Stochastic Choice Functions and Irresolute Choice Behavior

Edi Karni*
Johns Hopkins University

October 3, 2022

Abstract

This paper provides an axiomatic characterizations of stochastic choice functions that are rationalizable by the irresolute choice model of Karni (2022) and examines its applications to demand theory and portfolio selection.

.

.

.

Keywords: Stochastic choice functions, irresolute choice, random choice behavior, incomplete preferences, stochastic demand theory.

*Department of Economics. Email: karni@jhu.edu.

1 Introduction

It is standard practice in economics and decision theory to depict individual choice behavior by rational (i.e., transitive) preference relations on sets of alternatives whose interpretations are context dependent.[1] Formally speaking, a preference relation is a transitive and irreflexive binary relation, denoted by $\succ$, on a set A of alternatives, where $a \succ a'$ means that the alternative a is strictly preferred to a'.

The exact meaning of the last statement is open to interpretation. One interpretation is that the preference relation captures intrinsic characteristics of the decision maker that govern his choice behavior and make him choose alternative a whenever facing a choice between the distinct alternatives a and a'. An alternative interpretation takes the same statement to parsimoniously summarize the decision maker's observed choices. According to this revealed choice interpretation, $a \succ a'$ means that, other things, including his information, being the same, a decision maker repeatedly facing the need to choose between the alternatives a and a', consistently chooses alternative a.

The problem with both interpretations is that they are not testable with any finite number of observations. Put differently, the decision maker may choose randomly, and the fact that he has not yet been observed to have chosen a' from the set $\{a, a'\}$ may be because he rarely choses a'. By contrast, if observations of repeated choices reveal that either alternative has been chosen on occasion, the displayed choice behavior is not fully characterized by $\succ$. In other words, factors that are not captured by the primitives A and $\succ$ may affect the choice. These neglected factors may include unobserved psychological processes, such as boredom, variations in mood, changing needs, or inability to compare the alternatives that is resolved by deliberate randomization; by exogenous stimuli (e.g., imitation of others); or by subconscious neurological process (e.g., drift diffusion). To an observer, the decision maker's choice behavior appears to be stochastic.

Whatever the underlying causes, what one observes are the inputs (the sets of feasible alternatives) and the outputs (the alternatives chosen). Lacking the ability to discern what is going on in the decision maker's mind, one must, provisionally, settle on models that make sense of the observed choice

[1] Sometimes included in the definition of rational preference relation the condition of completeness (e.g., Mas-Collel, Whinston, and Green [1995]). However, there is nothing irrational in finding some alternatives noncomparable and displaying a preference relation that is incomplete.

patterns and derive their implications.[2]

In Karni (2022), I proposed a model, dubbed irresolute choice model, in which stochastic choice is expressed by a set of transitive and irreflexive binary relations $\succ^{\alpha}$ on A, referred to as random choice relations, where $a \succ^{\alpha} a'$ is interpreted to mean that, ceteris paribus, facing repeated choices from the set $\{a, a'\}$, the relative frequency with which a decision maker chooses a is at least α. More generally, according to the irresolute choice model, repeated choices under similar conditions from a feasible set of alternatives reveal that distinct alternatives are chosen with different frequencies.

A stochastic choice function assigns to every element in every feasible set of alternatives a probability of being selected. Stochastic choice functions are formal summaries of the relationships between the inputs and outputs that are constituents primitives of the model. The main purpose of this paper is to characterize the stochastic choice functions that are rationalizable as reflecting the choices depicted by the random choice relations and are represented by irresolute choice models.

The paper is organized as follows. The next section introduces the stochastic choice functions and analyzes their relations to the irresolute choice model. Section 3 discusses the representations of stochastic choice functions. Section 4 applies the irresolute choice model to the theories of demand and portfolio selection. Section 5 discusses the related literature and offers some concluding remarks.

2 Stochastic Choice Functions and the Irresolute Choice Model

2.1 Stochastic choice functions

Let A denote an arbitrary set with $\mid A \mid \geq 2$, referred to as the *choice set.* Elements of A are *alternatives*. Denote by $\mathcal{A}$ the set of all nonempty finite subsets of A. Elements of $\mathcal{A}$, dubbed *menus,* represent potential feasible sets of mutually exclusive alternatives that a decision maker may have to choose form.

A *stochastic choice function* (SCF) is a mapping $P : A \times \mathcal{A} \rightarrow [0, 1]$ such

[2] Improved understanding of the way the brain works may one day allow researchers to model the decision-making process at the neurological level.

that

$$\Sigma_{a\in M}P\left(a,M\right)=1,\ \text{for every } M\in\mathcal{A}$$

and

$$P\left(a',M\right)=0,\ \text{for every } a'\in A\backslash M.^{3}$$

I consider SCFs that feature two attributes. The first attribute is that the probability of choosing an alternative from a menu is (weakly) smaller the more inclusive the menu. This property restricts the structure of the SCF across menus, in the spirit of the weak axiom of revealed preference. In particular, it asserts that if an alternative a is revealed to be chosen from a menu M' with a certain frequency, then it is revealed to be chosen from a submenu $M\subset M'$ at lease as frequently. Formally,

(A.1) **Weak axiom of revealed stochastic choice (WARSC):** For all $M,\ M'\in\mathcal{A}$ such that $M\subset M'$ and $a\in M$, $P\left(a,M'\right)\leq P\left(a,M\right)$.[4]

The second attribute requires that the probabilistic choice relation depicted by SCF be transitive.

(A.2) **Transitivity:** For all $a,a',a''\in A$ and $M\in\mathcal{A}$, $P\left(a,M\right)>P\left(a',M\right)$ and $P\left(a',M\right)>P\left(a'',M\right)$ implies that $P\left(a,M\right)>P\left(a'',M\right)$.

2.2 Irresolute choice model

An *irresolute choice model* (ICM) is a set, $\{\succ^{\alpha}\mid\alpha\in[0,1]\}$, of binary relations on A, referred to as *probabilistic choice relations,* each of which is transitive and irreflexive, and jointly they satisfy set-inclusion monotonicity (i.e., for all $\alpha,\alpha'\in[0,1]$, $\succ^{\alpha}\subseteq\succ^{\alpha'}$ if and only if $\alpha'\leq\alpha$). For each $\alpha\in[0,1]$, the derived relations $\succcurlyeq^{\alpha},\sim^{\alpha}$, $\bowtie^{\alpha}$ and $\succsim^{\alpha}$ are defined as follows: $a\succcurlyeq^{\alpha}a'$ if, for all $a''\in A$, $a''\succ^{\alpha}a$ implies that $a''\succ^{\alpha}a'$; $a\sim^{\alpha}a'$ if $a\succcurlyeq^{\alpha}a'$ and $a'\succcurlyeq^{\alpha}a$; $a\bowtie^{\alpha}a'$ if and only if $\neg\left(a\succcurlyeq^{\alpha}a'\right)$ and $\neg\left(a'\succcurlyeq^{\alpha}a\right)$; $a\succsim^{\alpha}a'$ if $\neg\left(a'\succ^{\alpha}a\right)$.

