Stochastic Choice Functions and Irresolute Choice Behavior

Edi Karni*
Johns Hopkins University

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Abstract

This paper provides axiomatic characterizations of stochastic choice functions that are rationalizable by the irresolute choice model of Karni (2022) and examines its applications to demand theory and portfolio selection.

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Keywords: Stochastic choice functions**,** irresolute choice, random choice behavior, incomplete preferences, stochastic demand theory.

*Department of Economics. Email: karni@jhu.edu. I am grateful to Emel Filiz Ozbay for her comments on an earlier version of this paper.

1 Introduction

It is standard practice in economics and decision theory to depict individual choice behavior by rational (i.e., transitive) preference relations on sets of alternatives whose interpretations are context dependent.[1] Formally speaking, a preference relation is a transitive and irreflexive binary relation, denoted by $\succ$, on a set A of alternatives, where $a \succ a'$ means that the alternative a is strictly preferred to a'.

The exact meaning of the last statement is open to interpretation. One interpretation is that the preference relation captures intrinsic characteristics of the decision maker that govern his choice behavior and make him choose the alternative a whenever facing a choice between a and a'. An alternative interpretation takes the same statement to parsimoniously summarize the decision maker's revealed choices. According to this interpretation $a \succ a'$ means that, other things being the same, facing the need to choose between the alternatives a and a', repeatedly, the decision maker consistently chooses a.

Underlying both interpretations is the notion described by Block and Marschak (1960) as "absolute consistency of choices." Absolute consistency may depict accurately choice behavior in some situations (e.g., when the choice is between bets ranked by pointwise first-order stochastic dominance). However, in many situations (e.g., choice between dining in Chinese or Indian restaurants), repeated choices reveal that different alternatives are chosen on occasion, producing a pattern depicted by stable frequency distribution. Such behavior may be described by Block and Marschak (1960) as "stochastic consistency of choices." The stochastic pattern may be the manifestation of the effects of factors not captured by the primitives A and $\succ$. The neglected factors may include unobserved psychological processes, such as boredom, variations in mood, changing needs, or inability to compare the alternatives that is resolved by deliberate randomization; by exogenous stimuli (e.g., imitation of others); or by subconscious neurological process (e.g., drift diffusion). To an observer, the decision maker's choice behavior appears to be stochastic.

Whatever the underlying causes, what one observes are the inputs (the

[1] Sometimes included in the definition of rational preference relation the condition of completeness (e.g., Mas-Collel, Whinston, and Green [1995]). However, there is nothing irrational in finding some alternatives noncomparable and exhibiting incomplete preferences.

sets of feasible alternatives) and the outputs (the alternatives chosen). Lacking the ability to discern what is going on in the decision maker's mind, one must, provisionally, settle on models that make sense of the observed choice patterns and derive their implications.[2]

In Karni (2022), I proposed a theory, dubbed irresolute choice model, in which stochastic choice is expressed by a set of transitive and irreflexive binary relations $\succ^{\alpha}$ on A, referred to as random or probabilistic choice relations, where $a \succ^{\alpha} a'$ is interpreted to mean that, ceteris paribus, facing repeated choices from the set $\{a, a'\}$, the relative frequency with which a decision maker chooses alternative a is at least α. More generally, according to the irresolute choice model, repeated choices under similar conditions from a feasible set of alternatives reveal that distinct alternatives are chosen with stable frequencies.

A stochastic choice function assigns to every element in every feasible set of alternatives a probability of being selected. Stochastic choice functions are formal summaries of the relationships between the inputs and outputs that are constituents primitives of the model. The main purpose of this paper is to characterize the stochastic choice functions that are rationalizable as reflecting the choices depicted by the random choice relations and are represented by irresolute choice models.

The contribution of this work to the literature dealing with the modelling and analysis of stochastic choice behavior is better understood after the ideas and results of this work are presented. I therefore relegate the discussion of the related literature to the concluding section.

The paper is organized as follows. The next section introduces the stochastic choice functions and the irresolute choice model. Section 3 analyzes the relationships between stochastic choice functions and irresolute choice behavior. Section 4 discusses the representations of stochastic choice functions. Section 5 applies the irresolute choice model to the theories of consumer demand and portfolio selection. Section 6 discusses the related literature and offers some concluding remarks.

[2] Improved understanding of the way the brain works may one day allow researchers to model the decision-making process at the neurological level.

2 Stochastic Choice Functions and the Irresolute Choice Model

2.1 Stochastic choice functions

Let A denote an arbitrary set with $\mid A \mid \geq 2$, referred to as the *choice set.* Elements of A are *alternatives*. Denote by $\mathcal{A}$ the set of all nonempty finite subsets of A. Elements of $\mathcal{A}$, dubbed *menus,* represent potential feasible sets of mutually exclusive alternatives that a decision maker may have to choose form.

A *stochastic choice function* (SCF) is a mapping $P : A \times \mathcal{A} \rightarrow [0, 1]$ such that

$$\Sigma_{a \in M} P(a, M) = 1, \text{ for every } M \in \mathcal{A}$$

and

$$P(a', M) = 0, \text{ for every } a' \in A \backslash M.$$[3]

I consider SCFs that feature two attributes. The first attribute, *regularity*, asserts that the probability of choosing an alternative from a menu is (weakly) smaller the more inclusive the menu.[4] This property restricts the structure of the SCF across menus in the spirit of the weak axiom of revealed preference. In particular, it asserts that if an alternative a is revealed to be chosen from a menu M' with certain frequency, then it is revealed to be chosen from a submenu $M \subset M'$ at lease as frequently. Formally,

(A.1) **Regularity:** For all M, $M' \in \mathcal{A}$ such that $M \subset M'$ and $a \in M$, $P(a, M') \leq P(a, M)$.[5]

The second attribute requires that the restriction of the probabilistic choice relation depicted by SCF to be binary menus be stochastically transitive. Formally,

(A.2) **Stochastic Transitivity (ST):** For all $a, a', a'' \in A$ and $\alpha \in [0, 1]$, $P(a, \{a, a'\}) > \alpha$ and $P(a', \{a', a''\}) > \alpha$ imply $P(a, \{a, a''\}) > \alpha$.

The literature dealing with stochastic choice behavior contains distinct conceptions of stochastic choice transitivity[6] One such concept is Partial Stochastic Transitivity (PST). Formally, for all a, $a', a'' \in A$, $P(a, \{a, a'\}) >$

[3]See Ok and Tserenjigmid (2021).

[4]See Block and Marschak (1960).

[5]The axiom may be stated as follow: For all M, $M' \in \mathcal{A}$ and $a \in M \cap M'$, $\max\{P(a, M), P(a, M')\} \leq P(a, M \cap M')$.

[6]See Fishburn (1973) for a discussion of different notions of stochastic transitivities.

$1/2$ and $P(a', \{a', a''\}) > 1/2$ implies $P(a, \{a, a''\}) \geq \min\{P(a, \{a, a'\}), P(a', \{a', a''\})\}$.[7]

The strict version of ST is stronger than PST. Formally,

Proposition: Strict ST $\Rightarrow$ PST.

