On the Definition of States and the Choice Set in the Theory of Decision Making under Uncertainty

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Abstract

This paper discusses the definition of the state space and corresponding choice sets that figure in the theory of decision making under uncertainty. It elucidates an approach that overcomes some conceptual difficulties with the standard models and accommodates a procedure for expanding the state space in the wake of growing awareness.

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"If a tree falls in a forest and no one is around to hear it, does it make a sound?"

1 Introduction

Subjective probability quantifies a decision maker's degree of belief in the likelihoods of events about which reasonable decision makers might disagree. Ramsey (1926) proposed the key idea that subjective probabilities may be inferred from the odds a decision maker is willing to offer (or accept) when betting on events or the truth of certain proposition. Ramsey's approach requires that the event on which a bet is placed be *observable* so that the uncertainty is resolved and the payments can be affected.[1]

To formalize the idea of resolution of uncertainty, Savage (1954) introduced the notion of *state of nature*, "a description of the world so complete that, if true and known, the consequences of every action would be known" (Arrow [1971], p. 45). Implicit in this definition is the notion that there is a unique true state that is fully depicted by the consequences associated with every possible action. In practice, however, decision theorists and economists routinely specify a state space as a primitive constituent of the decision problem. Savage himself applied this "state first" approach when he wrote, "If two different acts had the same consequences in every state of the world, there would from the present point of view be no point in considering them different acts at all. An act may therefore be identified with its possible consequences. Or, more formally, an *act* is a function attaching a consequence to each state of the world" (Savage [1954], p. 14). However, treating the state space as a primitive ingredient of the model tends to conceal critical aspects of the notion of states, and imposes tacit and unnecessary restrictions on its usefulness.

The purpose of this note is twofold. First, it discusses and elaborates an approach to modeling the resolution of uncertainty described in Fishburn (1970), Schmeidler and Wakker (1987) and in Karni and Schmeidler (1991) and recently invoked in the works of Karni and Vierø (2013, 2015a, 2015b).[2]

[1] A similar idea appears in the work of de Finetti (1937) and was later incorporated into Savage's (1954) subjective expected utility theory.

[2] See related discussion in Gilboa, Postlewaite and Schmeidler (2009).

Second, it examines the implications of this approach to the modeling of the choice set in the theory of decision making under uncertainty.

2 States of Nature

2.1 Reality and perception

Albert Einstein is reported to have asked Niels Bohr whether he believed that the moon does not exist if nobody is looking at it. Bohr replied that however hard he (Einstein) might try, he would not be able to prove that it does. Perception is the organization, identification, and interpretation of sensory information in order to represent and understand the environment. An object cannot be perceived, cannot be known to exists. This concept of knowledge is pertinent to understanding Savage's idea of the state of nature. To paraphrase the epigram "if a state of nature obtains but leaves no perceived manifestations, does it obtain?" In what follows I argue that a state of nature is a perception and that the specific interpretation of sensory information lends a state of nature its meaning. Moreover, to serve as a meaningful ingredient of a decision model, states of nature must be observable, in the sense that independent observers must agree on and communicate what has been observed.[3]

2.2 The approach

According to the approach advanced here, the set of states of nature, or the state space, is constructed using two basic ingredients: A set, A, of *basic acts*, and a set, C, of *feasible consequences*.[4] Basic acts depict alternative courses of action that can be implemented, and feasible consequences are outcomes that may result from these actions. A *state of nature*, s, is a mapping from the set of basic acts to the set of feasible consequences. The *state space* is the set of all such mappings. Formally, the state space is C^A. In practice, however, some basic actions cannot possibly result in some consequences

[3] See discussion in Machina (2003).