Given any $a,a'\in A$, the interpretation of $a\succ^{\alpha}a'$ is as follows: Facing a choice from the menu $\{a,a'\}$, alternative a is strictly preferred and, hence, chosen, over a' with probability that is at least α. Thus, for all $\alpha'<\alpha$, $a\succ^{\alpha}a'$ implies that $a\succ^{\alpha'}a'$. Moreover, if $a\succcurlyeq^{\alpha}a'$ then, for all $\alpha'<\alpha$, $a\succ^{\alpha'}a'$.

Given any $a,a'\in A$, such that $\neg\left(a'\sim a\right)$, let $\bar{\alpha}\left(a,a'\right):=\sup\{\alpha\in[0,1]\mid$

[3] See Ok and Tserenjigmid (2021).

[4] The axiom may be stated as follow: For all $M,\ M'\in\mathcal{A}$ and $a\in M\cap M'$, $\max\{P\left(a,M\right),P\left(a,M'\right)\}\leq P\left(a,M\cap M'\right)$.

$a \succ^{\alpha} a'\}$.[5] Then, $a \succcurlyeq^{\bar{\alpha}(a,a')} a'$ implies that $\bar{\alpha}(a,a')$ is the exact probability that a is chosen from the set $\{a,a'\}$, and a' is chosen with probability $1-\bar{\alpha}(a,a')$. Consistent with the interpretation of the probabilistic choice relations, $a \succcurlyeq^1 a'$ and $\neg(a' \succcurlyeq^1 a)$ imply that a is chosen from the set $\{a,a'\}$ with a probability that is at least and, therefore, equal to, one.[6] If $a \sim^1 a'$, then insofar as the probability of a chosen over a' is concerned, the model is silent.

3 The Relationship between the ICM and SCF

3.1 Two questions

The depictions of the input (i.e., $M \in \mathcal{A}$) - output (i.e., $a \in M$) patterns by the ICM and SCF models raises two questions about the relationship between them:

(a) If a decision maker's choice behavior is described by an ICM, do his choices from menus necessarily generate an SCF that satisfies the weak axiom of revealed stochastic choice and transitivity?

(b) If a decision maker's choice behavior for a family of menus $\mathcal{A}$ is captured by an SCF that satisfies transitivity and the weak axiom of revealed stochastic choice is there an ICM that generates his choices?

To answer these questions, I introduce the following additional definitions and notations. An alternative $a \in M \in \mathcal{A}$ is said to be *dominated* if, for no $\alpha \in [0,1]$, it holds that $a \succcurlyeq^{\alpha} a'$, $\forall a' \in M \backslash \{a\}$. Let $D(M)$ denote the subset of dominated alternatives in M and let $UD(M) = M \backslash D(M)$ denote the subset of *undominated* alternatives in M. Formally, $UD(M) = \{a \in M \mid \exists \alpha \in [0,1]$, s.t. $a \succcurlyeq^{\alpha} a', \forall a' \in M\}$. Note that $UD(M)$ is not empty. For each $M \in \mathcal{A}$ and $a \in M$, I write $a \succcurlyeq^{\alpha} M$ if and only if $a \succcurlyeq^{\alpha} a'$, for all $a' \in M$.

Let $UD(M) = \{a_1, ..., a_m\}$ and, for each $a_i \in UD(M)$ define $\Lambda_i(M) = \{\alpha \in [0,1] \mid a_i \succcurlyeq^{\alpha} a', \forall a' \in UD(M)\}$. In words, $\Lambda_i(M)$ is the set of indices designating the random choice relations that rank the alternative a_i (weakly) higher than any other undominated alternative in the menu M.

[5] That the supremum exists follows from the fact that the set is bounded and that $\neg(a' \sim a)$ implies that there is $\alpha' \in [0,1]$ such that $a \succ^{\alpha'} a'$. Hence, the set is nonempty. Note that $a' \sim a$ implies that $a \sim^{\alpha} a'$ for all $\alpha \in [0,1]$.

[6] The $a \succ^1 a'$ is what is usually meant by the the strict preference for a over a'.

Define $\underline{\alpha}(a_i; M) = \inf \Lambda_i(M)$ and $\bar{\alpha}(a_i; M) = \sup \Lambda_i(M)$.[7] By definition, $\underline{\alpha}(a_i; M)$ and $\bar{\alpha}(a_i; M)$ are the indices of the probabilistic choice relations such that $\succcurlyeq^{\bar{\alpha}(a_i;M)} \subseteq \succcurlyeq^{\alpha} \subseteq \succcurlyeq^{\underline{\alpha}(a_i;M)}$, for all $\alpha \in \Lambda_i(M)$.

Without loss of generality and invoking monotonicity, rearrange the elements of $UD(M)$ in a ascending order of set inclusion (i.e., $\succcurlyeq^{\underline{\alpha}(a_1;M)} \subseteq \succcurlyeq^{\underline{\alpha}(a_2;M)} \subseteq, ..., \subseteq \succcurlyeq^{\underline{\alpha}(a_m;M)}$). If $\underline{\alpha}(a_1; M) = 1$ then a_1 is the only element of the undominated set and $D(M) = M \backslash \{a_1\}$. By definition, $\bar{\alpha}(a_1; M) = 1$, and $\bar{\alpha}(a_i; M) = \underline{\alpha}(a_{i-1}; M)$, for all $i = 2, ..., m$. Hence, in general, $1 = \bar{\alpha}(a_1; M) > \bar{\alpha}(a_2; M) >, ..., > \bar{\alpha}(a_m; M) > \bar{\alpha}(a_{m+1}; M) = 0$. Define $J_1(M) = [1, \bar{\alpha}(a_2; M)]$ and $J_i(M) = (\bar{\alpha}(a_i; M), \bar{\alpha}(a_{i+1}; M)]$, $i = 2, ..., m$. Then, $\mathcal{J}(M) := \{J_1(M), ..., J_m(M)\}$ is a partition of the unit interval.

3.2 SCFs generated by ICM

Given an ICM $\{\succ^{\alpha} | \ \alpha \in [0, 1]\}$, define a stochastic choice function $P : A \times \mathcal{A} \to [0, 1]$ by

$$P(a_i, M) = \begin{bmatrix} \bar{\alpha}(a_i; M) - \bar{\alpha}(a_{i+1}; M) & \text{if } a_i \in UD(M) \\ 0 & \text{if } a_i \notin UD(M) \end{bmatrix}. \tag{1}$$

The SCF $P(a_i, M)$ so defined is said to be *generated* by an ICM.

