Non-Cooperative Games with Many Players*

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1 Introduction

Shapiro-Shapley introduce their 1961 memorandum (published 17 years later as Shapiro-Shapley (1978)) with the remark that "institutions having a large number of competing participants are common in political and economic life," and cite as examples "markets, exchanges, corporations (from the shareholders viewpoint), Presidential nominating conventions and legislatures." They observe, however, that "game theory has not yet been able so far to produce much in the way of fundamental principles of "mass competition" that might help to explain how they operate in practice," and that it might be "worth while to spend a little effort looking at the behavior of existing n-person solution concepts, as n becomes very large." In this, they echo both von Neumann-Morgenstern (1944) and Kuhn-Tucker (1950),[1] and anticipate Mas-Colell (1998).[2]

Von Neumann-Morgenstern (1944) saw the number of participants in a game as a variable, and presented it as one determining the "total set" of variables of the problem. "Any increase in the number of variables inside a participant's partial set may complicate our problem technically, but only technically; something of a very different nature happens when the number of participants – i.e., of the partial sets of variables – is increased." After remarking that the complications arising from the "fact that every participant is influenced by the anticipated reactions of the others to his own measures" are "most strikingly the crux of the matter," the authors write:

> When the number of participants becomes really great, some hope emerges that the influence of every particular participant will become negligible, and that the above difficulties may recede and a more conventional theory become possible. Indeed, this was the starting point of much of what is best in economic theory. It is a well known phenomenon in many branches of the exact and physical sciences that very great numbers are often easier to handle than those of medium size.[3] This is of course due to the excellent possibility of applying the laws of statistics and probabilities in the first case.

Two further points are explicitly noted. First, a satisfactory treatment of such "populous games" may require "some radical theoretical innovations – a really fundamental reopening of [the] subject." Second, "only after the theory of moderate numbers has been satisfactorily developed will it be possible to decide whether extremely great numbers of participants will simplify the situation."[4] However, an optimistic prognosis is evident.[5]

Nash (1950) contains in the space of five paragraphs a definitive formulation of the theory of non-cooperative games with an arbitrary finite number of players. This "theory, in contradistinction [to that of von Neumann-Morgenstern], is based on the *absence* of coalitions in that it is assumed that each participant acts independently, without collaboration and communication from any of the others. The non-cooperative idea will be implicit, rather than explicit. The notion of an *equilibrium point* is the basic ingredient in our theory. This notion yields a generalization of the concept of a solution of a two-person zero sum game." In a treatment that is remarkably modern, Nash presented a theorem on the existence of equilibrium in an n-person game, where n is an arbitrary finite number of participants or players. In addition to the von-Neumann-Morgenstern book, the only other reference is to Kakutani's generalization of Brouwer's fixed point theorem.[6]

With Nash's theorem in place, all that an investigation into non-cooperative games with many players requires is a mathematical framework that fruitfully articulates "many" and the attendant notions of "negligibility" and "inappreciability." This was furnished by Milnor-Shapley (1961) in the context of cooperative game theory. They presented an idealized limit game with a "continuum of infinitesimal minor players ..., an 'ocean,' to emphasize the almost total absence of order or cohesion." The oceanic players were represented in measure-theoretic terms and their "voting power expressed as a measure, defined on the measurable subsets of the ocean." The authors did not devote any space to the justification of the notion of a continuum of players; they were clear about the "benefits of dealing directly with the infinite-person game, instead of with a sequence of finite approximants."[7]

With the presumption that "models with a continuum of *players* (traders in this instance) are a relative novelty,[8] [and that] the idea of a continuum of traders may seem outlandish to the reader," Aumann (1964) used such a model for a successful formalization of Edgeworth's 1881 conjecture on the relation of core and competitive allocations. Aumann's discussion proved persuasive because the framework yielded an equivalence between these two solution concepts, and thereby affected a qualitative change in the character of the resolution of the problem. Aumann argued that "the most natural model for this purpose contains a *continuum* of participants, similar to the continuum of points on a line or the continuum of particles in a fluid." After all, "continuous models are nothing new in economics or game theory, [even though] it is usually parameters such as price or strategy that are allowed to vary continuously." More generally, he stressed the "the power and simplicity of the continuum-of-players methods in describing mass phenomena in economics and game theory," and saw his work "primarily as an illustration of this method as applied to an area where no other treatment seemed completely satisfactory." In Aumann (1964) four methodological points are made explicit.

1. The continuum can be considered an approximation to the "true" situation in which there is a large but finite number of particles (or traders or strategies or possible prices). In economics, as in the physical sciences, the study of the ideal state has proved very fruitful, though in practice it is, at best, only approximately achieved.[9]

2. The continuum of traders is not merely a mathematical exercise; it is the expression of an economic idea. This is underscored by the fact that the chief result holds *only* for a continuum of traders – it is false for any finite number.

3. The purpose of adopting the continuous approximation is to make available the powerful and elegant methods of a branch of mathematics called "analysis," in a situation where treatment by finite methods would be much more difficult or hopeless.

4. The choice of the unit interval as a model for the set of traders is of no particular significance. In technical terms, T can be any measure space *without atoms.* The condition that T have no atoms is precisely what is needed to ensure that each individual trader have no influence.[10]

In their work on the elimination of randomization (purification) in statistics and game theory, Dvoretsky-Wald-Wolfowitz (1950) had already emphasized the importance of Lyapunov's theorem,[11] and explicitly noted that the "non-atomicity hypothesis is indispensable [and that] it is this assumption that is responsible for the possibility to disregard mixed strategies in games ... opposed to the finite games originally treated by J. von Neumann."[12] With the ideas of purification and the continuum-of-traders in place, a natural next step was an extension of Nash's theorem to show the existence of a pure strategy equilibrium. This was accomplished in Schmeidler (1973) in the setting of an arbitrary finite number of pure strategies. Since there does not exist such an equilibrium in general finite player games,[13] this result furnished another example of a qualitative change in the resolution of the problem. However, the analysis of situations with a continuum of actions – the continuous variation in the price or strategy variables referred to by Aumann[14] – eluded the theory. In this chapter, we sketch the shape of a general theory that encompasses, in particular, such situations. Our focus is on non-cooperative games, rather than on perfect competition, and primarily on how questions of the existence of equilibria for such games dictate, and are dictated by, the mathematical framework chosen to formalize the idea of "many" players. That being the case, we keep the methodological pointers delineated in this introduction constantly in view. The subject has a technical lure and it is important not to be unduly diverted by it. At the end of the chapter we indicate applications but leave it to the reader to delve more deeply into the relevant references. This is only because of considerations of space; of course, we subscribe to the view that there ought to be a constant interplay between the framework and the economic and game-theoretic phenomena that it aspires to address and explain.

2 Antecedent Results

We motivate the need for a measure-theoretic structure on the set T of players' names by considering a model in which no restrictions are placed on the cardinality of T. For each player t in T, let the set of actions be given by A_t, and the payoff function by $u_t : A \longrightarrow \mathbb{R}$, where A denotes the product $\Pi_{t \in T} A_t$. Let the partial product $\Pi_{t \in T, t \neq i} A_t$ be denoted by A_{-i}. We can present the following result.[15]

Theorem 1 *Let $\{A_t\}_{t \in T}$ be a family of nonempty, compact convex sets of a Hausdorff topological vector space, and $\{u_t\}_{t \in T}$ be a family of real-valued continuous functions on A such that for each $t \in T$, and for any fixed $a_{-t} \in A_{-t}$, $u_t(\cdot, a_{-t})$ is a quasi-concave function on A_t. Then there exists $a^* \in A$, such that for all t in T, $u_t(a^*) \geq u_t(a, a^*_{-t})$ for all a in A_t.*

Nash (1950, 1951) considered games with finite action sets, and his focus on mixed strategy equilibria led him to probability measures on these action sets and to the maximization of expected utilities with respect to these measures. Theorem 1 is simply an observation that if the finiteness hypothesis is replaced by convexity and compactness, and the linearity of the payoff functions by quasi-concavity, his basic argument remains valid.[16] Once the closedness of the "one-to-many mapping of the [arbitrary] product space to itself" is established, we can invoke the full power of Tychonoff's celebrated theorem on the compactness of the product of an arbitrary set of compact spaces, and rely on a suitable extension of Kakutani's fixed point theorem.[17] The upper semicontinuity result in Fan (1952), and the fixed point theorem in Fan (1952) and Glicksberg (1952) furnish these technical supplements.

However, with Theorem 1 in hand, we can revert to Nash's setting and exploit the measure-theoretic structure available on each action set. For each player t, consider the measurable space $(A_t, \mathcal{B}(A_t))$, where $\mathcal{B}(A_t)$ is the Borel σ-algebra generated by the topology on A_t. Let $\mathcal{M}(A_t)$ be the set of Borel probability measures on A_t endowed with the weak* topology.[18] Without going into technical details of how to manufacture new probability spaces $(A, \mathcal{B}(A), \Pi_{s\in T}\mu_s)$ and $(A_{-t}, \mathcal{B}(A_{-t}), \Pi_{s\neq t}\mu_s)$ from $\{(A_t, \mathcal{B}(A_t), \mu_t)\}_{t\in T}$, and the fine points of Fubini's theorem on the interchange of integrals,[19] we can deduce[20] the following result from Theorem 1 by working with the action sets $\mathcal{M}(A_t)$ and with an explicit functional form of the payoff functions u_t.

Corollary 1 *Let $\{A_t\}_{t\in T}$ be a family of nonempty, compact Hausdorff spaces, and $\{v_t\}_{t\in T}$ be a family of real-valued continuous functions on A. Then there exists $\sigma^* = (\sigma_t^* : t \in T) \in \Pi_{t\in T}\mathcal{M}(A_t)$ such that for all t in T,*

$$u_t(\sigma^*) = \int_A v_t(a) d\Pi_{t\in T}\sigma_t^* \geq u_t(\sigma, \sigma_{-t}^*) \text{ for all } \sigma \text{ in } \mathcal{M}(A_t).$$

The question is whether any substantive meaning can be given to the continuity hypothesis on the functions v_t? The following result[21] shows that it is not merely a technical requirement but has a direct implication for the formalization of player interdependence.

Definition 1 $v : A \longrightarrow \mathbb{R}$ *is finitely determined if for all $a, b \in A$, $v(a) = v(b)$ if there exists a finite subset F of T such that $a_t = b_t$ for all $t \notin F$. v is almost finitely determined if for every $\varepsilon > 0$, there exists a finitely determined function v_ε such that* $\sup_{a\in A} |v_\varepsilon(a) - v(a)| < \varepsilon$.

Proposition 1 *For a real-valued function $v : A \longrightarrow \mathbb{R}$, the following conditions are equivalent:*
(1) v is almost finitely determined.
(2) v is a continuous function on the space A endowed with the product topology.
(3) v is integrable with respect to any $\sigma \in \mathcal{M}(A)$ and its integral is a continuous function on $\mathcal{M}(A)$.

Thus, if we conceive of a finite set of players as a "negligible" set, the hypothesis of continuity in the product topology implies strong restrictions on how player interaction is formalized. If individual payoffs depend on the actions of a "non-negligible" set of players so that the continuity hypothesis is violated, there may not exist any Nash equilibrium in pure or in mixed strategies. The following example due to Peleg (1969) illustrates this observation.[22]

Example 1: Consider a game in which the set of players' names T is given by the set of positive integers $\mathbb{N}$, the action set A_t by the set $\{0, 1\}$, and the individual payoffs by functions on actions that equal the action or its negative, depending on whether the sum of the actions of all the other players is respectively finite or infinite. Note that these functions are not continuous in the product topology.[23]

There is no pure strategy Nash equilibrium in this game. If the sum of all of the actions is finite in equilibrium, all players could not be playing 1, and players playing 0 would gain by playing 1. On the other hand, if the sum of all of the actions is infinite, all players could not be playing 0, and players playing 1 would gain by playing 0.

The more interesting point is that this game does not have any mixed strategy Nash equilibrium either. If $\sigma^* \in \Pi_{s \in T} \mathcal{M}(\{0,1\})$ is such an equilibrium, there must exist a player t, and a mixed strategy $(1-p, p)$, $0 < p < 1$, such that her payoff in equilibrium is given by

$$u_t(\sigma, \sigma^*_{-t}) = \int_{A_{-t}} [\sigma(\{0\}) v_t(0, \sum_{s \neq t} a_s) + \sigma(\{1\}) v_t(1, \sum_{s \neq t} a_s)] d\sigma^*_{-t} = p \int_{A_{-t}} v_t(1, \sum_{s \neq t} a_s) d\sigma^*_{-t},$$

where a_s denotes the action of player s. Since an individual player's payoff depends on whether $\sum_{s \neq t} a_s$ converges or diverges, player t obtains p or $-p$ as a consequence of the following zero-one law.[24]

Proposition 2 *Let $(\Omega, \mathcal{F}, P)$ be a probability space, and X_n be a sequence of independent random variables. Then the series $\sum_{n \in \mathbb{N}} X_n(\omega)$ converges or diverges for P-almost all $\omega \in \Omega$.*

In either case, player t would gain by playing a pure strategy (1 or 0 respectively), and hence σ^* could not be a mixed strategy equilibrium. ■

While the exploitation of the independence hypothesis in Example 1 is fully justified in its non-cooperative context, the fact that an individual player does not explicitly randomize on whether the sum of others' actions converges or diverges is less justifiable.[25] The important question to ask, however, is whether the above formalization of a "large" game is merely of technical interest; or does it point to something that is false for the finite case but true for the ideal, and if so, to something that we can learn about the finite case from the ideal?