[4] In Schmeidler and Wakker (1987), Karni and Schmeidler (1991) and Karni and Vierø (2013, 2015a) the reference was to feasible acts. For reasons that will become clear, the term basic seems more appropriate.

implying that certain states are null. Formally, for each $a \in A$, denote by $C(a)$ the set of consequences that are feasible under a.[5] Then, the states $s \in C^A$ such that, for some $a \in A$, $s(a) \notin C(a)$ are null. Taking these "feasibility constraints" into account allows a more parsimonious depiction of the state space consists of the event of all nonnull states. Formally, the *parsimonious state space* is $S = \{s : A \to C \mid s(a) \in C(a), \forall a \in A\}$.

The definition of parsimonious state space presumes that all decision makers agree on the set of consequences that may result from any given basic act. If this is not the case, then the relevant state space is C^A, and $C(a)$ is a matter of subjective belief, giving rise to subjective parsimonious state space. A state that assigns an act a a consequence $c \notin C(a)$, is, by definition, *null*.

Our definition of the state space does not precludes the existence of salient background states. In other words, it does not contradict the idea that a tree might make noise when it falls even if there is nobody to hear it. It does presume, however, that, insofar as decision theory is concerned with modeling and characterizing choice behavior, such salient states are immaterial. To paraphrase Savage, if two different states have the same consequences for every basic act, there would from the present point of view be no point in considering them different states at all. In other words, it is irrelevant whether or not there are some states that differ if their difference is not act relevant. Two salient states are regarded as equivalent if and only if every basic act has equal consequences in both of them. Hence, the definition of the state space proposed here can be regarded as quotient state space embedded in some larger salient state space.

Because each state assigns a unique consequence to each basic act it constitutes, by definition, a complete resolution of uncertainty and the states are mutually disjoint. There are situations, however, in which the basic acts are mutually exclusive. For example, a patient who decides to undergo surgery at Johns Hopkins University Hospital cannot find out what would have been the outcome if he had chosen, instead, to undergo the same surgery at the Mayo clinic. In such cases, a complete resolution of uncertainty is inherently impossible. More generally, if the implementation of a basic act, a, excludes other basic acts, and an outcome, c, is observed, then the uncertainty is only partially resolved. Only the *event* (that is, a subset of S) consisting of all the states that assign the consequence c to the basic act a can be

[5]Note that $C = \cup_{a \in A} C(a)$.

known to have been obtained. Gilboa, Pestlewaite and Schmeidler (2009) argue that the lack of complete resolution of uncertainty undermines Bayesian rationality. According to Gilboa et. al. "The Bayesian approach calls for the generation of prior beliefs about the outcomes given each possible choice. ... But the decision maker will choose only one of her options. ... Importantly, this feature is inherent to every problem: each state has to describe the outcome of all acts, while only one act will actually be chosen." (Gilboa et. al. [2009] p. 295)[6]

2.3 Examples

The particular sets of basic acts and consequences that are used in their construction determine the meaning of states. The most obvious notion of a state of nature is a depiction of a natural phenomenon – tomorrow's temperature in Baltimore, the force of the next earthquake to hit San Francisco. These natural phenomena are perceived through measurements. Thus, the set of basic acts consists of measurements; the set of feasible consequences consists of the union of the sets of measurable values corresponding to each measurement. If only one measurement is taken, then the state space consists of all the possible values taken by the measurement. If several measurements of the same phenomenon are taken, the measurements might not agree (e.g., because of inaccuracy of the instrument or the conditions under which the measurements are taken). In this case, the state space consists of all the configurations of values taken by the different measurements. If measurements are taken of different phenomena then the state space consists of the Cartesian product of the set of the possible values of the measurements.