The following theorem asserts that the answer to the first question posed in the preceding section is affirmative.

Theorem 1. *A stochastic choice function P on $A \times \mathcal{A}$ generated by an irresolute choice model satisfies the weak axiom of revealed stochastic choice and transitivity.*

Proof. Given a ICM $\{\succ^{\alpha} | \ \alpha \in [0, 1]\}$ let P on $A \times \mathcal{A}$ be defined in (1). Then $a_i \succcurlyeq^{\alpha} M$, for all $\alpha \in J_i$. Let $M \subset M'$ and denote by $\mathcal{J}(M)$ and $\mathcal{J}(M')$ the corresponding partitions of the unit interval. Then $P(a_i, M') = \bar{\alpha}(a_i; M') - \bar{\alpha}(a_{i+1}; M')$, $a_i \in UD(M')$ is determined by the endpoints of the interval $J_i(M') \in \mathcal{J}(M')$ and $a_i \succcurlyeq^{\alpha} M'$, for all $\alpha \in [\bar{\alpha}(a_{i+1}, M'), \bar{\alpha}(a_i; M')]$. If $UD(M') = UD(M)$ then $\mathcal{J}(M) = \mathcal{J}(M')$ and $P(a_i, M') = P(a_i, M)$, for all $a_i \in M$. If $UD(M') \neq UD(M)$ then either $a_i \in UD(M) \backslash UD(M')$, in which case $P(a_i, M') = 0 \leq P(a_i, M)$, or $a_i \in UD(M) \cap UD(M')$. In the latter case, for some a_i, $a_i \succcurlyeq^{\alpha} M'$ for all $\alpha \in J_i(M') \cap J_i(M)$ in the meet

[7] That the infimum and supremum exist follows from the facts that the set $\Lambda_i(M)$ is bounded and, because a_i is undominated, $\Lambda_i(M)$ nonempty.

of $\mathcal{J}(M)$ and $\mathcal{J}(M')$. Let $\bar{\alpha}(a_i; M') - \bar{\alpha}\left(a'_{i+1}; M'\right)$ be the endpoints of the interval $J_i(M') \cap J_i(M)$. Since $J_i(M') \cap J_i(M) \subseteq J_i(M)$, we have that

$$P(a_i, M) = \bar{\alpha}(a_i; M) - \bar{\alpha}(a_{i+1}; M) \geq \bar{\alpha}(a_i; M') - \bar{\alpha}\left(a'_{i+1}; M'\right) = P(a_i, M').$$

Thus, $P(a_i, M)$ satisfies WARSC.

By definition, for all $a, a' \in A$, $a \succcurlyeq^\alpha a'$ if and only if $P(a, \{a, a'\}) = \bar{\alpha}(a; \{a, a'\}) - \bar{\alpha}(a'; \{a, a'\})$. But $\bar{\alpha}(a'; \{a, a'\}) = 0$ and $\bar{\alpha}(a; \{a, a'\}) = \alpha$. Hence, $P(a, \{a, a'\}) = \alpha$. Let $a, a', a'' \in A$ and suppose that $a \succcurlyeq^\alpha a'$ and $a' \succcurlyeq^\alpha a''$ then, by the argument above, $P(a, \{a, a'\}) = \alpha$ and $P(a', \{a', a''\}) = \alpha$. By transitivity of $\succ^\alpha$, $a \succcurlyeq^\alpha a''$. Hence, $P(a, \{a, a''\}) = \alpha$. Thus, P is transitive. ■

3.3 Rationalizable SCF

The next theorem asserts that the answer to the second question posed in the preceding section is affirmative and that the generating ICM is unique. An SCF P^* on $A \times \mathcal{A}$ is said to be *rationalized* by an ICM if $P^*(a, M) = P(a, M)$, for all $M \in \mathcal{A}$ and $a \in M$, where P is generated by an ICM.

Theorem 2. *If $P^* : A \times \mathcal{A} \to [0,1]$ is a transitive SCF satisfying the weak axiom of revealed stochastic choice, then there is a unique ICM that rationalizes it.*

Proof. Let P^* on $A \times \mathcal{A}$ be a transitive SCF satisfying WARSC. We need to show that there exists an ICM $\{\succ^\alpha | \ \alpha \in [0,1]\}$ such that $\succ^\alpha$ are transitive, irreflexive, and, for all $\alpha, \alpha' \in [0,1]$, $\succ^\alpha \subseteq \succ^{\alpha'}$ if and only if $\alpha' \leq \alpha$ and the SCF P generated by $\{\succ^\alpha | \ \alpha \in [0,1]\}$ satisfies $P(a, M) = P^*(a, M)$, for all $M \in \mathcal{A}$ and $a \in M$.

Define a binary relation $\succ^\alpha$ on A by $a \succ^\alpha a'$ if $P^*(a, \{a, a'\}) > \alpha$. Suppose that $a \succ^\alpha a'$ and $a' \succ^\alpha a''$. Then, by definition, $P^*(a, \{a, a'\}) > \alpha$ and $P^*(a', \{a', a''\}) > \alpha$. By transitivity of P^*, $P^*(a, \{a, a''\}) > \alpha$. Hence, by definition, $a \succ^\alpha a''$. Thus, $\succ^\alpha$ is transitive.

Since $\Sigma_{a \in M} P^*(a, M) = 1$, it holds that $P^*(a, \{a, a'\}) > \alpha$ if and only if $P^*(a', \{a, a'\}) < 1 - \alpha$. If $a = a'$ then, by definition, $P^*(a, \{a, a\}) = 1 > \alpha$ and $P^*(a, \{a, a\}) = 1 < 1 - \alpha$, for all $\alpha \in (0,1)$. By definition these inequalities imply that, for all $\alpha \in (0,1)$, $a \succ^\alpha a$ and $\neg(a \succ^\alpha a)$. A contradiction. Thus, $a \succ^\alpha a$ is irreflexive.

To see that the binary relations $\{\succ^\alpha | \ \alpha \in [0,1]\}$ satisfy set-inclusion monotonicity, let $\alpha, \alpha' \in [0,1]$ such that $\alpha' \leq \alpha$. Suppose that $a \succ^\alpha a'$

then, by definition, $P^*(a, \{a, a'\}) > \alpha$ implies $P^*(a, \{a, a'\}) > \alpha'$. Thus, by definition, $a \succ^{\alpha'} a'$. Consequently, $\succ^{\alpha} \subseteq \succ^{\alpha'}$.