3 Interactions based on Distributions of Individual Responses

In Example 1, the set of players' names can be conceived as an infinite (but σ-finite) measure space consisting of a counting measure on the power set of $\mathbb{N}$, but it is precisely this lack of finiteness that rules out consideration of situations in which a player's payoff depends in a well-defined way on the proportion of other players taking a specific action. Such an idea admits of a precise formulation if a measure-theoretic structure on the set of players' names is explicitly brought to the fore in the form of an atomless probability space $(T, \mathcal{T}, \lambda)$, with the atomless assumption formalizing the "negligible" influence of each individual player. However, what needs to be underscored is that λ is a countably additive, rather than a finitely additive, measure.[26]

A game is now simply a random variable from T to an underlying space of characteristics, and its Nash equilibrium another random variable from T to a common action set A.[27] We shall also adopt as a working hypothesis, until Section 10, Aumann's (1964) statement that the "measurability assumption is of technical significance only and constitutes no economic restriction. Nonmeasurable sets are extremely "pathological"; it is unlikely that they would occur in the context of an economic model."

3.1 A Basic Result

The set of players is divided into ℓ groups or institutions,[28] with $T_1, \ldots, T_\ell$ being a partition of T with positive λ-measures $c_1, \ldots, c_\ell$. For each $1 \leq i \leq m$, let λ_i be the probability measure on T_i such that for any measurable set $B \subseteq T_i$, $\lambda_i(B) = \lambda(B)/c_i$. We assume A to be a countable compact metric space.[29] Let $\mathcal{U}_A^d$ be the space of real-valued continuous functions on $A \times \mathcal{M}(A)^\ell$, endowed with its sup-norm topology and with $\mathcal{B}(\mathcal{U}_A^d)$ its Borel σ-algebra (the superscript d denoting "distribution"). This is the space of player characteristics, with the payoff function of each player depending on her action as well as on the distribution of actions in each of the ℓ institutions. We now have all the terminology we need to present.[30]

Theorem 2 *Let $\mathcal{G}^d$ be a measurable map from T to $\mathcal{U}_A^d$. Then there exists a measurable function $f : T \longrightarrow A$ such that for λ-almost all $t \in T$,*

$$u_t(f(t), \lambda_1 f_1^{-1}, \cdots, \lambda_\ell f_\ell^{-1}) \geq u_t(a, \lambda_1 f_1^{-1}, \cdots, \lambda_\ell f_\ell^{-1}) \text{ for all } a \in A,$$

where $u_t = \mathcal{G}^d(t) \in \mathcal{U}_A^d$, f_i the restriction of f to T_i, and $\lambda_i f_i^{-1}$ the induced distribution on A.

Since Theorem 2 is phrased in terms of distributions, it stands to reason that the most relevant mathematical tools needed for its proof will revolve around the distribution of a correspondence. What is interesting is that a theory for such an object can be developed on the basis of the "marriage lemma". We turn to this.

3.2 The Marriage Lemma and the Distribution of a Correspondence

Halmos-Vaughan (1950) introduce the marriage lemma by asking for "conditions under which it is possible for each boy to marry his acquaintance if each of a (possibly infinite) set of boys is acquainted with a finite set of girls?" A general answer going beyond specific counting measures is available in the following result.[31]

Proposition 3 *Let I be a countable index set, $(T_\alpha)_{\alpha\in I}$ a family of sets in $\mathcal{T}$, and $\Lambda = (\tau_\alpha)_{\alpha\in I}$ a family of non-negative numbers. There exists a family $(S_\alpha)_{\alpha\in I}$ of sets in $\mathcal{T}$ such that for all $\alpha,\ \beta \in I,\ \alpha \neq \beta$, one has $S_\alpha \subseteq T_\alpha,\ \lambda(S_\alpha) = \tau_\alpha,\ S_\alpha \cap S_\beta = \emptyset$ if and only if for all finite subsets I_F of I, $\lambda(\bigcup_{\alpha\in I_F} T_\alpha) \geq \sum_{\alpha\in I_F} \tau_\alpha$.*

We can use Proposition 3 to develop results on the non-emptiness, purification, convexity, compactness and upper semicontinuity of the distribution of a correspondence as is required for the application of fixed-point theorems.[32] However the countability hypothesis on the range of a correspondence deserves special emphasis; all of the results reported below are false for particular correspondences from the unit Lebesgue interval to an interval,[33] the former denoted in the sequel by $([0,1], \mathcal{B}([0,1]), \nu)$.

A correspondence F from T to A is said to be measurable if for each $a \in A$, $F^{-1}(\{a\}) = \{t \in T : a \in F(t)\}$ is measurable. A measurable function f from $(T, \mathcal{T}, \lambda)$ to X is called a measurable selection of F if $f(t) \in F(t)$ for all $t \in T$. F is said to be closed- (compact-) valued if $F(t)$ is a closed (compact) subset of X for all $t \in T$, and its distribution is given by

$$\mathcal{D}_F = \{\lambda f^{-1} : f \text{ is a measurable selection of } F\}.$$

We can now present a simple and direct translation of Proposition 3 into a basic result on the existence of selections.

Proposition 4 *If F is measurable and $\tau \in \mathcal{M}(A)$, then $\tau \in \mathcal{D}_F$ if and only if for all finite $B \subseteq A$, $\lambda(F^{-1}(B)) \geq \tau(B)$.*

Proposition 3 also yields a result on purification.[34] The integral is the standard Lebesgue integral and $\{a_i : i \in \mathbb{N}\}$ is the list of all of the elements of A.

Proposition 5 *Let g be a measurable function from T into $\mathcal{M}(A)$, and $\tau \in \mathcal{M}(A)$ such that for all $B \subseteq A$, $\tau(B) = \int_{t\in T} g(t)(B) d\lambda$. If G is a correspondence from T into A such that for all $t \in T$, $G(t) = supp\ g(t) = \{a_i \in A : g(t)(\{a_i\}) > 0\}$, then there exists a measurable selection $\bar{g}$ of G such that $\lambda \bar{g}^{-1} = \tau$.*

After a preliminary definition, we can present basic properties of the object $\mathcal{D}_F$.

Definition 2 *A correspondence G from a topological space Y to another topological space Z is said to be upper semicontinuous at $y_0 \in Y$ if for any open set U which contains $G(y_0)$, there exists a neighborhood V of y_0 such that $y \in V$ implies that $G(y) \subseteq U$.*

Proposition 6 *(i) For any correspondence F, $\mathcal{D}_F$ is convex. (ii) If F is closed-valued, then $\mathcal{D}_F$ is closed, and hence compact, in the space $\mathcal{M}(A)$. (iii) If Y is a metric space, and for each fixed $y \in Y$, $G(\cdot, y)$ is a closed-valued measurable correspondence from T to A such that $G(t, \cdot)$ is upper semicontinuous on the metric space Y for each fixed $t \in T$, then $\mathcal{D}_{G(\cdot,y)}$ is upper semicontinuous on Y.*

3.3 Sketch of Proofs

The convexity assertion in Proposition 6 is a simple consequence of Proposition 3. However, the other two assertions rely on what can be referred to as an analogue of Fatou's lemma, which is itself a direct consequence of Proposition 3.[35]

The proof of Theorem 2 follows Nash (1950) in its essentials; we now look for a fixed point in the product space $\mathcal{M}(A)^\ell$, and consider the one-to-many best-response (countering) mapping from $T \times \mathcal{M}(A)^\ell$ into A given by

$$(t, \mu_1, \cdots, \mu_\ell) \longrightarrow F(t, \mu_1, \cdots, \mu_\ell) = \text{Arg Max}_{a \in A} u_t(a, \mu_1, \cdots, \mu_\ell).$$

The continuity and measurability assumptions on u_t allow us to assert the upper semicontinuity of $F(t, \cdots)$ and guarantee the existence of a measurable selection from $F(\cdot, \mu_1, \cdots, \mu_\ell)$.[36] We focus on the objects $\mathcal{D}_{F^i(\cdot,\mu_1,\cdots,\mu_\ell)}$ and $G(\mu_1, \cdots, \mu_\ell) = \Pi_{i=1}^{\ell} \mathcal{D}_{F^i(\cdot,\mu_1,\cdots,\mu_\ell)}$, where $F^i(t, \mu_1, \cdots, \mu_\ell) = F(t, \mu_1, \cdots, \mu_\ell)$ for each $t \in T_i$, and finish the proof by applying the Fan-Glicksberg fixed-point theorem to the one-to-many mapping $G : \mathcal{M}(A)^\ell \longrightarrow \mathcal{M}(A)^\ell$.

4 Two Special Cases

Theorems 1 and 2 concern large non-anonymous games in that each player is identified by a particular name or index t belonging to a set T. In this section, we focus on Theorem 2 and draw out its implication for two specific contexts: one where a player is also parametrized by the information at his disposal; and another anonymous setting where a player has no identity other than his characteristics. The atomlessness assumption now formalizes "dispersed" or "diffused" characteristics rather than "numerical negligibility."

4.1 Finite Games with Independent Private Information

Building on the work of Harsanyi (1967-68, 1973) and of Dvoretsky, Wald and Wolfowitz already referred to above, Milgrom-Weber (1981, 1985) and Radner-Rosenthal (1982) use the hypothesis of independence to present a formulation of games with incomplete information.[37]

In this subsection, we show how the dependence of individual payoffs on induced distributions in this model allows us to invoke the purification and existence results furnished as Proposition 5 and Theorem 2 above.

A *game with private information* consists[38] of a finite set I of ℓ players, each of whom is endowed with an identical action set A,[39] an information space $(\Omega, \mathcal{F}, \mu)$ where $(\Omega, \mathcal{F})$ is constituted by the product space $(\Pi_{i \in I}(Z_i \times X_i), \Pi_{i \in I}(\mathcal{Z}_i \otimes \mathcal{X}_i))$, and a utility function $u_i : A^\ell \times X_i \longrightarrow \mathbb{R}$. For any point $\omega = (z_1, x_1, \cdots, z_\ell, x_\ell) \in \Omega$, let $\zeta_i(\omega) = z_i$ and $\chi_i(\omega) = x_i$.

A *mixed strategy* for player i is a measurable function from Z_i to $\mathcal{M}(A)$. If the players play the mixed strategies $\{g_i\}_{i \in I}$, the resulting expected payoff to the i^{th} player is given by

$$U_i(g) \equiv \int_{\omega \in \Omega} \int_{a_\ell \in A} \cdots \int_{a_1 \in A} u_i(a_1, \cdots, a_\ell, \chi_i(\omega)) g_1(\zeta_1(\omega); da), \cdots, g_\ell(\zeta_\ell(\omega); da) \mu(d\omega).$$

A *pure strategy* for player i is simply a measurable function from Z_i to A. An *equilibrium in mixed strategies* is a vector of mixed strategies $\{g_i^\star\}_{i \in I}$, such that $U_i(g^\star) \geq U_i(g_i, g_{-i}^\star)$ for any mixed strategy g_i for player i. An equilibrium $b^\star$ in pure strategies is a *purification* of an equilibrium b in mixed strategies, if for each player i, $U_i(b) = U_i(b^\star)$.

Corollary 2 *If, for every player i, (a) the distribution of ζ_i is atomless, and (b) the random variables $\{\zeta_j : j \neq i\}$ together with the random variable $\xi_i \equiv (\zeta_i, \chi_i)$ form a mutually independent set, then every equilibrium has a purification.*

Proof: Apply the change-of-variables formula and the independence hypothesis to rewrite the individual payoff functions in a form that satisfies the hypothesis of Proposition 5. Check that the pure strategy furnished by its conclusion yields a purification of the original equilibrium. ■

Corollary 3 *Under the hypotheses of Corollary 2, there exists an equilibrium in pure strategies if for every player i, (i) $u_i(\cdot, \chi_i(\omega))$ is a continuous function on A^ℓ for μ-almost all $\omega \in \Omega$, and (ii) there is a real-valued integrable function h_i on $(\Omega, \mathcal{F}, \mu)$ such that μ-almost all $\omega \in \Omega$, $\|u_i(a, \chi_i(\omega))\| \leq h_i(\omega)$ holds for every $(a_1, \cdots, a_\ell) \in A^\ell$.*

Proof: By an appeal to the change-of-variable formula and the independence hypothesis, rewrite the individual payoff functions in the form required in Theorem 2. Check that all of the hypotheses of this theorem are satisfied, and that the equilibrium furnished by its conclusion is also an equilibrium in pure strategies. ■

We conclude with the observation that the above results are false without the independence hypothesis or the cardinality restriction on the action set.[40]

4.2 Large Anonymous Games

Once the space of characteristics has been formalized as the measurable space, $(\mathcal{U}_A^d, \mathcal{B}(\mathcal{U}_A^d))$ in Section 3 with $\ell = 1$ for example, it is natural to consider a game as simply a probability measure on such a space.[41] In this section, we show how the non-anonymous setting of Section 3 sheds light on the anonymous formulation of Mas-Colell (1984a).[42] The hypothesis of a countable compact metric action set A remains in force in this subsection.