An important class of states of nature are an organism's states of health. These phenomena are perceived by direct sensations, diagnostic tests, and/or

[6] To illustrate their point, Gilboa et. al. invoke Ellsberg's (1961) two urns experiment. Specifically, they claim that the decision maker has to choose to bet on a color from either urn I *or* on urn II. Thus, she will never be able to tell which state obtains. If the decision maker chooses to bet on red from urn I, she will only know which element of the partition $\{\{(IB, IIR), (IB, IIB)\}\{(IR, IIR), (IR, IIB)\}$ obtains. In terms of the approach advocated in this paper, this assumes implicitly that there are two, mutually exclusive, basic acts, a_I- "draw a ball from urn I" and $a_{II}-$ "draw a ball from urn II" with the consedquences $C(a_I) = C(a_{II}) = \{R, B\}$. However, if the acts are not mutually exlusive, as is in fact the case, then there are four states and the decision maker can tell which state obtains.

response to treatments. Presumably, some underlying causes determine the organism's state of health and the corresponding symptoms. According to the approach described here, the only meaningful definition of a state of health is its perceived symptoms (that is, the results of diagnostic tests and/or response to treatments). The underlying causes considered salient states.

Other prevalent class are constitutes of states induced by competitive sporting events. Anscombe and Aumann (1963) chose the example of the outcomes of a horse race to illustrate their conception of the state space. In this case the basic acts correspond to the sets of horses that enter the race. The set of consequences, which is identical to the set of states, constitutes all the possible orders according to which the horses cross the finish line. According to our approach, running the race is a way of measuring the relative and/or absolute speed of the horses, which determine the states. Other contests, such as presidential elections, beauty contests and jury trials, should also be thought of as forms of measurement. In the case of presidential election, for instance, the outcome of the vote is a measurement of the support for the competing candidates and the platforms on which they run. In the case of a beauty contest the outcome is a measurement of the opinions of the panel of judges, and in the case of jury trial, the outcome measures the weight the evidence in the minds of the jurors.

2.4 Savage's omelet

To grasp the difference between the traditional approach to modeling decision making under uncertainty and the approach advocated here, it is instructive to compare Savage's analysis to the analysis according to the approach of this paper of the following scenario:

> Your wife has just broken five good eggs into a bowl when you come and volunteer to finish making the omelet. A sixth egg, which for some reason must either be used for the omelet or wasted altogether, lies unbroken beside the bowl. You must decide what to do with this unbroken egg. Perhaps it is not too great an oversimplification to say that you must decide among three acts only, namely, to break it into the bowl containing the other five, to break an egg into a saucer for inspection, or to throw it away without inspection. (Savage [1954] p.13)

Savage takes the state space and the set of consequences as primitives and define the acts to be the functions from the set of states to the set of consequences as follows:

Acts/States	Good	Rotten
Break into bowl	Six-egg omelet (c_1)	No omelet, five good eggs destroyed (c_2)
Break into saucer	Six-egg omelet, a saucer to wash (c_3)	Five-egg omelet, a saucer to wash (c_4)
Throw away	Five-egg omelet, one good egg destroyed (c_5)	Five-egg omelet (c_6)

The approach advanced here takes the sets of basic acts and feasible consequences as primitives and constructs the state space. The basic acts are: a_1- Inspect the egg in the bowl, a_2- Inspect the egg in the saucer, a_3- Do not inspect. The corresponding feasible consequences are: $C(a_1) = \{c_1, c_2\}$, $C(a_2) = \{c_3, c_4\}$, and, because under a_3 the only observable consequence is five-egg omelet, or c_6, the distinction between c_5 and c_6 not perceivable. Consequently, $C(a_3) = \{c_6\}$, and the state space is depicted in the following matrix.

$A\backslash S$	s_1	s_2	s_3	s_4
a_1	c_1	c_1	c_2	c_2
a_2	c_3	c_4	c_3	c_4
a_3	c_6	c_6	c_6	c_6

The inclusion of the states $s_2 = (c_1, c_4, c_6)$ and $s_3 = (c_2, c_3, c_6)$ suggests that the two forms of inspection might yield opposite conclusions, which is possible if the the inspections are subject to error. If the inspections are perfect, as is implicitly assumed in Savage's analysis, then the states s_2 and s_3 are inherently inconsistent, and the only remaining states are s_1 and s_4, in which the two inspection methods agree. These are, respectively, the good and rotten states in Savage's analysis. Notice, however, that according to Savage's analysis it is the observer who decides whether or not the inspections are perfect, and the observer's perception determines the state space. By contrast, according to the approach advocated the decision maker determines whether or not the inspections are infallible (that is, whether or not the event $E = \{s_2, s_3\}$ is null). Consequently, the state space is conceptualized by the decision maker and reflect his understanding of the environment.