Let P be generated by the ICM $\{\succ^{\alpha} \mid \alpha \in [0,1]\}$ defined above. Then P is given by (1). By Theorem 1, P is transitive and satisfies WARSC. Moreover, by (1), $P(a, \{a, a'\}) = \bar{\alpha}(a, a')$ and $a \succcurlyeq^{\bar{\alpha}(a,a')} a'$, for all doubleton sets $\{a, a'\} \in \mathcal{A}$ and $a \in \{a, a'\}$. By definition, $P^*(a, \{a, a'\}) = \bar{\alpha}(a, a')$. Thus, $P^*(a, \{a, a'\}) = P(a, \{a, a'\})$, for all doubleton sets $\{a, a'\} \in \mathcal{A}$ and $a \in \{a, a'\}$. But $P(a, M) = \min_{a' \in M} P(a', \{a, a'\})$ and $P^*(a, M) = \min_{a' \in M} P^*(a, \{a, a'\})$, for all $M \in \mathcal{A}$ and $a \in M$. Hence, $P^*(a, M) = P(a, M)$ for all $M \in \mathcal{A}$ and $a \in M$.

To establish uniqueness, it suffices to observe that, because $\mathcal{A}$ contains all the binary sets $\{a, a'\}$, the SCF P^* fully characterizes the binary relations $\succ^{\alpha}$, $\alpha \in [0,1]$, of the rationalizing ICM and that the ICM so characterized defines, by (1), a unique SCF P. ■

Corresponding to $\mathcal{J}$, define *choice function induced by ICM* as follows: Let $C : \mathcal{A} \to \mathcal{A}$ be a function defined by $C(M) = \{c(M, J_1), ..., c(M, J_m)\}$, where $c(M, J_i) = \{a \in M \mid a \succcurlyeq^{\alpha} M, \forall \alpha \in J_i\}$. Then, $\Pr\{c(M, J_i)\} = P(a_i, M)$, where the SCF P is generated by the ICM as in (1).

4 Representations and the Canonical Signal Spaces

4.1 Representations

In Karni (2022), I showed that the ICM in conjunction with the existing models of decision making under certainty, under risk, and under uncertainty are represented by sets of utility functions (in the cases of decision making under certainty and under risk) and sets of utility-probability pairs (in the case of decision making under uncertainty). To grasp this point, consider the case of decision making under certainty.

Let the choice set A be a nonempty topological space. A nonempty set $\mathcal{U}$ of real-valued functions on A is said to *represent* a transitive and irreflexive binary relation $\rhd$ on A if, for all $a, a' \in A$, $a \rhd a'$ if and only if $u(a) > u(a')$, for all $u \in \mathcal{U}$. The following is a corollary of Theorem 1 in Karni (2022).

Corollary: *Let A be a locally compact separable metric space and $\{\succ^{\alpha} \mid \alpha \in [0,1]\}$ a ICM, where $\succ^{\alpha}$are continuous, then there exists a collection $\{\mathcal{U}^{\alpha} \mid \alpha \in [0,1]\}$ of real-valued, continuous, strictly $\succ^{\alpha}$ −increasing, func-*

tions such that, for every $\alpha \in [0,1]$, $\mathcal{U}^\alpha$ represents $\succ^\alpha$, and $\alpha \geq \alpha'$ if and only if $\mathcal{U}^\alpha \supseteq \mathcal{U}^{\alpha'}$.

The uniqueness of the representation is as follows: Given any nonempty subset $\mathcal{U}^\alpha$ of $\mathbb{R}^A$, define the map $\Upsilon_{\mathcal{U}^\alpha} : A \to \mathbb{R}^{\mathcal{U}^\alpha}$ by $\Upsilon_{\mathcal{U}^\alpha}(a)(u) := u(a)$. Two nonempty subsets $\mathcal{U}^\alpha$ and $\mathcal{V}^\alpha$ of continuous real-valued functions on A represent the same preorder if, and only if, there exists an $f : \Upsilon_{\mathcal{U}^\alpha}(A) \to \Upsilon_{\mathcal{V}^\alpha}$ such that (i) $\Upsilon_{\mathcal{V}^\alpha} = f(\Upsilon_{\mathcal{U}^\alpha})$; and (ii) for every $b, c \in \Upsilon_{\mathcal{U}^\alpha}(A)$, $b > c$ if and only if $f(b) > f(c)$.[8]

4.2 Canonical signal spaces

The premise underlying the stochastic choice behavior depicted by the ICM is that choices are governed by a random signal–generating process that is not specified and, hence, not an explicit ingredient of the decision model. Consider the choice between two alternatives, a and a', such that $\neg(a \sim a')$. Then $P(a, \{a, a'\}) = \bar{\alpha}(a, a')$ may be interpreted as the probability of a signal that would resolve the indecision in favor of a. By Theorem 1 and the Corollary, this is the case if and only if $u(a) > u(a')$, for all $u \in \mathcal{U}^{\bar{\alpha}(a,a')}$.

Given SCF P, let $\{\succ^\alpha \mid \alpha \in [0,1]\}$ be the ICM that rationalizes it. Define a function $F : 2^{\mathcal{U}} \backslash \varnothing \to [0,1]$ as follows: For $\alpha \in [0,1]$, $F(\mathcal{U}^\alpha) = \alpha$. Then $P(a, \{a, a'\}) = F\left(\mathcal{U}^{\bar{\alpha}(a,a')}\right)$, for all $a, a' \in A$. In other words, facing a choice between two alternatives, a and a' that are not indifferent to one another, the decision maker behaves *as if* a function u is selected from $\mathcal{U}^1$ according to a probability distribution F and a is chosen if $u \in \mathcal{U}^{\bar{\alpha}(a,a')}$ and a' is chosen if $u \in \mathcal{U}^1 \backslash \mathcal{U}^{\bar{\alpha}(a,a')}$. Therefore, the set $\mathcal{U}^1$ may be taken to be the *canonical signal space.*

4.3 Representation of the SCF

Corresponding to the partition $\mathcal{J}(M)$ of $[0,1]$, define a partition of $\mathcal{U}^1$ as follows: Let $Q_i(M) := \{u \in \mathcal{U}^1 \mid u \in \mathcal{U}^{\bar{\alpha}(a_i;M)} / \mathcal{U}^{\bar{\alpha}(a_{i+1};M)}\}$, $i = 2, ..., m$.[9]

[8] See Evren and Ok (2011). Note that, in general, for arbitrary multi-utility representations, $\mathcal{V}^\alpha$ and $\mathcal{V}^{\alpha'}$, of two preorders, $\succcurlyeq^\alpha$ and $\succcurlyeq^{\alpha'}$, such that $\succcurlyeq^\alpha \subset \succcurlyeq^{\alpha'}$ does not imply that $\mathcal{V}^\alpha \supset \mathcal{V}^{\alpha'}$. Given $\succcurlyeq^\alpha$, the probability that the subject will choose a over a' when facing the choice from the set $\{a, a'\}$ does not depend on the representation. In other words, if $\mathcal{U}^\alpha$ and $\mathcal{V}^\alpha$ are two representations of $\succcurlyeq^\alpha$, then the functions in $\mathcal{V}^\alpha$ are given by the uniqueness of the representation.