A *large anonymous game* is a probability measure μ on the measurable space of characteristics, and it is *dispersed* if μ is atomless. A probability measure τ on the product space $(\mathcal{U}_A^d \times A)$ is a *Cournot-Nash equilibrium distribution (CNED)* of the large anonymous game μ if the marginal of τ on $\mathcal{U}_A^d$, $\tau_{\mathcal{U}}$, is μ, and if $\tau(B_\tau) = 1$ where

$$B_\tau = \{(u,a) \in (\mathcal{U}_A^d \times A) : u(a, \tau_A) \geq u(x, \tau_A) \text{ for all } x \in A\},$$

τ_A the marginal of τ on A. A CNED τ can be *symmetrized* if there exists a measurable function $f : \mathcal{U}_A^d \longrightarrow A$ and another CNED τ^s such that $\tau_A = \tau_A^s$ and $\tau^s(\text{Graph}_f) = 1$, where Graph_f is simply the set $\{(u, f(u)) \in (\mathcal{U}_A^d \times A) : u \in \mathcal{U}_A^d\}$. In this case, τ^s is a symmetric CNED.

We see that these reformulations[43] make heavy use of probabilistic terminology, and as in any translation, give rise to additional questions stemming from the new vocabulary. The fact that players' names are not a factor in the specification of the game, and only the statistical distribution of the types of players is given, is clear enough; what is interesting is that in the formalization of a symmetric CNED, one is asking for a "reallocated" equilibrium in which players with identical characteristics choose identical actions. Thus, an *ad hoc* assumption common to many models can be given a rigorous basis. In any case, the simple resolution of this question is perhaps surprising.[44]

Corollary 4 *Every CNED of a dispersed large anonymous game can be symmetrized.*

Proof: Let τ be a CNED of the game μ, and for each $a \in A$, let $W_a = \{u \in \mathcal{U}_A^d : (u,a) \in B_\tau\}$. $(W_a)_{a \in A}$ is a countable family of subsets of $\mathcal{B}(\mathcal{U}_A^d)$ such that for any finite subset A_F of A,

$$\begin{aligned} \mu(\bigcup_{a\in A_F} W_a) = \tau_{\mathcal{U}}(\bigcup_{a\in A_F} W_a) = \tau\left((\bigcup_{a\in A_F} W_a) \times A\right) \quad &\geq \quad \sum_{a\in A_F} \tau(W_a \times \{a\}) \\ &= \quad \sum_{a\in A_F} \tau(\mathcal{U}_A^d \times \{a\}) = \sum_{a\in A_F} \tau_A(\{a\}). \end{aligned}$$

Since μ is an atomless probability measure on $(\mathcal{U}_A^d, \mathcal{B}(\mathcal{U}_A^d))$, all the hypotheses of Proposition 3 are satisfied, and there exists a family $(T_a)_{a\in A}$ of sets in $\mathcal{B}(\mathcal{U}_A^d)$ such that $T_a \subseteq W_a$, $\mu(T_a) =$

$\tau_A(\{a\})$. Now define $h : \mathcal{U}_A^d \longrightarrow A$ such that $h(t) = a$ for almost all $t \in T_a$, all $a \in A$, and note that the measure $\mu(i, h)^{-1}$, i being the identity mapping on $\mathcal{U}_A^d$, is the required symmetrization. ∎

This yields the interesting characterization of symmetric equilibria as the extreme points of a set of equilibria.[45]

Corollary 5 *Let μ be a dispersed large anonymous game. Then a CNED τ of μ is a symmetric CNED if and only if τ is an extreme point of the set $\Lambda_\tau = \{\rho \in \mathcal{M}(\mathcal{U}_A^d \times A) : \rho_{\mathcal{U}} = \mu; \rho_A = \tau_A; \rho(B_\tau) = 1\}$.*

All that remains is the question of existence.

Corollary 6 *There exists a symmetric CNED for a dispersed large anonymous game μ.*

Proof: In Theorem 2, use $(\mathcal{U}_A^d, \mathcal{B}(\mathcal{U}_A^d), \mu)$ as the space of players' names, and the identity mapping i as the game. If f is the equilibrium guaranteed by the theorem, the measure $\mu(i, f)^{-1}$ is a symmetric CNED. ∎

We conclude with the observation that these results are false without the dispersedness hypothesis.[46]

5 Non-Existence of a Pure Strategy Nash Equilibrium

In this section, we present two examples of games without Nash equilibria, in which the set of actions is a compact interval. Apart from their intrinsic methodological interest, these examples are useful because they anchor the abstract treatment of Section 3 to concrete specifications that one can compute and work with. Both examples are predicated on the fact that it is impossible to choose from the correspondence[47] on the Lebesgue unit interval defined by $t \longrightarrow \{t, -t\}$, a measurable selection that induces a uniform distribution ν^* on $[-1, 1]$.[48]

5.1 A Nonatomic Game with Nonlinear Payoffs

The following example[49] is due to Rath-Sun-Yamashige (1995) who present it in the context of Corollary 6 above.

Example 2: Consider a game $\mathcal{G}_1$ in which the set of players $(T, \mathcal{T}, \lambda)$ is the Lebesgue unit interval $([0, 1], \mathcal{B}([0, 1]), \nu)$, A is the interval $[-1, 1]$, and the payoff function of any player $t \in [0, 1]$ is given by

$$u_t(a, \rho) = g(a, \beta d(\nu^*, \rho)) - |t - |a||, \quad 0 < \beta < 1, \quad a \in [-1, 1], \quad \rho \in \mathcal{M}([-1, 1]),$$

where $d(\nu^*, \rho)$ is the Prohorov distance between ν^* and ρ based on the natural metric on $[-1,1]$,[50] and $g : [-1,1] \times [0,1] \longrightarrow \mathbb{R}_+$. If $g(a,0) \equiv 0$ for any $a \in [-1,1]$, there is no Nash equilibrium that induces ν^*. The point is that one can choose the function $g(\cdot,\cdot)$ such that the best-response function based on a distribution $\rho \neq \nu^*$ induces a distribution different from ρ and therefore precludes the existence of a Nash equilibrium. An example of such a function is the periodic function, with period 2ℓ, $\ell \in (0,1]$, and defined on $[0, 2\ell]$ by

$$g(a,\ell) = \begin{cases} a/2 & \text{for } 0 \leq a \leq (\ell/2) \\ (\ell - a)/2 & \text{for } (\ell/2) \leq a \leq \ell \\ -g(a-\ell, \ell) & \text{for } \ell \leq a \leq 2\ell, \end{cases}$$

with $g(a,\ell) = -g(-a,\ell)$ for $a < 0$, and extended in both directions. Indeed, this specification of $g(\cdot,\cdot)$ furnishes an equicontinuous family of payoff functions.[51] ■

5.2 Another Nonatomic Game with Linear Payoffs

In Example 2, the distribution of societal responses enters an individual's payoff function in a non-linear way; here we present an example in which players maximize expected utilities with respect to this distribution and thereby lead to linear specification. This example is due to Khan-Rath-Sun (1997b), and unlike Example 2, does not involve the Prohorov metric.

Example 3: Consider a game different from that in Example 2 only in that the payoff function of player t is given, for $a \in [-1,1]$, $\rho \in \mathcal{M}([-1,1])$, by

$$u_t(a,\rho) = \int_{-1}^{1} v_t(a,x)\,d\rho(x), \quad \text{where} \quad v_t(a,x) = -|t - |a|| + (t-a)z(t,x),$$

and the function $z : [0,1] \times [-1,1] \longrightarrow \mathbb{R}$ is such that for all $t \in [0,1]$,

$$z(t,a) = \begin{cases} a & \text{if } \ 0 \leq a \leq t \\ t & \text{if } \ t < a \leq 1 \\ -z(t,-a) & \text{if } \ a < 0. \end{cases}$$

This game does not have a Nash equilibrium. A distribution that is not uniform cannot be an equilibrium.[52] On the other hand, for a uniform distribution ρ, the value of the summary statistic $\int_A z(t,a)d\rho$ is zero for ν-almost all $t \in [0,1]$, and hence requires that ρ be induced by a measurable selection from the best-response correspondence on the Lebesgue unit interval defined by $t \longrightarrow \{t, -t\}$, which we have seen to be impossible. It is easy to check that $u_t(a,\rho)$ is a jointly continuous function in its three arguments (t, a, ρ), and that the family of utility payoff functions indexed by the name t is an equicontinuous family. ■

5.3 New Games from Old

Now consider another game $\mathcal{G}_2$ manufactured from the game $\mathcal{G}_1$ in Example 2. In this game the set of players $(T, \mathcal{T}, \lambda)$ is again given by the Lebesgue unit interval $([0,1], \mathcal{B}([0,1])\nu)$, A by $[-1,1]$, and the payoff function of any player $t \in [0,1]$ by $v_t : A \times \mathcal{M}([-1,1]) \longrightarrow \mathbb{R}$ where

$$v_t(\cdot,\cdot) = \begin{cases} u_{2t}(\cdot,\cdot) & \text{if} \quad 0 \le t \le (1/2) \\ u_{2-2t}(\cdot,\cdot) & \text{if} \quad (1/2) < t \le 1. \end{cases}$$

It is clear that the new game $\mathcal{G}_2$ is formed from $\mathcal{G}_1$ by endowing each player with a "twin" and then by normalizing the space of players to the Lebesgue unit interval. It is clear that both games $\mathcal{G}_1$ and $\mathcal{G}_2$ induce identical distributions on the space of characteristics. Hence, in some essential macroscopic sense, the two games are identical. However, the point is that $\mathcal{G}_2$ has a Nash equilibrium whereas $\mathcal{G}_1$ does not! It is easy to check that $g : [0,1] \longrightarrow [-1,1]$ is a Nash equilibrium of the modified game $\mathcal{G}_2$, where

$$g(t) = \begin{cases} 2t & \text{if} \quad 0 \le t \le (1/2) \\ 2t-2 & \text{if} \quad (1/2) < t \le 1. \end{cases}$$

6 Interactions based on Averages of Individual Responses

In light of these counterexamples, the question arises as to how far one can proceed if a player's dependence on the distribution of societal responses is restricted to dependence on specific moments of the distribution. In this section, we focus on the first moment,[53] and are thereby led to integration, and a consequent linearity requirement on the action set. Since integration occurs with respect to players' names, what is important in this connection is Aumann's observation that the "chief mathematical tools are Lebesgue measure and integration [and that] Riemann integration can *not* be substituted."

6.1 A Basic Result

We shall follow the notation of Section 3.2 but with the difference that the common action set A is now conceived to be a nonempty compact subset of Euclidean space $\mathbb{R}^n$, and the space of player characteristics $(\mathcal{U}_A^{av}, \mathcal{B}(\mathcal{U}_A^{av}))$ is the space of real-valued continuous functions on $A \times (con(A))^\ell$, endowed with its sup-norm topology and the induced Borel σ-algebra (the superscript av denoting "distribution"). We can now present[54]

Theorem 3 *Let $\mathcal{G}^{av}$ be a measurable map from T to $\mathcal{U}_A^{av}$. Then there exists a measurable* *function $f : T \longrightarrow A$ such that for λ-almost all $t \in T$,*

$$u_t(f(t), \int_{s\in T_1} f_1(s)d\lambda_1, \cdots, \int_{s\in T_\ell} f_\ell(s)d\lambda_\ell) \geq u_t(a, \int_{s\in T_1} f_1(s)d\lambda_1, \cdots, \int_{s\in T_\ell} f_\ell(s)d\lambda_\ell)$$

for all $a \in A$, where $u_t = \mathcal{G}^{av}(t) \in \mathcal{U}_A^{av}$ and f_i is the restriction of f to T_i.