2.5 Expansion and contraction of the state space

An important advantage of the approach taken here is the flexibility it affords in allowing the state space to be redefined and expanded when new basic acts and/or consequences are discovered or the understanding of the links connecting acts and consequences is modified.[7] Karni and Vierø (2013, 2015a, 2015b) exploited this advantage to model reverse Bayesianism and decision makers' anticipation of discovery of consequences that, in their current state of ignorance, they cannot imagine and may even lack the language to describe. More specifically, the discovery that a basic act, say $a \in A$, resulted in an unfamiliar consequence $\hat{c} \notin C$, requires a redefinition of the state space. Formally, let $\hat{C} = C \cup \{\hat{c}\}$ then the new, expanded state space is $\hat{C}^A$. Similarly, the introduction and/or discovery of a new basic act, $\hat{a} \notin A$, for instance, taking an additional measurement of a natural phenomenon or the invention of a new financial asset (e.g., options), requires the redefinition of the state space. Formally, let $\hat{A} = A \cup \{\hat{a}\}$, then the new state space is $C^{\hat{A}}$.

It is worth emphasizing that while $\hat{C}^A$ constitutes a genuine expansion of the state space C^A, $C^{\hat{A}}$ is a refinement of C^A. Put differently, the event $\hat{C}^A \backslash C^A$ constitutes of states $s(a) = \hat{c}$, for some $a \in A$, which were not part of the description of the original state space. By contrast, if $\hat{a}$ may be associated with all the consequences $c \in C$, corresponding to each state, s, in the original state space C^A there is an event $E(s) := \{\Pi_{a \in A} s(a) \times c \mid c \in C\}$ in the state space $C^{\hat{A}}$. Thus, the state space $C^{\hat{A}}$ is a *uniform refinement* (filtration) of the original state space C^A. The sets $E(s)$, $s \in S$, described above constitute a partition of the state space $C^{\hat{A}}$.

Our approach to modelling the state space can be used to check avoid pitfalls in the analysis of decision making under uncertainty. For instance, Ahn and Ergin (2010) present a model in which the choice set consists of acts that are measurable with respect to partitions, interpreted as alternative descriptions of a fixed underlying state space. According to Ahn and Ergin, preference relations over measurable acts are partition dependent. In addition, Ahn and Ergin invoke the notion of filtration (that is, a uniform refinements of the partition of the state space) and gradual filtration (that

[7] Variations in the understanding of the links between acts and consequences is formalized by redefining the sets $C(a)$, $a \in A$.

is, a refinement of a partition that does not split all the nonnull cells of the original partition).[8] In terms of the model of this paper, every basic act, a, induces a partition of the state space C^A defined by $\{a^{-1}(c) \mid c \in C\}$. Viewed in this way, the descriptions of the state space in have concrete meanings, namely, they are the consequences of the basic acts. Moreover, filtration corresponds to refinements of the space in the wake of discovery of new basic acts. However, gradual filtration is not consistent with either the discovery of a new basic act or that of new consequences. This raises the question of how, and in what language, is the gradual filtration described?

2.6 Subjective states and coarse contingencies

Kreps (1979) introduced the notion of subjective state space derived from preferences over menus displaying 'preference for flexibility.' According to this approach, subjective states are resolutions of the uncertainty regarding choices from menus (i.e., nonempty sets of alternatives) having the interpretation of preference relations on alternatives in the menus.