[9] As indifference is not allowed, there is no ambiguity with regard to which element of the partition each utility function belongs to.

Then, $\alpha \in \Lambda_i(M)$ if and only if, for all $u \in Q_i(M)$, $u(a_i) \geq u(a')$, for all $\forall a' \in M \backslash \{a_i\}$. Since $\mathcal{U}^1$ is the canonical signal space, the probability of the signal $u \in Q_i(M)$ is $P(a_i, M)$. Consequently, given the SCF, P,

$$P(a_i, M) = F\left(\mathcal{U}^{\bar{\alpha}(a_i;M)}\right) - F\left(\mathcal{U}^{\bar{\alpha}(a_{i+1};M)}\right),\ i = 1, ..., m.$$

Thus, when facing a choice form a menu M, the decision maker behaves *as if* a utility function $u \in \mathcal{U}^1$ is selected according to the distribution F and $a_i \in M$ is chosen if $u \in \mathcal{U}^{\bar{\alpha}(a_i;M)} / \mathcal{U}^{\bar{\alpha}_{i+1}(a_i;M)}$, $i = 1, ..., m$.

5 Stochastic Demand and Portfolio Choice Theories

5.1 Stochastic demand functions

The application of the ICM to the theory of market demand is based on the following idea. When a consumer faces a menu consisting of commodity bundles, a utility function is selected at random from the canonical signal space according to some implicit probability measure, F, and the commodity bundle that maximizes this utility function is chosen. The empirical probability distribution on feasible sets of commodity bundles induced by F is an SCF P.

To model market demand, let $K = \{1, ..., K\}$ be the set of individuals in the market, and let $\mathbb{R}^n_+$ denote the set of alternatives representing commodity bundles. Menus are feasible budget sets, $B(p, I_k) = \{x \in \mathbb{R}^n_+ \mid x \cdot p \leq I_k\}$, where $p = (p_1, ..., p_n) \in \mathbb{R}^n_{++}$ denotes the price vector and I_k the income of individual k. Denote by $\mathcal{B}$ the set of budget sets. Assuming non-satiation, the undominated subset of $B(p, I_k) \in \mathcal{B}$ is $UB(p, I_k) = \{x \in \mathbb{R}^n_+ \mid x \cdot p = I_k\}$.

Denote by $\mathcal{U}^1_k$ the canonical signal space of individual k. Then, given a budget set $B(p, I_k)$ the realization of the random demands $\widetilde{x}^k(p, I_k)$ may be described as follows: For each $u \in \mathcal{U}^1_k$, let $x^*(p, I_k, u)$ be the solution to the program $\max u(x)$ subject to $x \in B(p, I_k)$, and denote by $x^*_i(p, I_k, u)$ its $i-th$ entry. Then, the stochastic commodity demands are driven by the random selection of a function $u \in \mathcal{U}^1_k$. Denote by F_k the probability measure on $\mathcal{U}^1_k$ of individual k and let $\widetilde{u}$ be the corresponding random utility function. Then, $\widetilde{x}^k(p, I_k) = x^*(p, I_k, \widetilde{u})$ is the observed random demands. For every $B(p, I_k) \in \mathcal{B}$ and $x \in B(p, I_k)$, let $U_k(x) = \{u \in \mathcal{U}^1_k \mid u(x) \geq u(x'), \forall x' \in$

$B(p, I_k)\}$. The revealed stochastic demand is an SCF $P : \mathbb{R}^n_+ \times \mathcal{B} \rightarrow [0,1]$ such that

$$P(x, B(p, I_k)) = F_k(U_k(x)). \tag{2}$$

The random demand for commodity i by individual k, $\widetilde{x}_i^k(p, I_k)$, whose support is $[0, I_k/p_i]$, is $x_i^*(p, I_k, \widetilde{u})$. Thus, given the budget set $B(p, I_k)$ the probability the individual k chooses x_i^k is $\Pr\left(x_i^k, B(p, I_k)\right) = P\left(x^k, B(p, I_k)\right)$. Given an income profile $I = (I_1, ..., I_K)$ and a price vector p, the market *stochastic demand function for commodity* i is: $\widetilde{X}_i(p, I) = \Sigma_{k=1}^K \widetilde{x}_i^k(p, I_k)$.

It is standard practice in economics to treat individual demands as independent variables.[10] The analogous assumption in the present context maintains that individual demands are stochastically independent random variables.[11] If $\widetilde{x}_i^k(p, I_k)$, $k \in K$, are stochastically independent, then the distribution, μ, of the market demand for commodity i, $\widetilde{X}_i(p, I)$, is given by the convolution $\mu_i^K = P(x_i^1, B(p, I_1)) * P(x_i^2, B(p, I_2)) * ... * P\left(x_i^K, B(p, I_K)\right)$. Expected demand is given by

$$E\left[\widetilde{X}_i(p, I)\right] = \Sigma_{k=1}^K \int_{U_k\left(x_i^*(p, I_k, u)\right)} x_i^*(p, I_k, u)\, dF_k(u). \tag{3}$$

Its variance is

$$Var\left(\widetilde{X}_i(p, I)\right) = \Sigma_{k=1}^K \int_{U_k\left(x_i^*(p, I_k, u)\right)} [x_i^*(p, I_k, u) - E\left[\widetilde{X}_i(p, I)\right]]^2 dF_k(u). \tag{4}$$

Standard practice notwithstanding, in many markets individual demands are correlated, possibly because of implicit social effects such as conformism and status seeking, for example. For instance, the demand for clothes is affected by fashion, the demand for vacation spots may be affected by the anticipated composition of the clientele, and demand for stocks may respond to information shared by many investors that respond to it in similar way. In these cases, the linearity of expectations implies that $E_k\left[\widetilde{X}_i(p, I)\right] = \Sigma_{k=1}^K E_k\left(\widetilde{x}_i^k(p, I_k)\right)$. The variance of market demand, however, depends on

[10] This assumption is reasonable when applied to commodities such as milk and gas; it is much less compelling when applied to other commodities.

[11] A collection of random avriables is said to be independent if every finite subcollection is independent.

the correlations among the individual demands and takes the form

$$Var\left(\widetilde{X}_i(p,I)\right) = \Sigma_{k=1}^{K} Var_k\left(\widetilde{x}_i^k(p,I_k)\right) + 2\Sigma_{j<k} Cov_k\left(\widetilde{x}_i^j(p,I_j),\widetilde{x}_i^k(p,I_k)\right). \tag{5}$$

In commodity markets in which individual demands are positively correlated, the individual stochastic choice behavior implied by the ICM induces greater demand fluctuation.