6.2 Lyapunov's Theorem and the Integral of a Correspondence

Let the integral of a correspondence $F : T \longrightarrow \mathbb{R}^n$ be defined as

$$\mathcal{I}_F = \{\int_T f(t)d\lambda : f \text{ is an integrable selection of } F\}.$$

Since the range of F is no longer countable, we need to modify the earlier definition of measurability to require that the graph of F be an element of $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^n)$. A measurable selection theorem and Lyapunov's theorem on the range of an atomless vector measure then yield the following analogue to Proposition 6.[55]

Proposition 7 *(i) $\mathcal{I}_F$ is nonempty and convex. (ii) If F is integrably bounded and compact-valued, then $\mathcal{I}_F$ is compact and $\mathcal{I}_F = \mathcal{I}_{con\ F}$. (iii) If Y is a metric space, G a closed-valued correspondence, and H an integrably bounded compact-valued correspondence, both from $T \times Y$ into $\mathbb{R}^n$ such that for each fixed $t \in T$, $G(t, \cdot)$ is upper semicontinuous on Y, and for each fixed $y \in Y$, $G(t, y) \subseteq H(t)$ for λ-almost all $t \in T$, then $\mathcal{I}_{G(\cdot,y)}$ is upper semicontinuous on Y.*

6.3 A Sketch of the Proof

With Proposition 7 in hand, one simply follows[56] all the guideposts laid out in the proof of Theorem 2, but with the Kakutani fixed-point theorem applied to a one-to-many countering or best-response mapping $G : T \times (con(A))^\ell$ into A given by

$$(t, a_1, \cdots, a_\ell) \longrightarrow F(t, a_1, \cdots, a_\ell) = \text{Arg Max}_{a\in A} u_t(a, a_1, \cdots, a_\ell).$$

7 An Excursion into Vector Integration

On comparing Theorems 2 and 3, the question arises as to why cardinality restrictions on action sets are crucial when player interactions are based on distributions, but play no role when they are based on averages. This discrepancy is more apparent than real: the induced distribution of a random variable is also an average, in a clearly defined sense, of a related random variable taking values in an infinite-dimensional space, and the cardinality restrictions simply shift over from the sets themselves to the dimensionality of the space in which they are located. Thus, even when the primary emphasis is explicitly on distributions,[57] and the parameters of the problem do not suggest it, the relevant backdrop is still that of vector integration.[58]

To see this, return to Proposition 5 and to the discussion of purification results in Section 4, and recall that the integral of a measurable function $g : T \longrightarrow \mathcal{M}(A)$ is obtained by fixing a particular element $B \in \mathcal{T}$, and then by integrating the resulting real-valued function $g(t)(B)$. However, g can also be reduced to a real-valued function by considering[59] $(x, g(t)) = \int_A x(a)g(t; da)$, where x is a particular element of $C(A)$, the space of continuous functions on the compact set A endowed with their sup norm. This procedure defines another integral, the so-called Gelfand integral $\int_T g(t)d\lambda(t)$, where

$$(x, \int_T g(t)d\lambda(t)) = \int_T (x, g(t))d\lambda(t) = \int_T \int_A x(a)g(t; da)d\lambda(t) \text{ for all } x \in C(A).$$

The point is that this integral is identical to the integral obtained by our first procedure, and furthermore, the Gelfand integral of $\delta_{\{g(\cdot)\}}$, the function obtained by "lifting" $g(\cdot)$ into the space of probability measures, is the same as the induced distribution λg^{-1} of g.[60]

This discussion for the specific space $C(A)$ and its dual $(C(A))^*$ can easily be transposed to the general setting of a separable Banach space[61] X and its dual X^*. We can now define the Gelfand integral[62] of any X^*-valued function $g(\cdot)$ by requiring that

$$(x, \int_T g(t)d\lambda(t)) = \int_T (x, g(t))d\lambda(t) \text{ for all } x \in X.$$

In addition to its norm topology, X^* is also endowed with the weak and weak* topologies, and it is the Borel σ-algebra generated by the latter that ensures that for any $x \in X$, the real-valued function $(x, g(\cdot))$ is measurable, and furnishes the relevant weak* measurability criterion for Gelfand integration.[63] However, we can also simply work with a separable Banach space X directly. In this case, we can be guided by Lebesgue integration, and focus on functions that are pointwise limits of simple functions. This furnishes us with the so-called strong measurability criterion, and a conventional notion of an integral, the so-called Bochner integral, based on the convergence of the sums of these simple functions.[64] With all of this terminology at our disposal, we can present the following consequence of the results of Section 3.2.[65]

Proposition 8 *Proposition 7 is valid for a correspondence F from T into a countably infinite subset of a separable Banach space or a dual of a separable Banach space. In the former case, we can work with the norm or weak topologies and the Bochner integral; and in the latter, with the weak* topology and the Gelfand integral.*

Proposition 8 is false without the cardinality restriction,[66] a fact that underlies Examples 2 and 3 above.

We conclude this section with a result that views a set of integrable functions as a productive object in its own right. Let $L_1(\lambda, X)$ be the space of equivalence classes of Bochner integrable functions equipped with the integral norm.[67] This is a Banach space, and its dual space $L^w_\infty(\lambda, X^*)$ consists of equivalence classes of weak* measurable functions with essentially bounded norm functions and equipped with the essential supremum norm.[68] We can now present analogues of results that play the same role in the sequel that Tychonoff's theorem played in Section 2.[69]

Proposition 9 *If A is a weakly compact subset of a separable Banach space X, $L_1(\lambda, A)$ is a weakly compact subset of the Banach space $L_1(\lambda, X)$. If A is a norm bounded, weak* closed subset of a dual Banach space X^*, $L^w_\infty(\lambda, A)$ is a weak* compact subset of the Banach space $L^w_\infty(\lambda, X^*)$.*

8 Interactions based on almost all Individual Responses

Even though in Theorem 1 payoff functions depend on the actions of each individual player rather than on some statistical summary of these actions, the hypothesis of continuity in the product topology reduces this apparent generality to dependence on the actions of an essentially finite number of players. In this section, we see how the measure-theoretic structure on the set of players' names can be exploited to yield results for situations where an individual player's payoff depends on the (equivalence class of the) entire function of individual responses.[70]

8.1 Results

Unlike earlier sections, we shall denote a game by the generic symbol $\mathcal{G}$. We continue to assume that product spaces are endowed with their product measurable or topological structures, and that the space $C(Y)$ of continuous functions on a compact set Y is equipped with its sup norm topology and the induced Borel σ-algebra. We reserve the symbol X for a separable Banach space, and except for the last result, work primarily with the weak and weak* topologies.[71]

Theorem 4 *Let $\mathcal{G}$ be a measurable map from T to $C(A \times L_1(\lambda, A))$, where $A \subseteq X$ is convex and weakly compact, and $\mathcal{G}(t)(\cdot, g) \equiv u_t(\cdot, g)$ is quasi-concave on A for any $g \in L_1(\lambda, A)$. Then there exists $f \in L_1(\lambda, A)$ such that for λ-almost all $t \in T$, $u_t(f(t), f) \geq u_t(a, f)$ for all $a \in A$.*

This statement is valid for $L_\infty^w(\lambda, A)$ substituted for $L_1(\lambda, A)$, and with $A \subseteq X^*$ convex and weak compact.*

It is clear in Nash (1950, 1951) that the question of the existence of pure-strategy equilibria precludes the assumptions of convexity on the action set and of quasi-concavity of the payoff function. Theorem 4 thus pertains to mixed-strategy equilibria, and one can ask whether it yields any implications for settings without these assumptions. We present three corollaries in this regard. Note that unlike the case of a finite action set, it now makes a difference whether pure strategies are conceived as extreme points of an action set or as Dirac measures on it.[72]

Corollary 7 *If A is a compact Hausdorff space, there exists a CNED for a large anonymous game.*[73]

Corollary 8 *Let $\mathcal{G}$ be a measurable map from T into $C(A \times L_\infty^w(\lambda, \mathcal{M}(A)))$, and A a compact metric space. Then for any $\epsilon > 0$, there exists $f \in L_\infty^w(\lambda, ext(\mathcal{M}(A)))$ and $K_\epsilon \in \mathcal{T}$, $\lambda(K_\epsilon) \geq (1-\epsilon)$ such that for all $t \in K_\epsilon$,*

$$\int_A v_t(a, f) f(t; da) \geq \int_A v_t(a, f) d\rho(a) - \epsilon \text{ for all } \rho \in \mathcal{M}(A), \text{ where } v_t \equiv \mathcal{G}(t).$$

Corollary 9 *Let $\mathcal{G}$ be a measurable map from T into $C(A \times A)$, A a convex and weakly compact subset of X. Then for any $\epsilon > 0$, there exists $f \in L_1(\lambda, ext(A))$ and $K_\epsilon \in \mathcal{T}$, $\lambda(K_\epsilon) \geq (1-\epsilon)$ such that for all $t \in K_\epsilon$,*

$$u_t(f(t), \int_T f(t) d\lambda(t)) \geq u_t(a, \int_T f(t) d\lambda(t)) - \epsilon \text{ for all } a \in A,$$

where $u_t \equiv \mathcal{G}(t)$ and the integral is the Bochner integral. The above statement is also true for $L_\infty^w(\lambda, ext(A))$ where $A \subseteq X^*$ and weak compact, with the integral the Gelfand integral.*

The following result reimposes cardinality restrictions on action sets to obtain exact equilibria in the Banach setting.

Theorem 5 *Theorem 3 is valid if A is a countable compact subset of X or of X^*, with the norm or weak topologies and the Bochner integral in the first case, and with the weak* topology and the Gelfand integral in the second.*

8.2 Uhl's Theorem and the Integral of a Correspondence

In the discussion of his existence theorem, Aumann (1966; p. 15) noted that in "the presence of a continuum of traders, the space of assignments is no longer a subset of a finite-dimensional Euclidean space, but of an infinite-dimensional function space. This necessitates the use of completely new methods ... of functional analysis (Banach spaces) and topology." There is the additional handicap that Lyapunov's theorem fails in the infinite-dimensional setting.[74] However, it can be shown that an approximate theory of integration can be developed on the basis that the closure of the range of a vector measure is convex and compact. For the weak or weak* topologies, this is a consequence of the finite-dimensional Lyapunov theorem; for the norm topology, the result is due to Uhl (1969).[75]

8.3 Sketch of Proofs

Once we have access to the mathematical tools discussed above, the proofs are a technical (functional-analytical) elaboration of the basic argument of Nash (1950), with the upper semi-continuity of the best-response correspondence being the essential difficult hurdle.[76]

9 Non-Existence: Two Additional Examples

On taking stock, we see that there exist exact pure strategy equilibria with cardinality restrictions on individual action sets (Theorem 5), and approximate pure strategy equilibria without such restrictions and even in situations where an individual player's dependence on societal responses is not limited to their distributions or averages (Corollary 8). In this section, we see that there cannot be progress on this score without additional measure-theoretic restrictions on the space of players' names.

9.1 A Nonatomic Game with General Interdependence

The following example is due to Schmeidler (1973), and it shows that Corollary 8 cannot be improved even in the setting of two actions.

Example 4: Consider a game in which the set of players $(T, \mathcal{T}, \lambda)$ is given by the Lebesgue unit interval $([0,1], \mathcal{B}([0,1]), \nu)$, A by $\{-1, 1\}$, and the payoff function of any player $t \in [0,1]$ by

$$u_t(a, f) = |\, a - \int_0^t f(x) d\nu \,|, \quad a \in \{-1, 1\}, \quad f \in L_1(\nu, \{-1, 1\}).$$

This game does not have a Nash equilibrium. For any equilibrium f, the value of the summary statistic $h(t) = \int_0^t f(x) d\nu$ must be zero for all $t \in [0,1]$. This implies that $f(t) = 0$ for ν-almost all t, which contradicts the fact that $f(t)$ is 1 or -1. ∎

9.2 A Nonatomic Game on the Hilbert Space ℓ_2

The following example is due to Khan-Rath-Sun (1997a) and it shows that Theorem 5 cannot be improved even when action sets are norm-compact. It is based on a function $f : [0,1] \longrightarrow \ell_2$ where

$$f(t) = \left(\frac{1 - W_n(t)}{2^n} \right)_{n=1}^{\infty}, \;\; W_1(t) \equiv -1, \; W_n(t) = (-1)^{[2^{n-1}t]} \text{ for } n \geq 2,$$

where $[x]$ is the integer part of x. It can be shown[77] that the range of f is norm compact, that it is Bochner integrable with integral $e \equiv (1, (2^{-n-1})_{n=1}^{\infty})$, and that $(e/2) \notin \int_0^1 \{0, f(t)\} d\nu(t)$.