Proof. Let $a, a', a'' \in A$, be such that $P(a, \{a, a'\}) > 1/2$ and $P(a', \{a', a''\}) > 1/2$. By ST, for all $\alpha \in \{\alpha \in [1/2, 1] \mid P(a, \{a, a'\}) > \alpha\} \cap \{\alpha \in [1/2, 1] \mid P(a', \{a', a''\}) > \alpha\}$ it holds that $P(a, \{a, a''\}) > \alpha$. Hence,

$$P(a, \{a, a''\}) \geq \sup\{\alpha \in [1/2, 1] \mid P(a, \{a, a'\}) > \alpha\} \cap \{\alpha \in [1/2, 1] \mid P(a', \{a', a''\}) > \alpha\}.$$

But

$$\sup\{\alpha \in [1/2, 1] \mid P(a, \{a, a'\}) > \alpha\} \cap \{\alpha \in [1/2, 1] \mid P(a', \{a', a''\}) > \alpha\} = \min\{P(a, \{a, a'\}), P(a$$

Hence, $P(a, \{a, a''\}) \geq \min\{P(a, \{a, a'\}), P(a', \{a', a''\})\}$. $\square$

2.2 Irresolute choice model

An *irresolute choice model* (ICM) is a set, $\{\succ^\alpha \mid \alpha \in [0, 1]\}$, of binary relations on A, referred to as *probabilistic choice relations,* each of which is transitive and irreflexive and jointly they satisfy set-inclusion monotonicity (i.e., for all $\alpha, \alpha' \in [0, 1]$, if $\alpha' \leq \alpha$ then $\succ^\alpha \subseteq \succ^{\alpha'}$).[8] For each $\alpha \in [0, 1]$, the derived relations $\succcurlyeq^\alpha, \sim^\alpha, \bowtie^\alpha$ and $\succcurlyeq^\alpha$ are defined as follows: $a \succcurlyeq^\alpha a'$ if, for all $a'' \in A$, $a'' \succ^\alpha a$ implies that $a'' \succ^\alpha a'$; $a \sim^\alpha a'$ if $a \succcurlyeq^\alpha a'$ and $a' \succcurlyeq^\alpha a$; $a \bowtie^\alpha a'$ if $\neg(a \succcurlyeq^\alpha a')$ and $\neg(a' \succcurlyeq^\alpha a)$; $a \succcurlyeq^\alpha a'$ if $\neg(a' \succ^\alpha a)$.

Given any $a, a' \in A$, the interpretation of $a \succ^\alpha a'$ is as follows: Facing a choice from the menu $\{a, a'\}$, alternative a is strictly preferred and, hence, chosen, over a' with probability that is at least α. Thus, for all $\alpha' < \alpha$, $a \succ^\alpha a'$ implies that $a \succ^{\alpha'} a'$. Moreover, if $a \succcurlyeq^\alpha a'$ then, for all $\alpha' < \alpha$, $a \succ^{\alpha'} a'$.

[7] Another concept, Moderate Stochastic Transitivity, is obtained from PST by replacing the strict inequalities in the hypothesis with weak inequalities. He and Natenzon (2022) show that a version of Moderate Stochastic Transitivity is necessary and sufficient for a binary stochastic choice rule, ρ, to have a moderate utility representations proposed by Halff (1976).

[8] Robert (1971) studied the relations between nested semiorders and the family $\{\succ^\alpha \mid \alpha \in [1/2, 1)\}$ induced by P (i.e., $a \succ^\alpha a'$ if and only if $P(a, \{a'a'\}) > \alpha$).

Given any $a, a' \in A$, such that $\neg (a' \sim^1 a)$, let $\bar{\alpha}(a, a') := \sup\{\alpha \in [0,1] \mid a \succ^\alpha a'\}$.[9] Then, $a \succcurlyeq^{\bar{\alpha}(a,a')} a'$ implies that $\bar{\alpha}(a, a')$ is the exact probability that a is chosen from the set $\{a, a'\}$, and a' is chosen with probability $1 - \bar{\alpha}(a, a')$. Consistent with the interpretation of the probabilistic choice relations, $a \succcurlyeq^1 a'$ and $\neg (a' \succcurlyeq^1 a)$ imply that a is chosen from the set $\{a, a'\}$ with a probability that is at least and, therefore, equal to, one.[10] If $a \sim^1 a'$ then, insofar as the probability of a chosen over a' is concerned, the model is silent. By definition, for all $a \in A$ and $\alpha \in [0,1]$, $a \succcurlyeq^\alpha a$ and $a \succcurlyeq^\alpha a$ implying that $a \sim^\alpha a$

3 The Relationship between the ICM and SCF

3.1 Two questions

The depictions of the input (i.e., $M \in \mathcal{A}$) - output (i.e., $a \in M$) patterns by the ICM and SCF models raises two questions about the relationship between them:

(a) If a decision maker's choice behavior is described by an ICM, do his choices from menus necessarily generate an SCF that satisfies regularity and stochastic transitivity?

(b) If a decision maker's choice behavior for a family of menus $\mathcal{A}$ is captured by an SCF that satisfies stochastic transitivity and regularity is there an ICM that generates his choices?

To answer these questions, I introduce the following additional definitions and notations. An alternative $a \in M \in \mathcal{A}$ is said to be *dominated* if, for no $\alpha \in [0,1]$, it holds that $a \succcurlyeq^\alpha a'$, $\forall a' \in M \backslash \{a\}$. Let $D(M)$ denote the subset of dominated alternatives in M and let $UD(M) = M \backslash D(M)$ denote the subset of *undominated* alternatives in M. Formally, $UD(M) = \{a \in M \mid \exists \alpha \in [0,1]$, s.t. $a \succcurlyeq^\alpha a', \forall a' \in M\}$. Note that $UD(M)$ is not empty. For each $M \in \mathcal{A}$ and $a \in M$, I write $a \succcurlyeq^\alpha M$ if and only if $a \succcurlyeq^\alpha a'$, for all $a' \in M$.

Let $UD(M) = \{a_1, ..., a_m\}$ and, for each $a_i \in UD(M)$ define $\Lambda_i(M) =$

[9] That the supremum exists follows from the fact that the set is bounded and that $\neg (a' \sim^1 a)$ implies that there is $\alpha' \in [0,1]$ such that $a \succ^{\alpha'} a'$. Hence, the set is nonempty. Note that $a' \sim^1 a$ implies that $a \sim^\alpha a'$ for all $\alpha \in [0,1]$.

[10] In terms of the ICM $a \succ^1 a'$ is what is usually meant by the strict preference for a over a'.

$\{\alpha \in [0,1] \mid a_i \succcurlyeq^{\alpha} a', \forall a' \in UD(M)\}$. In words, $\Lambda_i(M)$ is the set of indices designating the random choice relations that rank the alternative a_i (weakly) higher than any other undominated alternative in the menu M. Define $\underline{\alpha}(a_i, M) = \inf \Lambda_i(M)$ and $\bar{\alpha}(a_i, M) = \sup \Lambda_i(M)$.[11] By definition, $\underline{\alpha}(a_i; M)$ and $\bar{\alpha}(a_i; M)$ are the indices of the probabilistic choice relations such that $\succcurlyeq^{\bar{\alpha}(a_i;M)} \subseteq \succcurlyeq^{\alpha} \subseteq \succcurlyeq^{\underline{\alpha}(a_i;M)}$, for all $\alpha \in \Lambda_i(M)$.