The approach of this paper can be applied to the definition of subjective state space as follows: Analogous to the set of consequences is a finite set, F, of *alternatives* and corresponding to basic acts are nonempty subsets of F, dubbed *menus*.[9] Let $\mathcal{M}_F$ denote the set of all menus consisting of elements of F. By definition, the set of alternatives in each menu is the set of consequences that are feasible given the act represented by that menu. Let $c : \mathcal{M}_F \to \mathcal{M}_F$ be a choice function (i.e., $c(M) \subseteq M$, for all $M \in \mathcal{M}_F$). Then, by definition, $c(M)$, $M \in \mathcal{M}_F$, are non-empty, finite sets.

The subjective state space induced by $\mathcal{M}_F$ is the set of mappings $\Omega_F := \{\omega : \mathcal{M}_F \to F \mid \omega(M) \in c(M), \forall M \in \mathcal{M}_F\}$.[10] If alternatives are observable (i.e., agreed upon by distinct observers), then the derived state space is objective and is determined independently of the preferences of the decision

[8] Gradual filtration is required for the uniqueness of the represntation (see Ahn and Ergin (2010) Theorem 2).

[9] Alternatives may be consequences, in which case $F = C$.

[10] If $\mid F \mid= n$ then the number of states in Ω_F is $\prod_{i=1}^{n} i^{\binom{n}{i}}$. For example, if $F = \{f_1, f_2\}$ then the set of menus $\mathcal{M}_F = \{\{f_1\}, \{f_2\}, \{f_1, f_2\}\}$ and the set of states is $\Omega_F = \{\omega, \omega'\}$, where $\omega = (\omega(\{f_1\}) = f_1, \omega(\{f_2\}) = f_2, \omega(\{f_1, f_2\}) = f_1)$, and $\omega' = (\omega'(\{f_1\}) = f_1, \omega'(\{f_2\}) = f_2, \omega'(\{f_1, f_2\}) = f_2)$.,

maker.

In general, the states in Ω_F do not correspond to complete and transitive preference relations and, therefore, are of little interest. However, if c satisfies the weak axiom of revealed preference (that is, for any pair $f, f' \in F$ and $M, M' \in \mathcal{M}_F$, if $f, f' \in M \cap M'$, $f \in c(M)$ and $f' \notin c(M)$, then $f' \notin c(M')$), then, it is easy to show that each state correspond to a complete and transitive preference relation, $\succcurlyeq_\omega$ on F, and that $c(M) = \{f \in M \mid f \succcurlyeq_\omega f'$ for all $f' \in M\}$.[11] Moreover, if $f, f' \in c(M) \cap c(M')$ for some $M, M' \in \mathcal{M}_F$ then the preference relation corresponding to the states $\omega, \omega' \in \Omega_F$ such that $\omega(M) = f$, $\omega(M') = f'$, $\omega'(M) = f'$, $\omega'(M') = f$ and $\omega(M'') = \omega'(M'')$, for all $M'' \in \mathcal{M}_F \backslash \{M, M'\}$, satisfy $\succcurlyeq_\omega = \succcurlyeq_{\omega'}$. Hence, multiple states may be equivalent in the sense of corresponding to the same preference relation. By definition, equivalent states assign to different menus indifferent alternatives.

This approach may be useful for interpreting some results in the literature. In particular, Epstein, Marinacci and Seo (2007) presented axiomatic models based on menu choice with coarse contingencies. In their models, contingencies are subjective states and coarse contingencies are events in this space. According to Epstein et. al. a decision maker might be aware of her inability to describe in details all the contingencies that may affect her ex post behavior (i.e., choice from a menu). In terms of our definition of the state space, coarse contingencies arise when the decision maker neglects to consider certain menus in $\mathcal{M}_F$ when constructing the state space. In other words, the decision maker only consider menus in a subset $\mathcal{M} \subset \mathcal{M}_F$ and defines the state space $\Omega = \{\omega : \mathcal{M} \to F \mid \omega(M) \in c(M), \forall M \in \mathcal{M}\}$ on the restricted domain. If the decision maker knows F and, hence, possesses all the information necessary to construct the entire state space. The coarseness may be attributed to implicit cost associated with the complexity of detailed depiction of the entire state space. The decision maker is aware of, voluntarily, acting on the basis of incomplete articulation of the relevant alternatives to form the full set of contingencies.