Example 5: Consider a game in which the set of players $(T, \mathcal{T}, \lambda)$ is given by the Lebesgue unit interval $([0,1], \mathcal{B}([0,1])\nu)$, A is a norm compact subset of ℓ_2 containing $\{0\} \cup \{f(t) : t \in [0,1]\}$, and the payoff function of any player $t \in [0,1]$ is given by

$$u_t(a,b) = -h(t, a, f(t), \beta \,||b - (e/2)||) - \|a\| \cdot \|a - f(t)\|, \; 0 < \beta < 1, \; a \in A, \; b \in \overline{con}(A),$$

where $h : [0,1] \times \ell_2 \times \ell_2 \times \mathbb{R}_+ \to \mathbb{R}_+$. If $h(t, a, f(t), 0) = 0$, it is easy to see that there is no Nash equilibrium. The point is that one can choose the function h such that the best-response function based on an element $b \in \overline{con}(A)$ averages to a value b' such that $||b' - (e/2)|| \neq ||b - (e/2)||$, and therefore precludes the existence of a Nash equilibrium. An example of such a function is given by

$$h(t,a,b,\alpha) = \begin{cases} \alpha |\sin \frac{t}{\alpha} \pi| \cdot (\|a\| + 1 - (-1)^{\left[\frac{t}{\alpha}\right]}) \cdot (\|a - b\| + 1 + (-1)^{\left[\frac{t}{\alpha}\right]}) & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha = 0 \end{cases}.$$

Finally, we observe that the same example works for a game in which compactness and continuity are phrased in the weak rather than the norm topology on ℓ_2. ∎

We conclude with the observation that the isomorphism between ℓ_2 and any separable infinite dimensional L_2 space allows us to set Example 5 in the latter space. This is useful in light of the use of L_2 in models with information and uncertainty.

10 A Richer Measure-Theoretic Structure

In the light of the counterexamples presented above, the question arises whether additional measure-theoretic structure on the set of players' names will allow the construction of a more robust and general theory. In asking this, we are guided by the emphasis of Aumann (1964) that it is not the particularity of the measure space but its atomlessness that is important from a methodological point of view.[78] As a result of a particular class of measure spaces introduced by Loeb (1975), the so-called Loeb measure spaces, a richer structure is indeed available, and it is ideally suited for studying situations where non-atomic considerations such as strategic negligibility or diffuse information are an essential and substantive issue.

10.1 Atomless Loeb Measure Spaces and their Special Properties

A Loeb space $(\bar{T}, L(\bar{\mathcal{T}}), L(\bar{\lambda}))$ is a "standardization" of a hyperfinite internal probability space $(\bar{T}, \bar{\mathcal{T}}, \bar{\lambda})$, and constructed as a simple consequence of Caratheodory's extension theorem and the countable saturation property of the nonstandard extension.[79] It bears emphasizing that a Loeb measure space, even though constituted by nonstandard entities, is a standard measure space, and in particular, a result pertaining to an abstract measure space applies to it. For applications, its specific construction can usually be ignored,[80] and one simply focuses on its special properties not shared by Lebesgue or other measure spaces.[81] We now turn to those properties.[82]

Proposition 10 *If an atomless Loeb space $(\bar{T}, L(\bar{\mathcal{T}}), L(\bar{\lambda}))$ is substituted for $(T, \mathcal{T}, \lambda)$, Propositions 5, 6, and 8 are valid without any cardinality restrictions.*

Proposition 11 *Two real-valued random variables x and y on a Loeb counting probability space*[83] *have the same distribution iff there is an internal permutation that sends x to y.*

10.2 Results

The special properties of correspondences defined on a Loeb measure space delineated by Proposition 10 can be translated into results on exact pure-strategy equilibria for games with many players but with action sets without any cardinality restrictions.

Theorem 6 *If an atomless Loeb space $(\bar{T}, L(\bar{\mathcal{T}}), L(\bar{\lambda}))$ is substituted for $(T, \mathcal{T}, \lambda)$, Theorems 2 and 5 are valid without any cardinality restrictions on A.*

These results bring out the fact that the non-existence claims in Examples 2, 3 and 5 do not hold for the idealized Loeb setting. Indeed, one can show that in the finite-player versions of the games presented in these examples, approximate equilibria exist and that these approximations get finer as the number of players increases.[84] Thus, it is natural to ask what it is about the idealized Lebesgue setting that makes the exact counterparts of these equilibria disappear. Since Lebesgue measurability of a function can be represented in the Loeb setting under the condition that "infinitesimally close" points in the domain of the function have "infinitesimally close" values, the answer lies in the fact that in asking for an equilibrium that is by definition Lebesgue measurable, we have injected a cooperative requirement into an essentially non-cooperative situation. What is particularly interesting is that there may be no such equilibrium even in situations when Lebesgue measurability is fulfilled for the game itself, i.e., when players with "infinitesimally close" names have "infinitesimally close" payoff functions. Thus the use of Lebesgue measurability for the modelling of large finite game-theoretic phenomena fails at two levels: first, it restricts the types of large finite games to some special classes, and second, even with this restriction, the ideal limits of approximate equilibria cannot be modeled.

In referring to continuum-of-players methods in the context of non-cooperative game theory, Mas-Colell (1984a; p. 20) is clear that a "literal continuum of agents ... should be thought of only as a limit version of the Negligibility Hypothesis. It is an analytically useful limit because results come sharp and clean, unpolluted by ϵ's and δ's, but it is also the less realistic limit." Elsewhere, Mas-Colell-Vives hope that the "conclusions are not too misleading when applied to realistic situations."[85] We have already emphasized the distinction between two types of idealized limiting situations, and we now observe that results based on hyperfinite Loeb measure spaces can be "asymptotically implemented" for a sequence of large but finite games as a matter of course. One only needs to control the extent to which the characteristics are allowed to vary by focussing on a "tight" sequence of mappings.[86] The following result based on a compact metric action set[87] illustrates this particular advantage of the Loeb formulation.[88]

Corollary 10 *For each $n \geq 1$, let a finite game G^n be a mapping from $T^n = \{1, 2, \cdots, n\}$ into $\mathcal{U}_A^d$, and $\{T_1^n, \cdots, T_\ell^n\}$ be a partition of T^n. Assume that the sequence of finite games is tight and that there is a positive number c such that $|T_i^n|/n > c$ for all sufficiently large n and $1 \leq i \leq \ell$. Then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, there exists $g_n : T^n \longrightarrow A$ such that for all $t \in T^n$, and for all $a \in A$,*

$$u_t^n(g_n(t), \lambda_1^n g_n^{-1}, \cdots, \lambda_\ell^n g_n^{-1}) \geq u_t^n(a, \lambda_1^n g_n^{-1}, \cdots, \lambda_\ell^n g_n^{-1}) - \epsilon,$$

where $u_t^n \equiv G^n(t)$, and λ_i^n is the counting probability measure[89] *on T_i^n, $i = 1, \cdots \ell$.*

It is also one of the strengths of a Loeb counting measure space that anomalies presented in Section 5.3 cannot arise as illustrated by the following result.

Corollary 11 *Let $\mathcal{G}$ and $\mathcal{F}$ be measurable maps from T to $\mathcal{U}_A^d$, such that $\overline{\nu}_i \mathcal{G}_i^{-1} = \overline{\nu}_i \mathcal{F}_i^{-1}$ for $i = 1, \cdots, \ell$, where $\mathcal{G}_i$ and $\mathcal{F}_i$ are restrictions of $\mathcal{G}$ and $\mathcal{F}$ to T_i respectively. Then there exists automorphisms $\phi_i : (T_i, \overline{\nu}_i) \longrightarrow (T_i, \overline{\nu}_i)$ for each $i = 1, \cdots, \ell$ such that $\mathcal{G}_i(t) = \mathcal{F}_i(\phi_i(t))$ for almost all $t \in T_i$. Let $f : T \longrightarrow A$ be a Nash equilibrium of the atomless game $\mathcal{F}$ and define $g : T \longrightarrow A$ such that $g(t) = f(\phi_i(t))$ for all $t \in T_i$. Then g is a Nash equilibrium of the atomless game $\mathcal{G}$ and every Nash equilibrium of $\mathcal{G}$ is obtained in this way.*

10.3 Sketch of Proofs

Once we have access to the theory of distribution and integration of a correspondence defined on an atomless Loeb measure space, the proof of Theorem 6 is a straightforward consequence of the basic argument that we trace to Nash. Corollary 10 follows from the nonstandard extension,[90] and Corollary 11 from Proposition 11.

11 Large Games with Independent Idiosyncratic Shocks

When von Neumann-Morgenstern referred to the "excellent possibility of applying the laws of statistics and probabilities" to games with a large number of players, they did not have in mind the cancellation of individual (independent) risks through aggregation or diversification. Both "Crusoe" and a participant in a social exchange economy are "given a number of data which are "dead"; they are the unalterable physical background of the situation [and] even when they are apparently variable, they are really governed by fixed statistical laws. Consequently [these purely statistical phenomena] can be eliminated by the known procedures of the calculus of probabilities." Instead of these individual "uncontrollable factors [that] can be described by statistical assumptions", their primary focus was on "alien" variables that are the "product of other participants' actions and volitions" and which cannot be "obviated by a mere recourse to the devices of the theory of probability".[91]

Recent and ongoing work in economic theory, however, considers economic situations in which a continuum of (albeit identical) participants are exposed to individual chance factors.[92] This literature appeals to a basic intuition underlying the theory of insurance whereby the the classical law of large numbers is used to eliminate independent (idiosyncratic or non-systematic) risks. However, the difficulty in formalizing this intuition in the usual context of a continuum of random variables was pointed out early on by Doob (1937, 1953): the assumption of independence renders the sample function of the underlying stochastic process "too irregular to be useful". What is needed is a suitable analytical framework that renders the twin assumptions of independence and joint measurability compatible with each other. In this section, we discuss the nature of the difficulty, and show how the richer measure-theoretic structure discussed in Section 10, but now on a special product space, offers a viable solution to it.

11.1 On the Joint Measurability Problem

We couple the space of players' names $(T, \mathcal{T}, \lambda)$ with another probability space $(\Omega, \mathcal{A}, P)$ to represent the sample space. Let $(T\times\Omega, \mathcal{T}\otimes\mathcal{A}, \lambda\otimes P)$ be the usual product probability space. We shall refer to the functions $f(t,\cdot)$ on Ω and $f(\cdot,\omega)$ on T respectively as the random variables and the sample functions. The random variables f_t are said to be almost surely pairwise independent if for λ-almost all $t_1 \in T$, $f(t_1,\cdot)$ is independent of $f(t_2,\cdot)$ for λ-almost all $t_2 \in T$. The following result illustrates the joint measurability problem in a particularly transparent way.[93]

Proposition 12 *Let f be a jointly measurable function from the usual product space $(T \times \Omega, \mathcal{T}\otimes\mathcal{A}, \lambda\otimes P)$ to a complete separable metric space X. If the random variables f_t are almost surely pairwise independent, then for λ-almost all $t \in T$, f_t is a constant random variable.*

11.2 Law of Large Numbers for Continua

The difficulty that is brought to light in Proposition 12 is overcome in the context of a Loeb product space $(\bar{T}\times\bar{\Omega}, L(\bar{\mathcal{T}}\otimes\bar{\mathcal{A}}), L(\bar{\lambda}\otimes\bar{P}))$ constructed as a "standardization" of the hyperfinite internal probability space $(\bar{T}\times\bar{\Omega}, \bar{\mathcal{T}}\otimes\bar{\mathcal{A}}, \bar{\lambda}\otimes\bar{P})$. This special product space extends the usual product space $(\bar{T}\times\bar{\Omega}, L(\bar{\mathcal{T}})\otimes L(\bar{\mathcal{A}}), L(\bar{\lambda})\otimes L(\bar{P}))$, retains the crucial Fubini property, and is rich enough to allow many hyperfinite collections of random variables with any variety of distributions.[94] For simplicity, a measurable function on $(\bar{T}\times\bar{\Omega}, L(\bar{\mathcal{T}}\otimes\bar{\mathcal{A}}), L(\bar{\lambda}\otimes\bar{P}))$ will also be called a process. We also assume that both $L(\bar{\lambda})$ and $L(\bar{P})$ are atomless. We can now present a version of the law of large numbers for a hyperfinite continuum of random variables, and refer the reader to Sun (1996a, 1998a) for details and complementary results.[95]

Proposition 13 *Let f be a process[96] from $(\bar{T}\times\bar{\Omega}, L(\bar{\mathcal{T}}\otimes\bar{\mathcal{A}}), L(\bar{\lambda}\otimes\bar{P}))$ to a complete separable metric space X. Assume that the random variables $f(t,\cdot)$ are almost surely pairwise independent. Then for $L(\bar{P})$-almost all $\omega \in \bar{\Omega}$, the distribution μ_ω on X induced by the sample function $f(\cdot,\omega)$ on $\bar{T}$ is equal to the distribution μ on X induced by f viewed as a random variable on $\bar{T}\times\bar{\Omega}$.*

Since solutions to individual maximization problems are not unique, the need for a law of large numbers for set-valued processes arises in a natural way, where a set-valued process is a closed-valued measurable correspondence from a product space to X. Such a law can be derived from Proposition 13.[97] What is more difficult is the following result showing that possible widespread correlations can be removed from selections of a set-valued process.