Without loss of generality and invoking set-inclusion monotonicity, rearrange the elements of $UD(M)$ in a ascending order of set inclusion (i.e., $\succcurlyeq^{\underline{\alpha}(a_1;M)} \subseteq \succcurlyeq^{\underline{\alpha}(a_2;M)} \subseteq, ..., \subseteq \succcurlyeq^{\underline{\alpha}(a_m;M)}$). Define $J_i(M) = [\underline{\alpha}(a_i, M), \bar{\alpha}(a_i, M))$, $i = 1, 2, ..., m$. Then, $\mathcal{J}(M) := \{J_1(M), ..., J_m(M)\}$ is a partition of the unit interval.

3.2 SCFs generated by ICM

Given an ICM $\{\succ^{\alpha} \mid \alpha \in [0,1]\}$, define a stochastic choice function $P : A \times \mathcal{A} \rightarrow [0,1]$ by

$$P(a_i, M) = \begin{bmatrix} \bar{\alpha}(a_i; M) - \underline{\alpha}(a_i; M) & \text{if } a_i \in UD(M) \\ 0 & \text{if } a_i \notin UD(M) \end{bmatrix}. \tag{1}$$

The SCF $P(a_i, M)$ so defined is said to be *generated* by an ICM. Note that for binary menus, $M = \{a, a'\}$, if $\succcurlyeq^{\underline{\alpha}(a;\{a,a'\})} \subseteq \succcurlyeq^{\underline{\alpha}(a';\{a,a'\})}$ then $\underline{\alpha}(a; \{a, a'\}) = 0$. Hence, by definition, $P(a, \{a, a'\}) = \alpha$ if and only if $a \succcurlyeq^{\alpha} a'$ and $\neg (a \succ^{\alpha} a')$.

The following theorem asserts that the answer to the first question posed in the preceding section is affirmative.

Theorem 1. *A stochastic choice function P on $A \times \mathcal{A}$ generated by an irresolute choice model satisfies regularity and stochastic transitivity.*

Proof. Given a ICM $\{\succ^{\alpha} \mid \alpha \in [0,1]\}$ let P on $A \times \mathcal{A}$ be defined in (1). Then $a_i \succcurlyeq^{\alpha} M$, for all $\alpha \in \bar{J}_i(M)$, where $\bar{J}_i(M)$ is the closure of $J_i(M)$. Let $M \subset M'$ and denote by $\mathcal{J}(M)$ and $\mathcal{J}(M')$ the corresponding partitions of the unit interval. Then, for all $a_i \in UD(M')$, $P(a_i, M') = \bar{\alpha}(a_i; M') - \underline{\alpha}(a_i; M')$, and $a_i \succcurlyeq^{\alpha} M'$ for all $\alpha \in \bar{J}_i(M')$.

If $UD(M') = UD(M)$ then $\mathcal{J}(M) = \mathcal{J}(M')$ and $P(a_i, M') = P(a_i, M)$, for all $a_i \in M$. If $UD(M') \neq UD(M)$ then either $a_i \in UD(M) \cap D(M')$, or $a_i \in UD(M) \cap UD(M')$. In the former case $P(a_i, M') = 0 \leq P(a_i, M)$

[11] That the infimum and supremum exist follows from the facts that the set $\Lambda_i(M)$ is bounded and, because a_i is undominated, $\Lambda_i(M)$ nonempty.

and, in the latter case, $a_i \succcurlyeq^{\alpha} M'$ for all $\alpha \in \bar{J}_i(M')$. But $\bar{J}_i(M') \subseteq \bar{J}_i(M)$. Hence

$$P(a_i, M) = \bar{\alpha}(a_i; M) - \underline{\alpha}(a_i; M) \geq \bar{\alpha}(a_i; M') - \underline{\alpha}(a_i; M') = P(a_i, M'). \quad (2)$$

Thus, $P(a_i, M)$ satisfies regularity.

By definition, for all binary menus, $\{a, a'\} \in \mathcal{A}$, if $\succcurlyeq^{\underline{\alpha}(a;\{a,a'\})} \subseteq \succcurlyeq^{\underline{\alpha}(a';\{a,a'\})}$ then $\underline{\alpha}(a, \{a, a'\}) = 0$. Hence, $a \succ^{\alpha} a'$ if and only if $P(a, \{a, a'\}) = \bar{\alpha}(a; \{a, a'\}) > \alpha$. Let $a, a', a'' \in A$ and consider the binary menus $\{a, a'\}$, $\{a', a''\}$ and $\{a, a''\}$. Suppose that $P(a, \{a, a'\}) > \alpha$ and $P(a', \{a', a''\}) > \alpha$. Hence,

$$\min\{\bar{\alpha}(a, \{a, a'\}), \bar{\alpha}(a', \{a', a''\})\} > \alpha. \quad (3)$$

Without loss of generality let $\min\{\bar{\alpha}(a, \{a, a'\}), \bar{\alpha}(a', \{a', a''\})\} = \bar{\alpha}(a', \{a', a''\})$ then

$$\sup\{\alpha' \in [0, 1] \mid a \succ^{\alpha'} a' \text{ and } a' \succ^{\alpha'} a''\} = \bar{\alpha}(a', \{a', a''\}) > \alpha. \quad (4)$$

By the transitivity of the probabilistic choce relations, $\succ^{\alpha}$, $a \succ^{\alpha'} a'$ and $a' \succ^{\alpha'} a''$ imply $a \succ^{\alpha'} a''$. Thus,

$$\sup\{\alpha' \in [0, 1] \mid a \succ^{\alpha'} a''\} \geq \sup\{\alpha' \in [0, 1] \mid a \succ^{\alpha'} a' \text{ and } a' \succ^{\alpha'} a''\}.$$

Hence, by (4),

$$\sup\{\alpha' \in [0, 1] \mid a \succ^{\alpha'} a''\} \geq \bar{\alpha}(a, \{a, a''\}) > \alpha. \quad (5)$$

By definition, $P(a, \{a, a''\}) = \bar{\alpha}(a, \{a, a''\})$. Thus, P is stochastically transitive. ■

3.3 Rationalizable SCF

The next theorem asserts that the answer to the second question posed in the preceding section is affirmative, and that the generating ICM is unique. An SCF P^* on $A \times \mathcal{A}$ is said to be *rationalized* by an ICM if $P^*(a, M) = P(a, M)$, for all $M \in \mathcal{A}$ and $a \in M$, where P is generated by an ICM.

Theorem 2. *If $P^* : A \times \mathcal{A} \to [0, 1]$ is an SCF satisfying regularity and stochastic transitivity then there is a unique ICM that rationalizes it.*

Proof. Let P^* on $A \times \mathcal{A}$ be a stochastically transitive SCF satisfying regularity. We need to show that there exists an ICM $\{\succ^{\alpha} \mid \alpha \in [0, 1]\}$ such that $\succ^{\alpha}$ are transitive, irreflexive, and satisfy set-inclusion monotonicity, and that the SCF P the ICM generates satisfies $P(a,M) = P^*(a,M)$, for all $M \in \mathcal{A}$ and $a \in M$.