5.2 Comparative statics

Consider next the consequences of income and price variations on market demands. Suppose that, ceteris paribus, the income of individual k increases form I_k to I_k'. The supports of the random demands increase to $[0, I_k'/p_i]$, $i = 1, ..., n$. For each $u \in \mathcal{U}$, the optimal bundle changes from $x^*(p, I_k, u)$ to $x^*(p, I_k', u)$, and the corresponding change in the demand for commodity i is $x_i^*(p, I_k, u)$ to $x_i^*(p, I_k', u)$. For, each $u \in \mathcal{U}$, $x^*(p, I_k, u) \in \arg\max_{x \in B(p,I_k)} u(x)$ and $x^*(p, I_k', u) \in \arg\max_{x \in B\left(p,I_k'\right)} u(x)$, (2) implies that

$$P(x_i^*(p, I_k', u), B(p, I_k')) = P(x_i^*(p, I_k', u), B(p, I_k)).$$

The change in the demand distribution of commodity i depends on the income effects implies by the utility functions in the canonical signal space.

Similar considerations apply to relative price variations. Suppose that the price of commodity i increases from p_i to p_i'. Denote the new price vector by p'. Let $x^*(p', I_k, u)$ denote the optimal bundle given the budget set $B(p', I_k)$ corresponding to $u \in \mathcal{U}_k$ and let $x_i^*(p', I_k, u)$ denote its i entry. Then by the same argument as above,

$$P(x_i^*(p, I_k, u), B(p, I_k)) = P(x_i^*(p', I_k, u), B(p', I_k')).$$

The change in the market demand for commodity i is a random variable given by $\widetilde{X}_i(p', I) - \widetilde{X}_i(p, I) = \Sigma_{k=1}^{K}\left[\widetilde{x}_i^k(p', I_k) - \widetilde{x}_i^k(p, I_k)\right]$.

Example: Consider the case in which the set of utility functions of individual k consists of Cobb-Douglas utility functions (i.e., $\mathcal{U}_k = \{x_1^{\beta_1}x_2^{\beta_2}...x_n^{\beta_n} \mid \beta \in \left[\underline{\beta}_k, \bar{\beta}_k\right]^n, \underline{\beta}_k \geq 0, \Sigma_{i=1}^n \beta_i = 1\}$). Let $u_\beta = x_1^{\beta_1}x_2^{\beta_2}...x_n^{\beta_n}$ and denote by g_k the joint probability distribution function on $\left[\underline{\beta}_k, \bar{\beta}_k\right]^n$. Then, $x_i^*(p_i, I_k, u_\beta) = \beta_i I_k/p_i, i = 1, ..., n$. The stochastic demand for commodity i by individual

k, $\widetilde{x}_i^k(p, I_k)$ is depicted by g_k . Formally, let $\bar{g}_k(\beta_i)$ denote the marginal distribution of β_i then

$$P\left(x_i^*\left(p_i, I_k, u_\beta\right), B\left(p, I_k\right)\right) = \Pr\{\widetilde{x}_i^k\left(p, I_k\right) = x_i^*\left(p_i, I_k, u_\beta\right)\} = \bar{g}_k\left(\beta_i\right). \quad (6)$$

If the income of individual k increases from I_k to I_k', then the demand increases proportionally, (i.e.,for all $i = 1, ..., n$, $x_i^*\left(p_i, I_k', u_\beta\right) = \left(I_k'/I_k\right) x_i^*\left(p_i, I_k; u_\beta\right)$) and $P\left(\left(I_k'/I_k\right) x_i, B\left(p, I_k'\right)\right) = P\left(x_i, B\left(p, I_k\right)\right) = \bar{g}_k\left(\beta_i\right)$. Similarly, if the price of commodity 1 increase to p_1', then the demand decreases proportionally (i.e., $x_i^*\left(p_i', I_k; u_\beta\right) = \left(p_1/p_1'\right) x_1^*\left(p_i, I_k; u_\beta\right)$) and $P\left(\left(p_i'/I_k\right) x_i, B\left(p, I_k'\right)\right) = P\left(x_i, B\left(p, I_k\right)\right) = \bar{g}_k\left(\beta_i\right)$.

If the utility functions of all individuals $k \in K$ are Cobb-Douglas functions, then their demands are independent random variables. Consequently, given an income profile I and price vector p, the distribution of the market demand $\widetilde{X}_i(p, I)$ is the convolution of the distributions $\bar{g}_k$, $k = 1, ..., K$.

Let I' be another income profile. Then the change in the expected market demand is

$$E\left[\widetilde{X}_i(p, I)\right] = \Sigma_{k=1}^K\left[E_k\left(\widetilde{x}_i^k\left(p_i, I_k'\right)\right) - E_k\left(\widetilde{x}_i^k\left(p_i, I_k\right)\right)\right] = \Sigma_{k=1}^K\left[\left(I_k'/I_k\right) - 1\right] E_k\left(\widetilde{x}_i^k\left(p_i, I_k\right)\right),$$

where $E_k\left(\widetilde{x}_i^k\left(p_i, I_k\right)\right) = \int_{\underline{\beta}_k}^{\bar{\beta}_k} x_i^*\left(p_i, I_k; u_\beta\right) \bar{g}_k\left(\beta_i\right) d\beta_i$. The variance of individual demands increases by a factor $\left(I_k'/I_k\right)^2$ (i.e., $Var\left(\widetilde{x}_i^k\left(p, I_k'\right)\right) = \left(I_k'/I_k\right)^2 Var\left(\widetilde{x}_i^k\left(p, I_k\right)\right)$).

5.3 Stochastic portfolio choice

To apply the ICM to the theories of portfolio choice and financial markets, let $S = \{s_1, ..., s_n\}$ be a finite state space, and denote by $\{e^1, ..., e^n\}$ the corresponding set of Arrow securities.[12] The set of alternatives, $\mathbb{R}^n$, are portfolios of Arrow securities (i.e., portfolio is $y \in \mathbb{R}^n$, where y_i denotes the number of Arrow securities of type e^i in the portfolio). Denote by $\bar{y} = (1, ..., 1)$ the portfolio that consists of one Arrow security of each state. Then $\bar{y}$ is the riskless asset. Let $q = (q_1, ..., q_n)$ be the vector of prices of the Arrow securities then the price of $\bar{y}$ is $\bar{q} = \Sigma_{i=1}^n q_i$.

Let $\bar{y}_k$ denote the initial endowment of riskless asset of individual k whose value is $w_k = \bar{y}_k \cdot \bar{q}$. Then the budget set of individual k is $B(q, w_k) = \{y \in \mathbb{R}^n \mid y \cdot q = w_k\}$.

[12] An Arrow security $e^i = (0, 0, ..., 1, 0, ...0)$ pays off \$1 in the state s_i and nothing otherwise.