Proposition 14 *Let F be a set-valued process from $\bar{T} \times \bar{\Omega}$ to a complete separable metric space X. Assume that $F(t, \cdot)$ are almost surely pairwise independent. Let g be a selection of F with distribution μ. Then there is another selection f of F such that the distribution of f is μ, and $f(t, \cdot)$ are almost surely pairwise independent.*

11.3 A Result

We can now use this substantial machinery to present a result for a large non-anonymous game in which individual agents are exposed to idiosyncratic risks, but in equilibrium the societal responses do not depend on a particular sample realization, and each agent is justified in ignoring other players' risks.

Theorem 7 *Let $\mathcal{G} : \bar{T} \times \bar{\Omega} \longrightarrow \mathcal{U}_A^d$ be a game with individual uncertainty, i.e., the random payoffs $\mathcal{G}(t, \cdot)$ are almost surely pairwise independent.*[98] *Then there is a process $f : \bar{T} \times \bar{\Omega} \longrightarrow A$ such that f is an equilibrium of the game $\mathcal{G}$, the random strategies $f(t, \cdot)$ are almost surely pairwise independent, and for $L(\bar{P})$-almost all $\omega \in \bar{\Omega}$, $f(\cdot, \omega)$ is an equilibrium of the game $\mathcal{G}(\cdot, \omega)$ with constant societal distribution $L(\bar{\lambda} \otimes \bar{P})f^{-1}$.*

The basic idea for the proof of Theorem 7 is straightforward. On an appeal to Theorem 6, we know that there exists a measurable function $g : T \times \Omega \longrightarrow A$ such that

$$\mathcal{G}_{(t,\omega)}(g(t, \omega),\ L(\bar{\lambda} \otimes \bar{P})g^{-1}) \geq \mathcal{G}_{(t,\omega)}(a,\ L(\bar{\lambda} \otimes \bar{P})g^{-1}) \text{ for all } a \in A.$$

Now finish the proof by applying Proposition 14 to the set-valued process

$$(t, \omega) \longrightarrow F(t, \omega) = \text{Arg Max}_{a \in A} \mathcal{G}_{(t,\omega)}(a,\ L(\bar{\lambda} \otimes \bar{P})g^{-1}).$$

12 Other Formulations and Extensions

The formulation of a large game that we have explored in this chapter hinges crucially on the formalization of players' characteristics by a metric space $\mathcal{U}$, along with its Borel σ-algebra. A large non-anonymous game is then simply a measurable function with such a range space, and its anonymous counterpart the induced measure on it. Thus a random variable and its law constitute basic elements in the relevant vocabulary, and one can exploit this observation to incorporate a variety of additional aspects into the basic framework.

Two formulations deserve special mention. The first of these considers[99] "very large" or "thick" games based on a space of characteristics given $\mathcal{U} \times [0,1]$, where $\mathcal{U}$ is interpreted as the space of "types," and there is a continuum of each type. In such a setting, one can be explicit about the cardinality of each type through the so-called "mass revealing" function, and questions concerning symmetric equilibria in which player t of type u, where $(u,t) \in \mathcal{U} \times [0,1]$ plays an action independent of t, can be investigated.[100] The advantage of this formulation is that a correspondence defined on such a space of characteristics has a distribution with well-behaved properties of the kind we saw in Propositions 5, 6, and 10, even when the range space is compact metric, and without having to go into the Loeb setting.[101] The second formulation concerns dynamics, and a setting in which a game is constituted by an infinite sequence of distributions over a space of actions and states.[102] By using the "value function" and other techniques from stochastic dynamic programming, questions relating to the existence and stationarity of equilibria can be investigated in such a setting.

In terms of elaborations on the basic framework, there has been a substantial amount of work that investigates games with a richer space of characteristics: different actions sets,[103] upper semi-continuous payoffs[104] and more generally, non-ordered preferences,[105] uncertainty, imperfect information, differing beliefs and imperfect observability represent a selective list.[106] Issues of existence and of continuity of equilibria have both been investigated, and this work has led both to interesting technical issues, and to changes in viewpoint. For example, in the presence of non-ordered preferences, one is led to the problem of choosing selections of functions of two variables, continuous in one and measurable in the other.[107] Even without non-ordered preferences but with weakened continuity hypotheses on payoffs, it is fruitful to regard a player as a continuous function from societal responses to a space of preferences, and this leads to deeper topological questions.[108] Similar changes in viewpoint have proved useful in the case of uncertainty and imperfect information where the space of characteristics is enlarged to include a sub-σ-algebra for each player, which leads to the formulation of a measurable structure on the space of sub-σ-algebras of a given σ-algebra.[109] Refinements of the concept of Nash equilibrium are also considered in the setting of large games.[110] Yet another example is a focus on the space $L(T, \mathcal{M}(A))$ of so-called Young measures, and the use of this as a unifying framework.[111] Whereas it is incontestable that this work has incorporated a variety of additional considerations into one comprehensive framework, we leave it to the reader's judgement to determine what new game-theoretic phenomena have been brought to light.

13 A Catalogue of Applications

Any discussion of applications must begin with Cournot (1838).[112] As noted by Roberts, he was the "first to make the role of large numbers explicit in his analysis [of] quantity-setting noncollusive oligopoly, [and his] model yields prices in excess of marginal costs with this divergence decreasing asymptotically to zero as the number of firms increases."[113] In addition to numerical negligibility, Cournot also raised the question of product diversity.

> The effects of competition have reached their limit when each of the partial productions D_k is *inappreciable,* not only with respect to the total production $D = F(p)$ but also with respect to the derivative $F'(p)$ so that the partial production could be subtracted from D without any appreciable variation resulting in the price of the commodity. This hypothesis is the one which is realized, in social economy, for a multitude of products, and, among them, for the most important products. It introduces a great simplification into the calculations.[114]

A particularly vigorous aspect of the research program initiated by Cournot concerns what Mas-Colell (1982b) has termed the *Cournotian foundation of perfect competition.* "Under the Negligibility Hypothesis the Cournot quantity-setting equilibrium is identical with the Walras price-taking equilibrium. Every seller has an infinitesimal upper bound on how much it can sell. Therefore no seller can influence aggregate production; hence no seller can influence the price system." This non-cooperative justification of the price-taking assumption leads naturally to the question of the minimum efficient scale of production and, more generally, to the optimality properties of Nash equilibria.[115] Indeed, once one recognizes that "price-taking behavior is, in a mass market, the natural consequence of message-taking behavior," it is a short step to the result of Hammond (1979a, 1979b) that a competitive equilibrium is "incentive compatible" in a continuum economy in the sense that no single agent can influence the terms of trade by deviations from "straightforward behavior."[116] We are thus led to the imposition of game-theoretic structures and thereby to the literature on implementation and mechanism design for the allocation of resources in a large economy.[117]

A canonical example of an anonymous mechanism is of course the Walrasian equilibrium, and the relevance of a Nash equilibrium for the existence of a Walrasian equilibrium is well-understood.[118] Indeed, once one considers Walrasian equilibria in environments with widespread externalities,[119] player interaction is no longer limited to dependence only on the mean message, and we loose the so-called *aggregation axiom* which has played a crucial role in the convergence and implementation literature. Without it, we are led to monopolistic competition.[120] As observed by Samuelson (1967), it was "Chamberlin's contention that proliferation of numbers alone need not lead to perfection of competition. [It] does not mean that the limit as $N \to \infty$ is zero-market imperfection. Instead the limit may be at an irreducible positive degree of imperfection. It is an increase, in some sense, of the *density* of numbers that everybody recognizes to be the relevant situation that needs to be appraised." There is now a substantial literature that attempts a formulation of Chamberlin-Robinson imperfect competition as a large game.[121]

The inadequacy of numerical negligibility as a sole desideratum for optimality is most transparent in the case of information, and here one has to formalize what one means by the proposition that "when agents are informationally small, the inefficiency due to asymmetric information is small."[122] However, with a viable analytical framework for discussing both types of negligibility, there is an emerging literature on the microfoundations of macroeconomics.[123] This includes economics of search,[124] and of the foundation of the firm.[125] Indeed, the importance of economic environments with many agents is ubiquitous, and limited only by the imagination and technical competence of the investigator. A selective list would certainly include applications to the stock market,[126] stochastic rationing mechanisms,[127] design of tax and subsidy schemes for attaining second-best equilibria,[128] voting models,[129] evolutionary game theory,[130] and the economics of fashion and "social influences".[131]

14 Conclusion

There are two distinct motivations for the study of non-cooperative games with many players. On the one hand, they delineate qualitative changes in the resolution of particular problems,[132] and on the other, they allow formulation of questions that have resisted formal treatment. Thus, in opposition to finite games, Nash equilibria of large games without widespread externalities are efficient, their pure strategy versions exist, and in models with enough institutional features, these games are well-suited for studying incentives in a variety of industrial structures, particularly monopolistic competition. However, when we take stock and evaluate where we currently stand relative to the work of Cournot and Nash, it may be worthwhile to keep in mind the distinction between technical and conceptual advances emphasized by von Neumann-Morgenstern. On a technical level, we certainly have a more sophisticated understanding of the importance of the precise form of player-interdependence and of different kinds of measurable structures, and these become especially important with infinite action sets, widespread externalities, and independent shocks. In this case, distribution and integrals assume separate and distinct identities, and the importance of geometry and Lyapunov's theorem, already explicit in Dvoretsky, Wald and Wolfowitz, and brought to the fore by Aumann, shades into probability and the law of large numbers for a continuum of random variables. On a conceptual level, however, the extent to which the mathematical theory of large games currently offers a canonical model and an array of uniform techniques for handling a variety of important applications, remains to be seen in the future.

Footnotes

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[1]For the first two authors, see Section 2 in the Third (1953) edition of their book (in the sequel, all quotations are from this section). For the next two, see item 11 in Kuhn-Tucker (1950; p. x) – a list of problems that Aumann (1997; p. 6) terms "remarkably prophetic."

[2]In his Nancy Schwartz Lecture, Mas-Colell (1998) observes, "I bet that [results] built on the Negligibility Hypothesis are centrally located in the trade-off frontier for the extent of coverage and the strength of results of theories. This is, however, a matter of judgement based on the conviction that mass phenomena constitute an essential part of the economic world."

[3]"An almost exact theory of a gas, containing about 10^{25} freely moving particles, is incomparably easier than that of the solar system , made up of 9 major bodies; and still more than that of a multiple star of three or four objects of about the same size."

[4] von Neumann-Morgenstern are emphatic on this point. "There is no getting away from it: The problem must be formulated, solved and understood for small numbers of participants before anything can be proved about the changes of its character in any limiting case of large numbers such as free competition."

[5]"Let us say again: we share the hope – chiefly because of the above-mentioned analogy in other fields! – that such simplifications will indeed occur."

[6]We mention this to emphasize that Nash drew his inspiration from von-Neumann and Morgenstern (1944 edition) rather than from Cournot (1838); see Nash (1950), as well as the more detailed elaboration in Nash (1951). The quotation is from the introduction to the latter paper.

[7]Again, see the reprinted version in Milnor-Shapley (1978).

[8]After the statement that the "references can still be counted on the fingers of one hand," Aumann lists the Shapley and Milnor-Shapley memoranda referred to above, and the papers of Davis and Peleg on von Neumann-Morgenstern solutions and their bargaining sets.

[9]As in von Neumann-Morgenstern, a footnote refers to three ideal phenomena in the natural sciences: a freely falling body, an ideal gas, and an ideal fluid. "The individual consumer (or merchant) is as anonymous to [the policy maker in Washington] as the individual molecule is to the physicist."

[10]The quotations in this paragraph, including the four points listed above, are all taken from Aumann (1964; Section 1).

[11]We refer to this theorem at length in the sequel.

[12]See the detailed elaboration in Dvoretsky-Wald-Wolfowitz (1951 a and b) and in Wald-Wolfowitz (1951); also Chapter 21 of this Handbook.

[13]As is well-known, there are no pure-strategy equilibria in the elementary matching pennies game.

[14]Aumann (1987) singles out Ville as the first user of continuous strategy sets in game theory. See, for example, Rauh (2000a) for some very recent work using a continuous price space as the strategy

space in the setting of large games.

[15]Theorem 1 in its precise form is due to Ma (1969); see also Fan (1966). The hypothesis of quasi-concavity goes back at least to Debreu (1952).

[16]This existence proof is furnished in Nash (1950). Since the payoff function is a "polylinear form in the probabilities," the set of best responses – countering points – is convex, and the longest paragraph in the paper concerned the closedness of the graph of the "one-to-many mapping of the product space to itself." Perhaps it ought to be noted here that Nash ascribes the idea for the use of Kakutani's fixed point theorem to David Gale. An alternative proof based on the simpler Brouwer fixed-point theorem is furnished in Nash (1951); it was to prove equally influential both for game theory and for general equilibrium theory.