For every $\alpha \in [0,1]$, define a binary relation $\succ^{\alpha}$ on A by $a \succ^{\alpha} a'$ if $P^*(a,\{a,a'\}) > \alpha$. Let $\alpha, \alpha' \in [0,1]$ be such that $\alpha' \leq \alpha$. If $a \succ^{\alpha} a'$ then, by definition, $P^*(a,\{a,a'\}) > \alpha$. Hence, $P^*(a,\{a,a'\}) > \alpha'$. Thus, by definition, $a \succ^{\alpha'} a'$. Consequently, $\succ^{\alpha} \subseteq \succ^{\alpha'}$. Thus, $\{\succ^{\alpha} \mid \alpha \in [0,1]\}$ satisfies set-inclusion monotonicity.

Let $a, a', a'' \in A$ and suppose that $a \succ^{\alpha} a'$ and $a' \succ^{\alpha} a''$. By definition $P^*(a,\{a,a'\}) > \alpha$ and $P^*(a',\{a',a''\}) > \alpha$. By stochastic transitivity $P^*(a,\{a,a''\}) > \alpha$. Hence, by definition, $a \succ^{\alpha} a''$. Thus, $\succ^{\alpha}$ is transitive.

For all binary menus $\{a,a'\}$, $P^*(a,\{a,a'\}) + P^*(a',\{a,a'\}) = 1$. Hence, $P^*(a,\{a,a\}) > \alpha$ if and only if $P^*(a,\{a,a\}) < 1-\alpha$. Thus, by definition, for all $\alpha \in [0,1]$, $a \succ^{\alpha} a$ if and only if $\neg(a \succ^{1-\alpha} a)$. If $\alpha > 0.5 > 1-\alpha$ then $a \succ^{\alpha} a$ and $\neg(a \succ^{1-\alpha} a)$ contradic set-inclusion monotonicity. If $\alpha < 0.5 < 1-\alpha$ then, by set-inclusion monotonicity, $a \succ^{\alpha} a$ implies that $a \succ^{1-\alpha} a$, a contradiction. Hence, $\succ^{\alpha}$ is irreflexive.

Let P be generated by the ICM defined above. Then P is given by (1) and, by Theorem 1, it is stochastically transitive and satisfies regularity. Then $P^*(a,M)$, the probability of the event "a is chosen from the menu M," is the intersection of the events "a is chosen from the binary menus $\{a,a'\}$, for all $a' \in M$." By definition of $\succ^{\alpha}$, a is chosen from the binary menus $\{a,a'\}$ for all $\alpha \in [0,1]$ if $a \succ^{\alpha} a'$. Thus,

$$P^*(a,M) = \Pr \cap_{a' \in M} \{\alpha \in [0,1] \mid a \succ^{\alpha} a'\}. \tag{6}$$

But, by (1), $P(a,M) = \Pr \cap_{a' \in M} \{\alpha \in [0,1] \mid a \succ^{\alpha} a'\}$. Hence, $P(a,M) = P^*(a,M)$.

To establish uniqueness, it suffices to observe that, because $\mathcal{A}$ contains all the binary menus $\{a,a'\}$, the SCF P^* fully characterizes the binary relations $\succ^{\alpha}$, $\alpha \in [0,1]$, of the rationalizing ICM and that the ICM so characterized defines, by (1), a unique SCF P. ■

Corresponding to $\mathcal{J}$, define *choice function induced by ICM* as follows: Let $C : \mathcal{A} \to \mathcal{A}$ be a function defined by $C(M) = \{c(M,J_1), ..., c(M,J_m)\}$, where $c(M,J_i) = \{a \in M \mid a \succcurlyeq^{\alpha} M, \forall \alpha \in J_i\}$. Then, $\Pr\{c(M,J_i)\} = P(a_i,M)$, where the SCF P is generated by the ICM as in (1).

4 Representations and the Canonical Signal Spaces

4.1 Representations

In Karni (2022), I showed that the ICM in conjunction with the existing models of decision making under certainty, under risk, and under uncertainty are represented by sets of utility functions (in the cases of decision making under certainty and under risk) and sets of utility-probability pairs (in the case of decision making under uncertainty). To grasp this point, consider the case of decision making under certainty.

Let the choice set A be a nonempty topological space. A nonempty set $\mathcal{U}$ of real-valued functions on A is said to *represent* a transitive and irreflexive binary relation $\rhd$ on A if, for all $a, a' \in A$, $a \rhd a'$ if and only if $u(a) > u(a')$, for all $u \in \mathcal{U}$. The following is a corollary of Theorem 1 in Karni (2022).

Corollary: *Let A be a locally compact separable metric space and $\{\succ^{\alpha} \mid \alpha \in [0,1]\}$ an ICM, where $\succ^{\alpha}$ are continuous, then there exists a collection $\{\mathcal{U}^{\alpha} \mid \alpha \in [0,1]\}$ of real-valued, continuous, strictly $\succ^{\alpha}$ $-$increasing, functions such that, for every $\alpha \in [0,1]$, $\mathcal{U}^{\alpha}$ represents $\succ^{\alpha}$, and $\alpha \geq \alpha'$ if and only if $\mathcal{U}^{\alpha} \supseteq \mathcal{U}^{\alpha'}$.*

The uniqueness of the representation is as follows: Given any nonempty subset $\mathcal{U}^{\alpha}$ of $\mathbb{R}^{A}$, define the map $\Upsilon_{\mathcal{U}^{\alpha}} : A \to \mathbb{R}^{\mathcal{U}^{\alpha}}$ by $\Upsilon_{\mathcal{U}^{\alpha}}(a)(u) := u(a)$. Two nonempty subsets $\mathcal{U}^{\alpha}$ and $\mathcal{V}^{\alpha}$ of continuous real-valued functions on A represent the same preorder if, and only if, there exists an $f : \Upsilon_{\mathcal{U}^{\alpha}}(A) \to \Upsilon_{\mathcal{V}^{\alpha}}$ such that (i) $\Upsilon_{\mathcal{V}^{\alpha}} = f(\Upsilon_{\mathcal{U}^{\alpha}})$; and (ii) for every $b, c \in \Upsilon_{\mathcal{U}^{\alpha}}(A)$, $b > c$ if and only if $f(b) > f(c)$.[12]

4.2 Canonical signal spaces

The premise underlying the stochastic choice behavior depicted by the ICM is that choices are governed by unspecified random signal–generating process. Consider the choice between two alternatives, say a and a', such that $\neg(a \sim$

[12]See Evren and Ok (2011). Note that, in general, for arbitrary multi-utility representations, $\mathcal{V}^{\alpha}$ and $\mathcal{V}^{\alpha'}$, of two preorders, $\succcurlyeq^{\alpha}$ and $\succcurlyeq^{\alpha'}$, such that $\succcurlyeq^{\alpha} \subset \succcurlyeq^{\alpha'}$ does not imply that $\mathcal{V}^{\alpha} \supset \mathcal{V}^{\alpha'}$. Given $\succcurlyeq^{\alpha}$ and facing a choice from a binary set $\{a, a'\}$, the probability that the decision maker chooses the alternative a is independent of the representation. In other words, if $\mathcal{U}^{\alpha}$ and $\mathcal{V}^{\alpha}$ are two representations of $\succcurlyeq^{\alpha}$, then the functions in $\mathcal{V}^{\alpha}$ are given by the uniqueness of the representation.

a'). Then $P(a, \{a, a'\}) = \bar{\alpha}(a, a')$ may be interpreted as the probability of a signal that would resolve the indecision in favor of a. By Theorem 1 and the Corollary, this is the case if and only if $u(a) > u(a')$, for all $u \in \mathcal{U}^{\bar{\alpha}(a,a')}$.

