedure that is aggregation invariant and monotonic.

**Proof:** We know that the Aumann–Shapley price mechanism is incentive-compatible. That it is aggregation invariant follows from Corollary 4 in [3].

Conversely, suppose that \( \varphi \) is aggregation invariant and monotonic. Consider the case where the cost function \( f(x) \) is polynomial in \( x_1, \ldots, x_n \) and \( f(0) = 0 \). As shown in [1], \( f(x) \) can be written in the form

\[
f(x) = \sum_{k=1}^{I} c_k P_k(x) \quad \text{where } P_k(x) = \left( \sum_{j=1}^{n} b_{kj} x_j \right)^{r_k},
\]

where \( r_k \) is a positive integer for all \( k \) and all \( b_{kj} \geq 0 \).

Let the index \( I \) of \( f \) be the number of nonzero terms in any shortest expression for \( f(x) \) of form (8). Assuming the theorem is false for polynomial \( f \), let \( I^* \) be
the least index \( I^* \geq 0 \) for which there exists a pair \((f, \bar{x})\), \( \bar{x} > 0 \) and \( f \) polynomial with index \( I^* \) such that \( \varphi(f, \bar{x}) \neq \varphi^{AS}(f, \bar{x}) \).

If \( I^* = 0 \), then \( f \) has index zero and \( f \) is identically zero. Let \( y = \sum_{i=1}^{n} x_i \) and \( g(y) = f(x) \). Then \( g \) is identically zero and by cost-sharing, \( y \varphi(g, \bar{y}) = g(\bar{y}) = 0 \), so for all \( \bar{y} > 0 \), \( \varphi(g, \bar{y}) = 0 \). By aggregation invariance it follows that for all \( i \), \( \varphi_i(f, \bar{x}) = 0 = \varphi_i^{AS}(f, \bar{x}) \).

With \( I^* > 0 \), consider an expression (8) for \( f \). Let \( J^+ \) be the subset of indices \( i \) for which \( x_i \) has a positive coefficient in every term \( c_k P_k(x) \), and let \( J^0 \) be the set of all other indices. If \( i \in J^0 \), then by deleting the terms \( c_k P_k(x) \) in which \( x_i \) has coefficient 0 we obtain a polynomial function \( g(x) \) with index less than \( I^* \) such that for this particular \( i \)

\[
(9) \quad \frac{\partial g}{\partial x_i}(x) = \frac{\partial f}{\partial x_i}(x) \quad \text{for all } x \in D(\bar{x}).
\]

Thus by the induction assumption \( \varphi_i(g, \bar{x}) = \varphi_i^{AS}(g, \bar{x}) \).

Incentive-compatibility (3) implies in particular that for given \( \bar{x} \), \( \varphi_i(f, x) \) depends only on the function \( \partial f(x)/\partial x_i \); that is, if (9) holds for some \( i \), then \( \varphi_i(f, \bar{x}) = \varphi_i(g, \bar{x}) \). Thus conclude that for all \( i \in J^0 \),

\[
(10) \quad \varphi_i(f, \bar{x}) = \varphi_i(g, \bar{x}) = \varphi_i^{AS}(g, \bar{x}) = \varphi_i^{AS}(f, \bar{x}).
\]

If \( J^+ \) is empty, it follows that \( \varphi(f, \bar{x}) = \varphi^{AS}(f, \bar{x}) \) contrary to assumption. If \( J^+ \) consists of a single index, say \( i \), then the fact that \( \varphi \) and \( \varphi^{AS} \) must both satisfy (1) implies (since \( \bar{x}_i > 0 \)) that \( \varphi_i(f, \bar{x}) = \varphi_i^{AS}(f, \bar{x}) \).

There remains the case \( |J^+| \geq 2 \). Without loss of generality let \( 1 \in J^+ \), so \( b_{k_1} > 0 \) for all \( k \), \( 1 \leq k \leq I^* \). Define an aggregate product \( y_1 \) as follows:

\[
y_1 = x_i + \sum_{j=2}^{n} a_{ij}x_j \quad \text{where } a_{ij} = \min_{1 \leq k \leq I^*} \{b_{k_j}/b_{k_1}\} \geq 0.
\]

Let \( y_j = x_j \) for \( 2 \leq j \leq n \), and let \( A \) be the nonnegative, \( n \times n \) matrix such that \( y = Ax \).

Let

\[
(11) \quad g(y) = \sum_{k=1}^{I^*} c_k Q_k(y)
\]

where

\[
Q_k(y) = \left( b_{k_1}y_1 + \sum_{j=2}^{n} (b_{kj} - b_{k_1}a_{ij})y_j \right)^r_k \quad \text{for } 1 \leq k \leq I^*.
\]

Then \( g(Ax) = f(x) \). By choice of the coefficients \( a_{ij} \), \( y_1 \) is the only variable that has positive coefficients in every term of (11). Hence by the preceding argument conclude that for all \( \bar{y} > 0 \) and all \( i \),

\[
\varphi_i(g, \bar{y}) = \varphi_i^{AS}(g, \bar{y}).
\]
For all \( \bar{x} > 0 \) let \( \bar{y} = A\bar{x} > 0 \) and conclude by aggregation invariance for \( \varphi \) and \( \varphi^{AS} \) that

\[
\varphi(f, \bar{x}) = \varphi(g, \bar{y}) A = \varphi^{AS}(g, \bar{y}) A = \varphi^{AS}(f, \bar{x}),
\]

a contradiction. This establishes the theorem for polynomial \( f \) and the proof is completed for \( C^1 \)-functions \( f \) just as in the proof of Theorem 1. Q.E.D.

5. CONCLUSION

The desirable feature usually emphasized about the Aumann–Shapley price mechanism as a method of cost allocation is its independence of the way costs are broken down into different categories for accounting purposes. While appealing as an accounting principle it is less compelling on economic grounds. This paper demonstrates that the Aumann–Shapley price mechanism can be characterized by another property (monotonicity) that does have considerable economic significance. Namely, in a decentralized firm where innovations occur through the initiative of individual product or division managers, the Aumann–Shapley price mechanism is essentially the only one that consistently rewards more efficient processes of production by imputing to them lower unit costs. This does not say, of course, that the Aumann–Shapley allocation cannot in theory be manipulated: it might be possible for individual decision-makers to misrepresent privately held information about the production technology to their own advantage, or to artificially raise the apparent cost of other products to their own advantage. Rather, it says that it is the only method that attaches no penalty to diligence, and no reward to negligence.

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