[17]The setting of a Hausdorff topological vector space is a technical flourish whereby the two operations underlying the convexity hypothesis are abstracted and assumed to be continuous. The reader loses nothing of substance by thinking of each A_t as the unit interval. However, since we are no longer dealing with finite action sets, the compactness hypothesis needs to be made explicit.

[18]The use of the weak* topology in this context goes back to Glicksberg (1952); for details, see Billingsley (1968) and Parthasarthy (1967). Note also that Glicksberg did not utilize any metric hypothesis on the action sets; see Khan (1989) for dispensing with this hypothesis in another context.

[19]For measures on infinite product spaces, see, for example, Ash (1972; Section 2.7 and 4.4) or Loéve (1977; Sections 4 and 8). For Fubini's theorem, see, in addition to these references, Rudin (1974; Chapter 7).

[20]The compactness of $\mathcal{M}(A_t)$ is a basic property of the weak* topology known as Prohorov's theorem; see, for example, Billingsley (1968) or Parthasarathy (1967). The quasi-concavity of u_t is straightforward, and its continuity follows from Proposition 1 below. One can also furnish a direct proof based on the Schauder-Tychonoff theorem along the lines of Nash (1951), as in Peleg (1969).

[21]Definition 1 and Proposition 1 are due to Peleg (1969), who should be referred to for a proof.

[22]It is worth pointing out here that there are both countably additive and non-countably additive correlated equilibria in the example below; see Hart-Schmeidler (1989) for a discussion and references.

[23]If e denotes an infinite sequence of 1's and e_n the sequence with 1 in the first n places and 0 everywhere else, the sequence $\{e_n\}_{n\in\mathbb{N}}$ converges to e, but $u_t(e_n)$ is 1 for all $n \geq 1$ while $u_t(e)$ is -1.

[24]See Ash (1972; Section 7.2) or Loéve (1977; Section 16.3).

[25]If each player is allowed to attach equal probability to the outcome under the zero-one law, there would be a mixed strategy Nash equilibrium. For the implications of introducing individual subjective mappings from the probabilities formalizing societal responses to the space of probabilities on probabilities, see Chakrabarti-Khan (1991).

[26]As discussed in the introduction, this is necessitated by the needs of mathematical analysis. For interpretive difficulties, and even absurd results, that follow from finitely additive measures, see, for example, Hart-Schmeidler (1989) and Sun (1999c).

[27]There is little doubt that an extension to different action sets can be obtained by working with the hyperspace of closed subsets of a complete separable metric space. This idea is standard in general equilibrium theory; see Hildenbrand (1974; particularly Section B. II). For the use of this hyperspace in another relevant context, see Sun (1996b and 1999b).

[28]The motivation for this will become apparent when we turn to the special case of finite games with private information. One may draw a contrast here with Chakrabarti-Khan (1991) where the parameter ℓ is allowed to vary with each player deciding for herself how to conceive of societal stratification.

[29]This assumption shall also remain in force throughout the next section. However, nothing of substance is lost if the reader thinks of the set A, at first pass, as consisting of only two elements.

[30]As noted in the introduction, Schmeidler (1973) considered the case that $\ell = 1$ and A is finite.

[31]This is in itself a special case of the result in Bollobás-Varopoulos (1974), whose paper should be referred to for a proof based on the theorems of Hall and of Krein-Milman. For the proof of the special

case used here, see Rath (1996a). For the proof of the case where A is a finite set, see Hart-Kohlberg (1974; pp. 170-171) and Hildenbrand (1974; p. 74).

[32]As discussed above, Nash (1950) is the relevant benchmark. The analogy with the theory of integration of a correspondence reported in Aumann (1965) should also be evident to the reader.

[33]See Artstein (1983), Hart-Kohlberg (1974), and Sun (1996b). For approximate results, see these papers and Hart-Hildenbrand-Kohlberg (1974) and Hildenbrand (1974). For an expositional overview, see Khan-Sun (1997) and Sun (1999b).

[34]The proof is an exercise, but one needs the metrizability property of the weak* topology on $\mathcal{M}(A)$; see Khan-Sun (1996b) for details. As emphasized above, this purification result is a generalization of the corresponding result of Dvoretsky, Wald, and Wolfowitz to the case of countable actions.

[35]See Lemma 1 and its proof in Khan-Sun (1995b). For full details of the proof of Proposition 6, see Khan-Sun (1995b; Section 3).

[36]The first assertion is Berge's (1959; Section III.6) maximum theorem, and the second is its measure-theoretic version; see Castaing-Valadier (1977; Theorems III.14 and III.39) and Debreu (1967). An exposition of these results in the context of game theory is available in Khan (1986b). For the full details of the proof of Theorem 2, see Khan-Sun (1995b).

[37]A detailed elaboration of this subject is beyond the scope of this chapter; see Chapter 43 in this Handbook titled "Games of incomplete information."

[38]We confine ourselves to settings where all the players have an identical action set, and there are no information or type variables that are common to all of the players. For these essentially notational complications, as well as for the details of the computations involved in the proofs below, see Khan-Sun (1995b).

[39]Recall that the hypothesis of a countable compact metric action set is in force.

[40]For the first, see Aumann et al. (1983); and for the second, Milgrom-Weber (1985; Footnote 18), Khan-Rath-Sun (1999), and Khan-Sun (1999). The possibility of a positive result without the severe cardinality restrictions is suggested in Fudenberg-Tirole (1991; Theorem 6.2, p. 236).

[41]This idea is explicit in general equilibrium theory; see Kannai (1970; Section 7), Hart-Hildenbrand-Kohlberg (1974), and Hildenbrand (1975).

[42]Also see the formulations of Milgrom-Weber (1981, 1985) and Green (1984).

[43]This reformulation is due to Mas-Colell (1984a), and comes into non-cooperative game theory via general equilibrium theory; see Hart-Hildenbrand-Kohlberg (1974).

[44]The proof of Corollaries 4 and 6 has a somewhat tortured lineage. Corollary 4 in the case of finite A was first proved directly by Khan-Sun (1987), and the general case in Khan-Sun (1995a, 1995c). Corollary 6 in the case of finite A was proved by Mas-Colell (1984a) as a consequence of Kakutani's fixed-point theorem via results in Aumann (1965). The proof given here is due to Khan-Sun (1994).

[45]For two different proofs, one based on the Douglas-Lindenstrauss theorem and the other on a direct construction of suitable measures, see Khan-Sun (1995a).

[46]See Examples 1 and 2 in Rath-Sun-Yamashige (1995). The fact that they are also false without the cardinality assumption on A will be dealt with in detail below.

[47]This correspondence has a canonical status in general equilibrium theory and goes back at least to Debreu; see Kannai (1970; Section 7).

[48]Let f be such a measurable selection, and let $E = \{t \in [0,1] : f(t) \in [0,1]\}$. Then $\lambda^*(E) = \lambda(f^{-1}(E)) = \lambda(E)$. Since $\lambda^* = (1/2)\lambda$, $\lambda(E) = 0$, and hence $\lambda^*([-1,0]) = \lambda f^{-1}([-1,0]) = \lambda(\{t \notin E\}) = 1$, a contradiction.

[49]This is Example 3 in Rath-Sun-Yamashige (1995); see also Section 2 in Khan-Rath-Sun (1997a).

[50]See Billingsley (1968; p. 237-238) or Hildenbrand (1974; p. 49) for a definition.

[51]This furnishes the possible interpretation that there is a "bound on the diversity of payoffs." At any rate, it shows that nonexistence has nothing to do with the measurability of the function specifying the game.

[52]This assertion, though elementary, requires some delicate computations; see Khan-Rath-Sun (1997b) for details.

[53]As in Schmeidler (1973), but see Rauh (1997) for formulations involving higher moments.

[54]When ℓ equals one and the Lebesgue interval $([0,1], \mathcal{B}([0,1]), \nu)$ is the space of players' names, Schmeidler (1973) presents this result with A being the unit simplex in $\mathbb{R}^n$, and Rath (1992) proves the general case. Note that Remark 4 in Schmeidler (1973) does not cover this general case since a general utility function on a compact set A is usually not relevant to any quasi-concave function on the convex hull of A.

[55]For Lyapunov's theorem, see Lyapunov (1940) for the original statement; and Jamison (1974), Castaing-Valadier (1977), Diestel-Uhl (1977), and Loeb-Rashid (1987) for modern treatments. For a proof of Proposition 7, see Aumann (1965, 1976) and Hildenbrand (1974).

[56]This direct proof is due to Rath (1992). The proof in Schmeidler (1973) for the case that A is the unit simplex in $\mathbb{R}^n$ is based on a a purification argument.

[57]Distributions are emphasized, for example, in Milgrom-Weber (1981, 1985), Mas-Colell (1984a) and, in the context of general equilibrium theory, in Hart-Hildenbrand-Kohlberg (1974) and Hildenbrand (1975).

[58]We follow the terminology of the mathematical literature in reserving the term "vector integration" for integration of functions taking values in an infinite-dimensional space; see Diestel-Uhl (1977) and their references.

[59]Note the abuse of notation in expressing the value of the probability measure $g(t)$ at $B \in \mathcal{T}$, by $g(t; B)$.

[60]See Khan-Rath-Sun (1997a; Section 3) for details of these claims.

[61]For the basic definitions, see, for example, Royden (1968; Chapter 10) or Rudin (1973).

[62]See Diestel-Uhl (1977; Chapter II). For early applications of this integral in mathematical economics and game theory, see Bewley (1973) and Khan (1985a, 1986a).

[63]For basic properties of the weak and weak* topologies, see Diestel (1984). For different notions of measurability and their interrelationships, see Talagrand (1984).

[64]Since we are ignoring the predual, the above discussion can really be phrased in terms of any separable Banach space X. We may also point out here that there is yet another integral based on the reduction of a vector function to a real-valued function by evaluating it with respect to elements of its dual rather than its predual. This leads to the Pettis integral, a more elusive notion about which we shall have nothing to say here; see Talagrand (1984) for details and further references.

[65]See Khan-Sun (1996a) for details of proof.

[66]See Sun (1997) for counterexamples, and Khan-Sun (1997) and Sun (1999b) for an expositional overview; also Rustichini (1989).

[67]For any $f \in L_1(\lambda, X)$, $||f||_1 = \int_T ||f(t)|| d\lambda$; see Diestel-Uhl (1977; Chapter II) for details.

[68]For any $f \in L^w_\infty(\lambda, X^*)$ $||f||_\infty = \text{ess sup}_{t \in T} ||f(t)||$; see Dincleanu (1973).

[69]The first statement in the proposition below is due to Diestel (1977); see Khan (1984) for an alternative proof based on James' theorem. The second statement is a consequence of the Banach-Alaoglu theorem; see Castaing-Valadier (1977; Chapter V). For a leisurely discussion of the importance of these results for large games, see Khan (1986b).

[70]Thus, in contrast to the model of Section 2, instead of an arbitrary product of the players' actions, we assume measurable "profiles" of such actions. In contrast to the models of Sections 3 and 6, the situations considered here can be conceived as one of "widespread" externalities. Note, however, that this terminology has a different meaning in the work of Hammond (1995, 1998, 1999).

[71]The reason for this is that compact sets in the norm topology sets are "small"; a norm compact set in any normed linear space is contained in the closed convex hull of a sequence converging to zero; see Diestel (1984; Theorem 5; p. 3). However, see Toussaint (1984). We may also point out here that the hypothesis of weakly compact action sets guarantees that weakly measurable profiles are strongly

measurable as a consequence of the Pettis measurability theorem; see Diestel-Uhl (1977; Chapter II) and Uhl (1980). Also see Balder (1999a) for a synthetic treatment based on the so-called *feeble* topology.

[72]We shall use the notation $\text{ext}(A)$ to denote the set of extreme points of a set A. Note that in Corollary 8 below, $\text{ext}(\mathcal{M}(A))$ is the set of Dirac measures on A.

[73]See Section 4.2 above for definitions and comparisons.

[74]See Diestel-Uhl (1977; Chapter IX) for discussion and counterexamples. In particular, they observe that the examples "suggest that nonatomicity may not be a particularly strong property of vector measures, particularly from the point of view of the Lyapunov theorem in the infinite dimensional context."

[75]We leave it to the reader to develop the approximate analogues of Proposition 7; for details of the theory, see Hiai-Umegaki (1977), Artstein (1979), Khan (1985a), Khan-Majumdar (1986), Papageorgiou (1985, 1987, 1990), Yannelis (1991a).

[76]For a detailed proof of Theorem 4, see Khan (1985b, 1986a). The proof of Corollary 7 exploits the convexity hypothesis of $\mathcal{M}(A)$; see Mas-Colell (1984a) and Khan (1989) for a direct proof. For the details of proofs of Corollaries 8 and 9, see Schmeidler (1973), Khan (1986a), Pascoa (1988b, 1993b). Alternative direct proofs of these corollaries based on the ideas of Rath (1992) and the approximate integration theory discussed in Section 8.2 can also be furnished. The proof of Theorem 5 is a routine consequence of Proposition 8; see Khan-Rath-Sun (1997a) for details.