Given SCF P^*, let $\{\succ^\alpha | \ \alpha \in [0,1]\}$ be the ICM that rationalizes it. Define a probability measure $F : 2^{\mathcal{U}} \backslash \varnothing \to [0,1]$ as follows: For $\alpha \in [0,1]$, $F(\mathcal{U}^\alpha) = \alpha$. Then $P^*(a, \{a, a'\}) = F\left(\mathcal{U}^{\bar{\alpha}(a,a')}\right)$, for all $a, a' \in A$. In other words, facing a choice between two alternatives, a and a' that are not indifferent to one another, the decision maker behaves *as if* a function u is selected from $\mathcal{U}^1$ according to a probability measure F and a is chosen if $u \in \mathcal{U}^{\bar{\alpha}(a,a')}$ and a' is chosen if $u \in \mathcal{U}^1 \backslash \mathcal{U}^{\bar{\alpha}(a,a')}$. Therefore, the set $\mathcal{U}^1$ may be taken to be the *canonical signal space.*

4.3 Representation of the SCF

Corresponding to the partition $\mathcal{J}(M)$ of $[0,1]$, define a partition of $\mathcal{U}^1$ as follows: For each $a_i \in M$ let

$$Q(a_i, M,) := \{u \in \mathcal{U}^1 \mid u(a_i) > u(a'), \forall a' \in M \backslash \{a_i\}\}. \tag{7}$$

Then, $\alpha \in \Lambda_i(M)$ if and only if $u \in Q(a_i, M)$. Since $\mathcal{U}^1$ is the canonical signal space, the probability of the signal $u \in Q(a_i, M)$ is $P(a_i, M)$. Consequently, given an SCF P^* rationalized by a ICM we have

$$P^*(a_i, M) = F(Q(a_i, M,)), \ \forall a' \in M. \tag{8}$$

Thus, the random choice behavior depicted by an SCF P^* may be interpreted as follows: When facing a choice from a menu M, the decision maker behaves *as if* a utility function $u \in \mathcal{U}^1$ is selected according to the probability measure F and $a_i \in M$ is chosen if $u \in Q(a_i, M)$.

5 Stochastic Demand and Portfolio Choice Theories

5.1 Stochastic demand functions

The application of the ICM to the theory of market demand is based on the following idea. When a consumer faces a menu consisting of commodity bundles, a utility function is selected at random from the canonical signal space according to some implicit probability measure and the commodity bundle that maximizes this utility function is chosen. In this context the two questions of section 3.1 correspond to two issues concerning stochastic demand. First, what is the nature of the stochastic individual and market demands induced by irresolute choice behavior? Second, can the data induced by stochastic individual and market demands, be rationalized by irresolute choice behavior?

To model market demand, let $K = \{1, ..., K\}$ be the set of individuals in the market, and let $\mathbb{R}^n_+$ denote the set of alternatives representing commodity bundles. Menus are feasible budget sets, $B(p, I_k) = \{x \in \mathbb{R}^n_+ \mid x \cdot p \leq I_k\}$, where $p = (p_1, ..., p_n) \in \mathbb{R}^n_{++}$ denotes the price vector and I_k the income of individual k. Denote by $\mathcal{B}$ the set of budget sets. Assuming non-satiation, the undominated subset of $B(p, I_k) \in \mathcal{B}$ is $UB(p, I_k) = \{x \in \mathbb{R}^n_+ \mid x \cdot p = I_k\}$.

To answer the first question, let $\mathcal{U}^1_k$ denotes the canonical signal space corresponding to an ICM depicting the behavior of individual k. Then, given a budget set $B(p, I_k)$ the realization of the random demands $\widetilde{x}^k(p, I_k)$ may be described as follows: For each $u \in \mathcal{U}^1_k$, let $x^*(p, I_k, u)$ be the solution to the program

$$\max u(x) \text{ subject to } x \in B(p, I_k),$$

and denote by $x^*_i(p, I_k, u)$ its $i-th$ entry. Then the stochastic commodity demands are driven by the random selection of a function $u \in \mathcal{U}^1_k$. Denote by F_k the probability measure on $\mathcal{U}^1_k$ representing the stochastic choice behavior of individual k, and let $\widetilde{u}$ be the corresponding random utility function. Then, $\widetilde{x}^k(p, I_k) = x^*(p, I_k, \widetilde{u})$ is the observed random demands.

For every $B(p, I_k) \in \mathcal{B}$ and $x \in B(p, I_k)$, let $U_k(x) = \{u \in \mathcal{U}^1_k \mid u(x) \geq u(x'), \forall x' \in B(p, I_k)\}$. The revealed stochastic demand is an SCF $P : \mathbb{R}^n_+ \times \mathcal{B} \to [0, 1]$ given by

$$P(x, B(p, I_k)) = F_k(U_k(x)). \tag{9}$$

The random demand for commodity i by individual k, $\widetilde{x}^k_i(p, I_k)$, whose support is $[0, I_k/p_i]$, is $x^*_i(p, I_k, \widetilde{u})$. Thus, given the budget set $B(p, I_k)$ the probability that the individual k chooses x^k_i is $\Pr\left(x^k_i, B(p, I_k)\right) = P\left(x^k, B(p, I_k)\right)$. Given an income profile $I = (I_1, ..., I_K)$ and a price vector p, the market *stochastic demand function for commodity* i is: $\widetilde{X}_i(p, I) = \Sigma^K_{k=1}\widetilde{x}^k_i(p, I_k)$.