Denote by Π_k a set of subjective probability distributions on S representing the possible beliefs of individual k about the likely realizations of the states, and let u_k be a real-valued function on $\mathbb{R}$, representing his risk-attitudes. A preference relation, $\succ_k$, of individual k is said to exhibit *Knightian uncertainty* if, for all $y, y' \in \mathbb{R}^n$, $y \succ_k y'$ if and only if $\Sigma_{i=1}^n u_k(y_i)\pi(s_i) > \Sigma_{i=1}^n u_k(y'_i)\pi(s_i)$, for all $\pi \in \Pi_k$.[13] Note that, in this instance, the canonical signal space implied by the ICM is Π_k. Let $\mathcal{V}_k := \{\Sigma_{i=1}^n u_k(y_i)\pi(s_i) \mid \pi \in \Pi_k\}$ with generic element v_k.

Let G_k be a probability measure on Π_k induced by the ICM $\{\succ_k^\alpha \mid \alpha \in [0,1]\}$. Define

$$\Pi_k(y) = \{\pi \in \Pi_k \mid u_k(y)\cdot\pi \geq u_k(y')\cdot\pi, \forall\pi \in \Pi_k, y' \in B(q, w_k)\}.$$

The portfolio choice is decided by a selection of $\pi \in \Pi_k$ as follows: For each $\pi \in \Pi_k$, define the optimal portfolios of Arrow securities of individual k by

$$\widetilde{y}^k(q, w_k)(\pi) = (\widetilde{y}_i^k(q, w_k), ..., \widetilde{y}_n^k(q, w_k)(\pi) = \arg\max_{y \in B(q,w_k)} u_k(y)\cdot\pi.$$

Then $\widetilde{y}^k(q, w_k)$ is a random variable whose distribution (and that of $\widetilde{y}_i^k(q, w_k)$) induced by G_k. Formally, $\Pr\{\widetilde{y}^k(q, w_k) = y\} = G_k\{\Pi_k(y)\}$. Then the SCF representing the random portfolio choices of individual k is:

$$P_k(y, B(q, w_k)) = G_k\{\Pi_k(y)\}.$$

Clearly, $P_k(y_i, B(q, w_k)) = P_k(y, B(q, w_k))$.

Let $w = (w_1, ..., w_K)$ be a profile of initial endowments. The market demand $\widetilde{Y}_i(q, w)$ is the sum of individual demands, and the distribution of $\widetilde{Y}_i(q, w)$ is the convolution of the distributions of the individual demands, $P(y_i \mid q, w) = P_1(y_i, B(q, w_1)) * ... * P_K(y_i, B(q, w_K))$. The expected demand for Arrow security i is

$$E[Y_i] = \Sigma_{k=1}^K E[y_i, B(q, w_k) \mid P_k(\cdot, B(q, w_k))]$$

and it variance is given by

$$\Sigma_{k=1}^K Var\left(\widetilde{y}_i^k(q, w_k) \mid P_k(\cdot, B(q, w_k))\right) + 2\Sigma_{j<k} Cov\left(\widetilde{y}_i^j(p, w_j), \widetilde{x}_i^k(p, w_k) \mid P(y_i \mid q, w)\right).$$

Financial assets traded on the stock exchange are vectors of Arrow securities $\hat{y} = (y_1, ..., y_n)$ that pay off y_i in the state s_i, $i = 1, ..., n$, and are

[13] See Bewley (2002) and Galaabaatar and Karni (2013).

priced at $q_{\hat{y}} = \hat{y} \cdot q$. Consequently,from the viewpoint of individual k the stock expected value is $E_k [Y] = \int_{\Pi_k} \hat{y} \cdot \pi dG_k (\pi)$ and the variance is

$$Var_k (\hat{y} \cdot \pi) = \Sigma_{k=1}^{K} Var_k (\widetilde{y}_i (q, w_k)) + 2\Sigma_{i<j} Cov_k (\widetilde{y}_j (p, w_j), \widetilde{y}_i (p, w_k)).$$

6 Related Literature and Concluding Remarks

6.1 Related literature

Luce (1959) pioneered he study of random choice behavior. As in this paper, a primitive concept of Luce's model and those of more recent studies that extended it is a stochastic choice function summarizing the observed frequencies of choice of alternatives in the feasible sets.[14] Luce considers when the choice probabilities may be rationalized by a random utility model. At the individual level, random choice behavior may reflect the decision maker's indifference among feasible alternatives or his inability to compare them because of their complexity or the lack of familiarity with their consequences, which makes them difficult to evaluate. Ok and Tserenjigmid (2020) model these aspects of random choice behaviors by invoking the concept of stochastic choice functions. They characterize stochastic choice functions that assign positive probabilities solely to alternatives that constitute maximal elements of the feasible sets. They do not study the probability distributions on the sets of maximal elements.

The problem of revealed stochastic preference deals with a similar question – namely – whether the distribution of observed choices from variety of feasible sets of alternatives is consistent with preference maximization. Applied to a population, the distributions of observed choices arise because of heterogeneity of tastes and/or beliefs. Applied to individuals, the distribution is a reflection of stochastic variables underlying individual preferences. McFadden and Richter (1971, 1990), Falmagne (1978), Fishburn (1978), Stoye (2019) addressed the question of consistency of the distribution of observed choices with optimizing behavior.[15] The ICM may be regarded as a contribution to the part of this literature that deals with individual stochastic choice behavior. However, unlike the literature on stochastic preference

[14] See Echenique and Saito (2019), Ahumada and Ulku (2018), and Horan (2021) for extensions of Luce's seminal work.

[15] McFadden (2005) synthesizes and extends the literature on stochastic preference. He also provides an extensive reference list to this literature.

in which the set of utility functions is a primitive ingredient of the models, the primitive of the ICM is a set of incomplete probabilistic choice relations, that admit random utility representation. This difference is reflected in the axiomatic structures of the models.

Karni (2022) introduced and studied the ICM and the notion of canonical signal space that constitutes the foundation of the SCF. Karni and Safra (2016) axiomatized the representation of decision makers' perceptions of the stochastic process underlying the selection of their state of mind which, in turn, govern their choice behavior. This work may be regarded as providing axiomatic foundations of a probability measure F on the canonical signal space based on the decision makers' introspections.

Recently, there has been a revival of interest in random choice behavior (Gul, Natenzon, and Pesendorfer 2014; Fudenberg, Iijima, and Strzalecki 2015; Frick, Iijima, and Strzalecki 2019). Danan (2010), Agranov and Ortoleva (2017), and Cettolin and Riedl (2019) examined random choice behavior that is result of preference for deliberated randomization.

Becker (1962) argued that some economic theorems, such as the law of demand, do not depend on agents in the market behaving rationally. He showed that even if consumers choose their consumption bundles without attempting to optimize of some objective function, the change in the budget set caused by the relative price changes will force them to respond in a way that, in the aggregate, produces a downward-slopping demand functions. In Becker's analysis, households' choices may be irrational but not stochastic. Consequently, unlike in the market demand theory implied by the ICM, market demand is nonstochastic.