[77]For details as to these properties, see Khan-Rath-Sun (1997a). This function can be traced to Lyapunov; see Diestel-Uhl (1977; Chapter IX).

[78]In hindsight, one sees the same emphasis in the papers of Dvoretsky, Wald and Wolfowitz. It is also worth mentioning that Aumann concludes his paper by deferring to subsequent work a discussion of "the economic significance of the Lebesgue measure of a coalition." However, Debreu (1967) did point out "the identification of economic agents with points of an analytic set seems artificial".

[79]In addition to Loeb (1975), see Cutland (1983), Hurd-Loeb (1985), Lindstrom (1988), Anderson (1991, 1992), and Khan-Sun (1997).

[80]The relevant analogy is to the situation when a a user of Lebesgue measure spaces can afford to ignore the construction of a Lebesgue measure, and the Dedekind set-theoretic construction of real numbers on which it is based.

[81]The importance of Loeb measure spaces for game theory and mathematical economics is discussed, in particular, in Anderson's Chapter 14 in this Handbook; also see Rashid (1987), Anderson (1991), Khan-Sun (1997), and Sun (1999b).

[82]The first proposition is due to Sun (1996b, 1997) and the second to Keisler (1988; Theorems 2.1 and 2.3); see Khan-Sun (1997) and Sun (1999b) for expositional overviews. The fact that the second proposition is false for Lebesgue spaces was known to von-Neumann (1932).

[83]This is a specific instance of a Loeb measure space based on the hyperfinite set $N_\omega = (1, \cdots, \omega)$, where ω is an "infinitely large" integer, endowed with the counting probability measure on the set $\mathcal{N}_\omega$ of all internal subsets of N_ω. It is obtained by taking the "standard part" of the values of this finitely additive measure to obtain a countably additive measure and then its extension to the completion of the smallest σ-algebra containing $\mathcal{N}_\omega$. For details, see the references in Footnote 79 above.

[84]See Khan-Sun (1999; Section 7) for details.

[85]See Mas-Colell-Vives (1993; paragraph 2), who begin with the statement, "We have argued elsewhere (see Mas-Colell (1984 a and b), Vives (1988)) that strategic games with a continuum of players constitute a useful technique in economics."

[86]For a definition, see Billingsley (1968) or Parthasarathy (1967), and in the context of economic application, Hildenbrand (1974).

[87]Thus, as in Section 3 but now without countability requirements on A, $\mathcal{U}_A^d$ is the space of real-valued continuous functions on $A \times \mathcal{M}(A)^\ell$, endowed with its sup-norm topology and with $\mathcal{B}(\mathcal{U}_A^d)$ its Borel σ-algebra.

[88]For a detailed treatment of the asymptotic theory, see Khan-Sun (1996b, 1999). Note that this theory furnishes approximate results for the large but finite case rather than for an idealized limit setting as in Milgrom (1981, 1985), Aumann et al. (1983), Khan (1986a), Hausman (1987), Pascoa (1988, 1993b). Note also that these results have nothing to say about the rate of convergence problem as in Mas-Colell (1998), Kumar-Satterthwaite (1985), Gresik-Satterthwaite (1989), Satterthwaite-Williams (1989). Also see Rashid's (1983, 1985, 1987) work based on the Shapley-Folkman theorem for games with a finite number of players and a common finite action set.

[89]Note that $\lambda_i^n g_n^{-1}$ is the distribution on A induced by the restriction of g_n on T_i^n and is given for any $i = 1, \cdots, \ell$, by $(1/|T_i^n|) \sum_{s \in T_i^n} \delta_{g_n(s)}$ where for any $a \in A$, δ_a denotes the Dirac measure at a.

[90]This particular advantage of the nonstandard model, stemming from the simultaneous exploitation of finite and continuous methods, is by now well-understood; see Rashid (1987), Anderson (1991), Sun (1999b) and their references.

[91]See Section 2.2 titled *"Robinson Crusoe" Economy and Social Exchange Economy* in von Neumann-Morgenstern (1953; pp. 9-12).

[92]See, for example, the references in Feldman-Gilles (1985), Aiyagari (1994), Sun (1998a, 1999a), and Barut (2000).

[93]See Sun (1998b; Proposition 1) for details of a proof based on Fubini's theorem and the uniqueness of Radon-Nikodym derivatives. Earlier versions of this result are shown in Doob (1953). For additional complementary results, as well as an expositional overview, see Sun (1999c) and Hammond-Sun (2000).

[94]The first result is in Anderson (1976), the second in Keisler (1977) and the third in Sun (1998a).

[95]For an expositional overview, see Sun (1999b).

[96]In the context at hand, this implies by a version of Fubini's theorem due to Keisler (1977, 1984, 1988), the measurability of $f(\cdot, \omega)$ for $L(\bar{P})$-almost all $\omega \in \bar{\Omega}$, and of $f(t, \cdot)$ for $L(\bar{\lambda})$-almost all $t \in \bar{T}$.

[97]See Sun (1999 a and b). The same papers contain the details pertaining to Proposition 14 below. Note that the extension of the classical law of large numbers to correspondences is well-understood; see Arrow-Radner (1979), Artstein-Hart (1981) and their references.

[98]Here $\mathcal{U}_A^d$ is the space in Section 3.1 with $\ell = 1$.

[99]This formulation is due to Green (1984), with further work by Hausman (1987, 1988) and Pascoa (1993a, 1997). As we have seen in Sections 4.2, 8.2, and 10.3, Mas-Colell's (1984a) formulation of an anonymous game dispenses with the unit interval and focusses solely on $\mathcal{U}$.

[100]See Pascoa (1993b) for an approximate theorem in this context.

[101]The driving force behind this is the fact that any probability measure on the space $\mathcal{U} \times A$ can be represented as the induced distribution of a function $(i, f) : \mathcal{U} \times [0, 1] \longrightarrow \mathcal{U} \times A$; see Hart-Hildenbrand-Kohlberg (1974; pp. 164-165) and also Aumann (1964), Hausman (1987), Rustichini (1993) and Khan-Sun (1994) for related arguments. Indeed Hausman (1987) uses this fact as the basis for a definition of large games that are "thick".

[102]This formulation is due to Jovanovic-Rosenthal (1988) with further work by Massó-Rosenthal (1989), Bergin-Bernhardt (1992), Massó (1992), and Chakrabarti (2000).

[103]This is an important consideration especially in the context of applications to the existence of competitive equilibrium, and constitutes so-called "generalized games" or "abstract economies"; see Hausman (1987, 1988), Khan-Sun (1990), Tian (1992 a and b), and Toussaint (1984); the last one is set in the context of an arbitrary index of players.

[104]See Balder (1991), Khan (1989) and Rath (1996b).

[105]For action sets in a finite-dimensional space, see Khan-Vohra (1984). For general action sets, see Khan-Papageorgiou (1987a and b), Khan-Sun (1990), Kim-Prikry-Yannelis (1989), and Yannelis (1987). For difficulties of interpretation in the context of non-anonymous games, see Balder (2000).

[106]For the last four aspects, see Balder (1991, 1996), Chakrabarti-Khan (1991), Khan-Rustichini (1991, 1993), Kim-Yannelis (1997), and Shieh (1992).

[107]These are the so-called "Caratheodory selections"; see Artstein-Prikry (1987), Khan-Papageorgiou (1987a), Kim-Prikry-Yannelis (1987, 1988), and Yannelis (1991b).

[108]See Khan (1989) and Khan-Sun (1990), where the space of upper semi-continuous functions on the action set are topologized by the hypotopology of Dolecki-Salonetti-Wets, and the space of players by the compact-open topology.

[109]See Khan-Rustichini (1991), Chakrabarti-Khan (1991), and Balder-Rustichini (1994).

[110]See Rath (1994, 1998).

[111]See Balder (1995a and b) who makes an "externality mapping" an integral part of the definition of the game; also Valadier (1993), and Balder (1999a and b).

[112]Indeed, what we have been referring to as Nash equilibria are also termed Cournot-Nash equilibria, Nash-Cournot equilibria, or simply Cournot equilibria: Dubey et al. (1980), Allen (1994), and Novshek-Sonnenschein (1983) are respective examples.

[113]The quote is taken from the two different entries listed under Roberts (1987).

[114]See the first paragraph of Chapter VIII titled *Of Unlimited Competition* in Cournot (1838).

[115]The quote is from Mas-Colell (1998); also see Jaynes et al. (1978), Mas-Colell (1980, 1983, 1984b), Novshek (1980), Novshek-Sonnenschein (1978, 1980, 1983), Postlewaite-Schmeidler (1978), and Roberts-Postlewaite (1976). Questions of the rate of convergence are explored in Gresik-Satterthwaite (1989) and Satterthwaite-Williams (1989). Price-setting competition is explored in Allen (1994), Allen-Hellwig (1986 a and b, 1989), Gabszewicz-Vial (1972), and Roberts (1980).

[116]An intuitive suggestion of such a result was given by Hurwicz (1972). See also Dubey et al. (1980; p.340).

[117]See Myerson's Chapter 24 in this Handbook. The literature stemming from Vickrey (1945) and Mirrlees (1971) is voluminous but the basic references for environments with a continuum of agents are Dasgupta-Hammond (1980), Dubey et al. (1980), Hammond (1979a, 1979b), Champsaur-Laroque (1982), Mas-Colell-Vives (1993), and Hervés-Beloso et al. (1999). Makowski-Ostroy (1987, 1992) discuss the importance of large numbers in the context of a specific mechanism; see Roberts (1976), Roberts-Postlewaite (1976), and Cordoba-Hammond (1998) for some asymptotic results. For an overview, see Sonnenschein (1998) and his references.

[118]The basic insight is of course that of Arrow-Debreu (1954) and has engendered the literature on abstract economies; see Debreu (1952), Shafer-Sonnenschein (1975). Khan-Vohra (1984), Balder-Yannelis (1991), Yannelis (1987) consider this in an environment with a continuum of agents; see Balder (2000) for a critique.

[119]As in McKenzie (1955), Chipman (1970), Shafer-Sonnenschein (1976), Kaneko-Wooders (1986, 1989, 1994), and Hammond (1995, 1998, 1999). For the specific form of externality stemming from public goods in the sense of Samuelson (1954), see Khan-Vohra (1985), Sonnenschein (1998), and their references.

[120]See Dubey et al. (1980; p. 346), in particular, for a defence of this axiom. However, as they observe, there are environments where "the concept of mean ... is ... irrelevant to the equilibrium problem. It may not even be defined."

[121]See Hart (1979a, 1980, 1982, 1984), and Mas-Colell (1984b) for an asymptotic setting, and Pascoa (1988, 1993a) for the continuum.

[122]We have already seen the importance of diffuse information in Section 4.1. One can alternatively consider the set-up of a large exchange economy, as in Gul-Postlewaite (1992), or of bargaining, as in Mailath-Postlewaite (1990), or of an industry, as in Rob (1987). More recent investigations of Levine-Pesendorfer (1995), and Fudenberg-Levine-Pesendorfer (1998) are also relevant here.

[123]See Jovanovic-Rosenthal (1988) for a sketch of such a research program.

[124]See Lucas-Prescott (1974), Rauh (1996, 2000a, 2000b), and the McMillan-Rothschild Chapter 27 in this Handbook.

[125]As Hart (1979b) observes, "If each firm is negligible relative to the aggregate economy – a firm's shareholders will want the firm to maximize the (net) market value of its shares." Also see Kelsey-

Milne (1996), Kihlstrom-Laffont (1979), and Lucas (1978). It is of interest that Kelsey-Milne rely on results established for an abstract economy in Khan-Vohra (1984).

[126]Constantinedes-Rosenthal (1984), Hart (1979a), Haller (1988b), Kihlstrom-Laffont (1992), Nti (1988).

[127]See Gale (1979), Weinrich (1979) and their references.

[128]In addition to the relevant references in footnote 117, see Guesnerie (1981, 1995) and Dierker-Haller (1990); also Mas-Colell-Vives (1993).

[129]We began this chapter with a statement of Shapiro-Shapley on the relevance, in principle, of large games to the study of voting behavior in large societies; for recent work, see Chapters 29 and 30 in this Handbook, and the relevant section in Jovanovic-Rosenthal (1988). Also see Khan (1998) for the relevance of large games to questions of a more interdisciplinary nature.

[130]See Wieczorek (1996) and the Hammerstein-Selten Chapter 28 in this Handbook.

[131]See Karni-Schmeidler (1990), and for an application to restaurant pricing, Karni-Levin (1994).

[132]This motivation was already stressed in another context by Aumann (1964), where he is less than enthusiastic about generalizations where such changes do not obtain. Referring to the examination of Edgeworth's conjecture in the context of infinite commodities, he writes "This would serve no useful purpose. Our result holds for any number of commodities, many or few, so there is nothing to be gained by considering only the case of 'many' commodities."

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