It is standard practice in economics to treat individual demands as independent variables.[13] The analogous assumption in the present context

[13] This assumption is reasonable when applied to commodities such as milk and gas; it

maintains that individual demands are stochastically independent random variables.[14] If $\widetilde{x}_i^k(p, I_k)$, $k \in K$, are stochastically independent, then the distribution, μ, of the market demand for commodity i, $\widetilde{X}_i(p, I)$, is given by the convolution $\mu_i^K = P(x_i^1, B(p, I_1)) * P(x_i^2, B(p, I_2)) * ... * P\left(x_i^K, B(p, I_K)\right)$. Expected demand is given by

$$E\left[\widetilde{X}_i(p, I)\right] = \Sigma_{k=1}^K \int_{U_k\left(x_i^*(p, I_k, u)\right)} x_i^*(p, I_k, u)\, dF_k(u) . \tag{10}$$

Its variance is

$$Var\left(\widetilde{X}_i(p, I)\right) = \Sigma_{k=1}^K \int_{U_k\left(x_i^*(p, I_k, u)\right)} [x_i^*(p, I_k, u) - E\left[\widetilde{X}_i(p, I)\right]]^2 dF_k(u) . \tag{11}$$

Standard practice notwithstanding, in many markets individual demands are correlated, possibly because of implicit social effects such as conformism and status seeking. For instance, the demand for clothes is affected by fashion, the demand for vacation spots may be affected by the anticipated composition of the clientele, and demand for stocks may respond to information shared by many investors that respond to it in similar way. In these cases, the linearity of expectations implies that $E_k\left[\widetilde{X}_i(p, I)\right] = \Sigma_{k=1}^K E_k\left(\widetilde{x}_i^k(p, I_k)\right)$. The variance of market demand, however, depends on the correlations among the individual demands and takes the form

$$Var\left(\widetilde{X}_i(p, I)\right) = \Sigma_{k=1}^K Var_k\left(\widetilde{x}_i^k(p, I_k)\right) + 2\Sigma_{j<k} Cov_k\left(\widetilde{x}_i^j(p, I_j), \widetilde{x}_i^k(p, I_k)\right) . \tag{12}$$

In commodity markets in which individual demands are positively correlated, the individual stochastic choice behavior implied by the ICM induces greater demand fluctuation.

If the data summarizing individual demand behavior constitute SCFs satisfying regularity and stochastic transitivity, then by Theorem 2, it is ratioanlizable by ICMs.

5.2 Comparative statics

Consider next the consequences of income and price variations on market demands. Suppose that, ceteris paribus, the income of individual k increases

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is much less compelling when applied to other commodities.

[14] A collection of random avriables is said to be independent if every finite subcollection is independent.

form I_k to I'_k. The supports of the random demands increase to $[0, I'_k/p_i]$, $i = 1, ..., n$. For each $u \in \mathcal{U}$, the optimal bundle changes from $x^*(p, I_k, u)$ to $x^*(p, I'_k, u)$, and the corresponding change in the demand for commodity i is from $x^*_i(p, I_k, u)$ to $x^*_i(p, I'_k, u)$. For, each $u \in \mathcal{U}$, $x^*(p, I_k, u) \in \arg\max_{x \in B(p, I_k)} u(x)$ and $x^*(p, I'_k, u) \in \arg\max_{x \in B\left(p, I'_k\right)} u(x)$, (9) implies that

$$\Pr(x^*_i(p, I'_k, u)) = \Pr(x^*_i(p, I_k, u)) = F_k(u).$$

The change in the demand distribution of commodity i depends on the income effects implies by the utility functions in the canonical signal space.

Similar considerations apply to relative price variations. Suppose that the price of commodity i increases from p_i to p'_i. Denote the new price vector by p'. Let $x^*(p', I_k, u)$ denote the optimal bundle given the budget set $B(p', I_k)$ corresponding to $u \in \mathcal{U}_k$ and let $x^*_i(p', I_k, u)$ denote its i entry. Then by the same argument as above,

$$\Pr(x^*_i(p, I_k, u)) = \Pr(x^*_i(p', I_k, u)) = F_k(u).$$

The change in the market demand for commodity i is a random variable given by $\widetilde{X}_i(p', I) - \widetilde{X}_i(p, I) = \Sigma_{k=1}^K \left[\widetilde{x}^k_i(p', I_k) - \widetilde{x}^k_i(p, I_k)\right]$.

Example: Consider the case in which the set of utility functions of individual k consists of Cobb-Douglas utility functions (i.e., $\mathcal{U}_k = \{x_1^{\beta_1} x_2^{\beta_2} ... x_n^{\beta_n} \mid \beta \in \left[\underline{\beta}_k, \bar{\beta}_k\right]^n, \underline{\beta}_k \geq 0, \Sigma_{i=1}^n \beta_i = 1\}$). Let $u_\beta := x_1^{\beta_1} x_2^{\beta_2} ... x_n^{\beta_n}$ and denote by g_k the joint probability distribution function on $\left[\underline{\beta}_k, \bar{\beta}_k\right]^n$. Then, $x^*_i(p_i, I_k, u_\beta) = \beta_i I_k / p_i, i = 1, ..., n$. The stochastic demand for commodity i by individual k, $\widetilde{x}^k_i(p, I_k)$ is depicted by g_k. Formally, let $\bar{g}_k(\beta_i)$ denote the marginal distribution of β_i then

$$\Pr\{\widetilde{x}^k_i(p, I_k) = x^*_i(p_i, I_k, u_\beta)\} = \bar{g}_k(\beta_i). \tag{13}$$

If the income of individual k increases from I_k to I'_k, then the demand increases proportionally, (i.e.,for all $i = 1, ..., n$, $x^*_i(p_i, I'_k, u_\beta) = (I'_k/I_k)\, x^*_i(p_i, I_k; u_\beta)$) and $\Pr((I'_k/I_k)\, x^*_i(p_i, I_k, u_\beta)) = \Pr(x^*_i(p_i, I_k, u_\beta)) = \bar{g}_k(\beta_i)$. Similarly, if the price of commodity i increase to p'_i, then the demand decreases proportionally (i.e., $x^*_i(p'_i, I_k; u_\beta) = (p_i/p'_i)\, x^*_i(p_i, I_k; u_\beta)$) and $\Pr((p'_i/I_k)\, x^*_i(p_i, I_k; u_\beta)) = \Pr(x^*_i(p_i, I_k; u_\beta)) = \bar{g}_k(\beta_i)$.

If the utility functions of all individuals are Cobb-Douglas functions, then their demands are independent random variables. Consequently, given an income profile I and price vector p, the distribution of the market demand $\widetilde{X}_i(p, I)$ is the convolution of the distributions $\bar{g}_k$, $k = 1, ..., K$.

Let I' be another income profile. Then the change in the expected market demand is

$$\Sigma_{k=1}^{K}\left[E_k\left(\widetilde{x}_i^k(p_i, I'_k)\right) - E_k\left(\widetilde{x}_i^k(p_i, I_k)\right)\right] = \Sigma_{k=1}^{K}\left[(I'_k/I_k) - 1\right] E_k\left(\widetilde{x}_i^k(p_i, I_k)\right), \tag{14}$$

where $E_k\left(\widetilde{x}_i^k(p_i, I_k)\right) = \int_{\underline{\beta}_k}^{\bar{\beta}_k} x_i^*(p_i, I_k; u_\beta)\, \bar{g}_k(\beta_i)\, d\beta_i$. The variance of individual demands increases by a factor $(I'_k/I_k)^2$ (i.e., $Var\left(\widetilde{x}_i^k(p, I'_k)\right) = (I'_k/I_k)^2 Var\left(\widetilde{x}_i^k(p, I_k)\right)$).