6.2 Consistency with violations of the weak axiom of revealed preference

According to the revealed preference approach, stochastic choice functions are empirical manifestations of random choice behavior that may be governed by decision makers' indifference among feasible alternatives, their inability to compare and rank the alternatives, variations in their moods, and/or changing needs. Whatever the underlaying motivations, the reasons for the observed stochastic choice may not be directly observable to an outsider; if they are driven by subconscious impulses they may not be accessible even to the decision maker. It is necessary in such cases to build theories that make sense of observations that consist of feasible alternatives and actual choices. The irresolute choice model is a way of making sense of observed random choices in repeated decision situations involving the same feasible set of alternatives summarized by stochastic choice functions.

The model is also consistent with some violations of the weak axiom of revealed preference (WARP).[16] To see why, let $x^0 \in \mathbb{R}^n_+$ denote the initial endowment of a decision makers commodities. Let p and p' be two price vectors and consider the budget sets $B(p, x^0 \cdot p)$ and $B(p', x^0 \cdot p')$. The corresponding undominated sets are

$$UD\left(B\left(p, x^0 \cdot p\right)\right) = \{x \in B\left(p, x^0 \cdot p\right) \mid \exists u \in \mathcal{U} \text{ s.t. } u(x) \geq u(x'), \forall x' \in B\left(p, x^0 \cdot p\right)\}$$

and

$$UD\left(B\left(p', x^0 \cdot p'\right)\right) = \{x \in B\left(p', x^0 \cdot p'\right) \mid \exists u \in \mathcal{U} \text{ s.t. } u(x) \geq u(x'), \forall x' \in B\left(p', x^0 \cdot p'\right)\}.$$

If $x^0 \in UD(B(p, x^0 \cdot p)) \cap UD(B(p', x^0 \cdot p'))$ then $UD(B(p, x^0 \cdot p)) \cap intB(p', x^0 \cdot p')$ and $UD(B(p', x^0 \cdot p')) \cap B(p, x^0 \cdot p)$ are nonempty.[17] Let $x^* \in UD(B(p, x^0 \cdot p)) \cap intB(p', x^0 \cdot p')$ and $x^{**} \in UD(B(p', x^0 \cdot p')) \cap intB(p, x^0 \cdot p)$ then the choices x^* from $B(p', x^0 \cdot p')$ and x^{**} from $B(p, x^0 \cdot p)$ constitute a violation of WARP.[18] Such choice is consistent with the irresolute choice model.

[16] I am grateful to Yujian Chen for calling my attention to this point.

[17] $intB\left(p, x^0 \cdot p\right)$ and $intB\left(p', x^0 \cdot p'\right)$ are the interiors of the corresponding budget sets in the $\mathbb{R}^n$ topology.

[18] In the case of complete preferences, $\mathcal{U}$ is a singleton set.

References

[1] Ahumada, Alonso and Ulku, Levent (2018) "Luce Rule with Limited Consideration," *Mathematical Social Sciences*, 93, 52-56.

[2] Agranov,Marina and Ortoleva, Pietro (2017) "Stochastic Choice and Preferences for Randomization," *The Journal of Political Economy*, 125, 40-68.

[3] Becker, Gary S. (1962) "Irretional Behavior and Economic Theory," *The Journal of Political Economy,* LXX, 1-13.

[4] Bewley, Truman F. (2002) "Knightian Decision Theory: Part I," *Decision in Economics and Finance,* 25, 79-110.

[5] Cettolin, Elena and Riedl, Arno (2019) "Revealed Preferences under Uncertainty: Incomplete Preferences and Preferences for Randomization," *Journal of Economic Theory,* 181, 547-585.

[6] Danan, Eric (2010) "Randomization vs. Selection: How to Choose in the Absence of Preference?" *Management Science* 56, 503–518.

[7] Echenique, Federico and Saito, Kota (2019) "General Luce Model," *Economic Theory*, 68, 811-826.

[8] Eliaz, Kfir and Ok, Efe, A. (2006) "Indifference or indecisiveness? Choice-theoretic Foundations of Incomplete Preferences," *Games and Economic Behavior*, 56, 61–86.

[9] Evren, Özgür and Ok, Efe, A. (2011) "On the Multi-Utility Representation of Preference Relations," *Journal of Mathematical Economics*, 47, 554-563.

[10] Falmagne, J. (1978) "A Representation Theorem for Finite Random Scale Systems." *Journal of Mathematical Psychology* 18, 52-72.

[11] Fishburn, Peter (1978) "Choice probabilities and choice functions." *Jounal of Mathematical Psychology* 10, 327-352

[12] Frick, Mira, Iijima, Ryota, and Strzalecki, Tomasz (2019) "Dynamic Random Utility," *Econometrica*, 87, 1941–2002.

[13] Fudenberg, Drew, Iijima, Ryota, and Strzalecki, Tomasz (2015) "Stochastic Choice and Revealed Perturbed Utility," *Econometrica*, 83, 2371–2409.

[14] Galaabaatar, Tsogbadral and Karni Edi (2013) "Subjective Expected Utility with Incomplete Preferences," *Econometrica,* 81, 255–284.

[15] Gul, Faruk, Natenzon, Paulo and Pesendorfer, Wolfgang (2014) "Random Choice as Behavioral Optimization," *Econometrica*, 82, 1873–1912.

[16] Horan, Sean (2021) "Stochastic Semi-Orders," *Journal of Economic Theory*, 192, Article 105171.

[17] Karni, Edi (2022) "Incomplete Preferences and Random Choice Behavior," unpublished manuscript.

[18] Karni, Edi and Safra, Zvi (2016) "A Theory of Stochastic Choice under Uncertainty," *Journal of Mathematical Economics* 63, 164-173.

[19] Luce, Duncan, R. (1959) *Individual Choice Behavior: A Theoretical Analysis.* New York: Wiley.

[20] Mas-Colell, Andreu, Whinston, Michael D. and Green, Jerry, R. (1995) *Microeconokic Theory.* Oxford University Press, Oxford.

[21] McFadden, Daniel L., Richter, Marcel, K. (1991) "Stochastic Rationality and Revealed Stochastic Preference." Boulder: Westview Press In Chipman, J., McFadden, D., Richter, K. (eds.) *Preferences, uncertainty, and rationality*, 161-186.

[22] McFadden, Daniel L., (2005) "Revealed Stochastic Preference; A Synthesis." *Economic Theory*, 26, 245-264.

[23] Ok, Efe A. and Tserenjigmid, Gerelt (2020) "Indifference, Indecisiveness, Experimentation and Stochastic Choice," *Theoretical Economics* (Forthcoming).

[24] Stoye, Jörg (2019) "Revealed Stochastic Preference: A One-Paragraph Proof and Generalization." *Economic Letters,* 177, 66-68.