5.3 Stochastic portfolio choice

Consider next the application of the ICM to the theories of portfolio choice and financial markets. Let $S = \{s_1, ..., s_n\}$ be a finite state space, and denote by $\{e^1, ..., e^n\}$ the corresponding set of Arrow securities.[15] The set of alternatives, $\mathbb{R}^n$, are portfolios of Arrow securities (i.e., portfolio is $y \in \mathbb{R}^n$, where y_i denotes the number of Arrow securities of type e^i in the portfolio). Denote by $\bar{y} = (1, ..., 1)$ the portfolio that consists of one Arrow security of each state. Then $\bar{y}$ is a unit of a riskless asset. Let $q = (q_1, ..., q_n)$ denote the vector of prices of the Arrow securities then the price of $\bar{y}$ is $\bar{q} = \Sigma_{i=1}^{n} q_i$.

Let $\bar{y}_k$ denote the initial endowment of riskless asset of individual k whose value is $w_k = \bar{y}_k \cdot \bar{q}$. Then the budget set of individual k is $B(q, w_k) = \{y \in \mathbb{R}^n \mid y \cdot q^\tau = w_k\}$, where q^τ is the transposed of q.

Denote by Π_k a set of subjective probability distributions on S representing the possible beliefs of individual k about the likely realizations of the states, and let u_k be a real-valued function on $\mathbb{R}$, representing the individual's risk-attitudes. A preference relation, $\succ_k$, of individual k is said to exhibit *Knightian uncertainty* if, for all $y, y' \in \mathbb{R}^n$, $y \succ_k y'$ if and only if $\Sigma_{i=1}^{n} u_k(y_i)\, \pi(s_i) > \Sigma_{i=1}^{n} u_k(y'_i)\, \pi(s_i)$, for all $\pi \in \Pi_k$.[16] Note that, in this instance, Π_k the individual k's canonical signal space corresponding to the ICM. Let $\mathcal{V}_k := \{\Sigma_{i=1}^{n} u_k(y_i)\, \pi(s_i) \mid \pi \in \Pi_k\}$ with generic element v_k.

Let G_k denote a probability measure on Π_k induced by the ICM $\{\succ_k^\alpha \mid \alpha \in [0, 1]\}$. Define

$$\Pi_k(y) = \{\pi \in \Pi_k \mid u_k(y) \cdot \pi \geq u_k(y') \cdot \pi, \forall y' \in B(q, w_k)\}. \tag{15}$$

[15] An Arrow security $e^i = (0, 0, ..., 1, 0, ...0)$ pays off \$1 in the state s_i and nothing otherwise.

[16] See Bewley (2002) and Galaabaatar and Karni (2013).

The the optimal portfolios of Arrow securities of individual k corresponding to $\pi \in \Pi_k$ is

$$\widetilde{y}^k (q, w_k) (\pi) = (\widetilde{y}_i^k (q, w_k), ..., \widetilde{y}_n^k (q, w_k) (\pi) = \arg \max_{y \in B(q,w_k)} u_k (y) \cdot \pi. \quad (16)$$

Then $\widetilde{y}^k (q, w_k)$ is a random variable whose distribution (and that of $\widetilde{y}_i^k (q, w_k)$) induced by G_k. Formally, $\Pr\{\widetilde{y}^k (q, w_k) = y\} = G_k\{\Pi_k (y)\}$. Then the SCF induced by the ICM that represent the random portfolio choices of individual k is:

$$P_k (y, B (q, w_k)) = G_k\{\Pi_k (y)\}. \quad (17)$$

Clearly, $P_k (y_i, B (q, w_k)) = P_k (y, B (q, w_k))$. The market demand for Arrow security e^i is the sum of individual demands, whose distribution is the convolution of the distributions of the individual demands.

6 Related Literature and Concluding Remarks

6.1 Related literature

Luce (1959) pioneered the study of random choice behavior. As in this paper, a primitive of Luce's model is a stochastic choice function summarizing the observed frequencies of choice of alternatives in the feasible sets in a variety of situations encountered in psychology and economics. Luce explored (sufficient) conditions on the choice probabilities that admit a numerical scale that represents individual stochastic choice behavior. Invoking the notations of this paper, for a finite set of alternatives, say $A = \{a_1, ..., a_n\}$, Luce's proposed structure of the stochastic choice function is represented by (strictly positive) utility vector, unique up to positive scalar multiplication, such that $p (a_i, M) = u (a_i) / \Sigma_{a \in M} u (a)$, for all nonempty $M \subseteq A$ and $a_i \in M$.

Luce's model implies (and is implied by) a constancy of probability ratios condition. Formally, $p (a_i; M) > 0$ for every $a_i \in M$ and, for every $a_i, a_j \in A$, the ratio $p (a_i, M) / p (a_j, M)$ is constant over all menus $M \subseteq A$ that contain a_i and a_j. Neither of these conditions seems natural, nor are they intuitively compelling.[17] Therefore it is worth underscoring that neither of these conditions is required by the ICM and the corresponding SCFs. According to the ICM $p (a_i; M) = 0$ for all $a_i \in D (M)$, and while adding alternatives

[17] Recent research including Ahumada and Ulku (2018), Echenique and Saito (2019), and Horan (2021) extend Luce's seminal work to address these weaknesses of the model.

to a menu may decrease the probabilities of choosing existing alternatives, the decreases are not necessarily equiproportional. Consequently, the ICM induces and can rationalize a richer set of SCF (i.e. richer set of random choice behaviors) that Luce's model.

The notion of random choice governed by random selection of utility functions was explored by Block and Marschak (1960). In particular, Block and Marschak address the possibility of estimating a probability distribution on the space of random utility functions using the (estimated) probabilities $p(a, M)$. More specifically, Block and Marschak treat the (finite) set of utilities as primitive and postulate the existence of a random utility vector $U = (u(a_1), ..., u(a_n))$ (unique up to increasing monotone transformation) which induces random rankings of the elements of the alternatives such that, for all $a_i \in M$, $p(a_i, M)$ is equal to the probability of the set of rankings whose elements rank a_i above every other alternative in the menu M. They show that this condition requires that no two alternatives can be assigned the same rank. Formally, for all $a_i \neq a_j$, $\Pr\{u(a_i) = u(a_j)\} = 0$. They also show that the existence of probability distribution on rankings consistent with the probabilities $p(a_i, M)$ implies regularity. Unlike the random utility model of Block and Marschak (1960) in which the utility functions are primitives, the utility functions that constitute the signal space in the ICM model are derived from the underlying set of probabilistic choice relations, and the regularity condition is derived from the set inclusion monotonicity of the ICM. Moreover, the ICM admits infinite sets of utility functions and does not require that distinct alternatives are assigned the different utilities.

The problem of revealed stochastic preference deals with a similar question – namely – whether the distribution of observed choices from variety of feasible sets of alternatives is consistent with preference maximization. Applied to a population, the distributions of observed choices arise because of heterogeneity of tastes and/or beliefs. Applied to individuals, the distribution is a reflection of stochastic variables underlying individual preferences. McFadden and Richter (1971, 1990), Falmagne (1978), Fishburn (1978), Stoye (2019) addressed the question of consistency of the distribution of observed choices with optimizing behavior.[18] The ICM may be regarded as a contribution to the part of this literature that deals with individual stochastic choice behavior. However, unlike the literature on stochastic preference

[18] McFadden (2005) synthesizes and extends the literature on stochastic preference. He also provides an extensive reference list to this literature.

in which the set of utility functions is a primitive ingredient of the models, the primitive of the ICM is a set of incomplete probabilistic choice relations, that admit random utility representation. This difference is reflected in the axiomatic structures of the models. Recently, there has been a revival of interest in random utility models of choice and random choice behavior that reflects preference for deliberated randomization.[19]

The ICM model is closely related to Roberts (1971) analysis of homogeneous families of semiorders.[20] Roberts considers a set of binary relations $\{\succ^{\lambda} \mid \lambda \in \Lambda\}$, where Λ is an index set (e.g., $\Lambda = [0, 1/2]$) on a set A of alternatives induced by a binary probability function $p : A \times A \to [0, 1]$ that satisfies $p(a, a') + p(a', a) = 1$, for all $a, a' \in A$. Formally, $a \succ^{\lambda} a'$ is defined by $p(a, a') > \lambda$. Roberts (Theorem 4) shows that, for A finite, $\{\succ^{\lambda} \mid \lambda \in \Lambda\}$ is induced by binary choice probabilities if and only if it satisfies the following axioms: For all $a, a' \in A$ and $\lambda, \lambda' \in \Lambda$, (a) $a \succ^{\lambda} a'$ implies $\neg(a' \succ^{\lambda'} a)$ (b) Either $\succ^{\lambda} \subseteq \succ^{\lambda'}$ or $\succ^{\lambda'} \subseteq \succ^{\lambda}$. Roberts main result is the identification of conditions required for $\{\succ^{\lambda} \mid \lambda \in \Lambda\}$ to be induced by binary choice probabilities that satisfy Strong Stochastic Transitivity (in the notations of this paper, for all a, $a', a'' \in A$, $P(a, \{a, a'\}) \geq 1/2$ and $P(a', \{a', a''\}) \geq 1/2$ implies $P(a, \{a, a''\}) \geq \max\{P(a, \{a, a'\}), P(a', \{a', a''\}))$.[21] In particular, each of the relations in $\{\succ^{\lambda} \mid \lambda \in \Lambda\}$ is a semiorder and the set itself must be homogeneous in the sense that a common weak order on A underlies (i.e., is compatible with) every semiorder. Strong Stochastic Transitivity is distinct from ST and the set of binary relations of ICM is not required to be homogeneous. Roberts (1971) investigates conditions of probabilistic consistency in data that consists of the function p. By contrast, a main objective of this paper is study of the axiomatic foundations that rationalizes the revealed probabilistic choice behavior depicted by stochastic choice functions.

At the individual level, random choice behavior may reflect the decision maker's indifference among feasible alternatives or his inability to compare them because of their complexity or the lack of familiarity with their consequences, which makes them difficult to evaluate. Ok and Tserenjigmid (2020) model these aspects of random choice behaviors by stochastic choice

[19] Various aspects of random utility models of choice behavior have been studies by Gul, Natenzon, and Pesendorfer (2014), Fudenberg, Iijima, and Strzalecki (2015), and Frick, Iijima, and Strzalecki (2019). Danan (2010), Agranov and Ortoleva (2017), and Cettolin and Riedl (2019) examined random choice based on deliberate randomization.

[20] See also Fishburn (1973).

[21] See also Block and Marschak (1960).

functions. They characterize stochastic choice functions that assign positive probabilities solely to alternatives that constitute maximal elements of the feasible sets. They do not study the probability distributions on the sets of maximal elements.

Karni (2022) introduced and studied the ICM and the notion of canonical signal space that constitutes the foundation of the SCF. Karni and Safra (2016) axiomatized the representation of decision makers' perceptions of the stochastic process underlying the selection of their state of mind which, in turn, govern their choice behavior. This work may be regarded as providing axiomatic foundations of a probability measure F on the canonical signal space based on the decision makers' introspections.

Becker (1962) argued that some economic theorems, such as the law of demand, do not depend on agents in the market behaving rationally. He showed that even if consumers choose their consumption bundles without attempting to optimize of some objective function, the change in the budget set caused by the relative price changes will force them to respond in a way that, in the aggregate, produces a downward-slopping demand functions. In Becker's analysis, households' choices may be irrational but not stochastic. Consequently, unlike in the market demand theory implied by the ICM, market demand is nonstochastic.

6.2 Consistency with violations of the weak axiom of revealed preference

According to the revealed preference approach, stochastic choice functions are empirical manifestations of random choice behavior that may be governed by decision makers' indifference among feasible alternatives, their inability to compare and rank the alternatives, variations in their moods, and/or changing needs. Whatever the underlaying motivations, the reasons for the observed stochastic choice may not be directly observable to an outsider; if they are driven by subconscious impulses they may not be accessible even to the decision maker. It is necessary in such cases to build theories that make sense of observations that consist of feasible alternatives and actual choices. The irresolute choice model is a way of making sense of observed random choices in repeated decision situations involving the same feasible set of alternatives summarized by stochastic choice functions.

The model is also consistent with some violations of the weak axiom of revealed preference (WARP).[22] To see why, let $x^0 \in \mathbb{R}^n_+$ denote the initial endowment of a decision makers commodities. Let p and p' be two price vectors and consider the budget sets $B(p, x^0 \cdot p)$ and $B(p', x^0 \cdot p')$. The corresponding undominated sets are

$$UD\left(B\left(p, x^0 \cdot p\right)\right) = \{x \in B\left(p, x^0 \cdot p\right) \mid \exists u \in \mathcal{U} \text{ s.t. } u(x) \geq u(x'), \forall x' \in B\left(p, x^0 \cdot p\right)\}$$

and

$$UD\left(B\left(p', x^0 \cdot p'\right)\right) = \{x \in B\left(p', x^0 \cdot p'\right) \mid \exists u \in \mathcal{U} \text{ s.t. } u(x) \geq u(x'), \forall x' \in B\left(p', x^0 \cdot p'\right)\}.$$

If $x^0 \in UD(B(p, x^0 \cdot p)) \cap UD(B(p', x^0 \cdot p'))$ then $UD(B(p, x^0 \cdot p)) \cap intB(p', x^0 \cdot p')$ and $UD(B(p', x^0 \cdot p')) \cap B(p, x^0 \cdot p)$ are nonempty.[23] Let $x^* \in UD(B(p, x^0 \cdot p)) \cap intB(p', x^0 \cdot p')$ and $x^{**} \in UD(B(p', x^0 \cdot p')) \cap intB(p, x^0 \cdot p)$ then the choices x^* from $B(p', x^0 \cdot p')$ and x^{**} from $B(p, x^0 \cdot p)$ constitute a violation of WARP.[24] Such choice is consistent with the irresolute choice model.

[22] I am grateful to Yujian Chen for calling my attention to this point.

[23] $intB\left(p, x^0 \cdot p\right)$ and $intB\left(p', x^0 \cdot p'\right)$ are the interiors of the corresponding budget sets in the $\mathbb{R}^n$ topology.

[24] In the case of complete preferences, $\mathcal{U}$ is a singleton set.

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