An alternative interpretation of coarseness is that the decision maker is aware only of a proper subset of the alternatives known to the modeler. This can be related to the effect of discovery of new alternatives. If a decision maker becomes aware of new alternative, $\bar{f}$, then the set of menus becomes $\mathcal{M}_{F'}$, where $F' = F \cup \{\bar{f}\}$. The subjective state space state induced by

[11] See Kreps (2013) Proposition 1.2.

$\mathcal{M}_{F'}$ is $\Omega_{F'} = \{\omega : \mathcal{M}_{F'} \to F' \mid \omega(M) \in c(M), \forall M \in \mathcal{M}_{F'}\}$. Note that $\Omega_F \subset \Omega_{F'}$. Thus, $\Omega_{F'}$ represents a refinement of the original subjective state space, as the new alternative expends the domain of the definition of states. Each state or preference relations in Ω_F, constitutes an event in the new state space consisting of states that agree on $\mathcal{M}_F$ and differ in the set of subset menus $\{\bar{M} \in \mathcal{M}_{F'} \mid \bar{M} = M \cup \{\bar{f}\}, M \in \Omega_F\}$. Interpreting states as preference relations that satisfy the weak axiom of revealed preference, the refinement of the state space due to discovery of new alternatives does not affect the preference relations derived from the menus in $\mathcal{M}_F$. Consequently, it has not implication for models of menu choice.

3 The Choice Set

3.1 Grand world small worlds

Two different approaches can be used to model decision making under uncertainty. The first approach envisions a framework that includes the set of all basic acts and the corresponding feasible consequences to construct a grand state space along the lines described above. According to this approach, decision makers entertain beliefs about the likely realizations of the events (subsets) of this grand state space and act on these beliefs when facing specific decision problems. This approach imposes consistency of beliefs across decisions problems.

According to the second approach, when facing specific decision problems, decision makers construct the relevant "small world" state space by listing the relevant basic acts and consequences, and defining the relevant states to be the mapping from the set of relevant feasible act to the set of relevant feasible consequences. This approach does not require that a decision maker's beliefs across decision problems be consistent.[12]

The two approaches to formulating the decision problem are essentially the same. The difference is the definition of the relevant primitive sets of basic acts and consequences.

[12] For a more detailed discussion, see Karni and Vierø (2015b).

3.2 Feasible and conceivable acts

Once the state space is fixed, the choice set may be defined. In Schmeidler and Wakker (1987); Karni and Schmeidler (1991); and Karni and Vierø (2013, 2015a, 2015b) the choice space consists of the original basic acts and the set of *conceivable acts* (that is, all the mappings from the set of states to the set of feasible consequences). The set of conceivable acts needs to be understood for the model to be useful, and for the idea of Ramsey (1926) to be implemented. Perhaps the best way to illustrate this point is to model betting on the outcome of a horse race.

Taking the small world approach suppose, for the sake of simplicity of exposition, that only two horses enter the race, Incumbent and Challenger. There are three possible outcomes: o_1 (Incumbent wins), o_2 (Challenger wins) and o_3, (dead heath). Let the set of consequences include the outcomes of the race, $O = \{o_1, o_2, o_3\}$, as well as sums of money represented by M. Thus, the set of consequences is $O \cup M$. The relevant set, A, of basic acts is a singleton, a, "running the race once between Incumbent and Challenger." Because elements of M are not outcomes associated with a, the set of feasible consequences is $C(a) = O$. Hence, according to the approach described above, the parsimonious state space consists of three states, $S = \{s_1, s_2, s_3\}$.

Conceivable acts include all the mappings from S to $O \cup M$. Some conceivable acts are hypothetical. For example, the conceivable constant act whose image is o_1 has the interpretation of running a race under the condition that Challenger cannot possibly win. By contrast, conceivable acts whose payoffs are sums of money (that is, elements of M) are feasible and correspond to betting on the outcome of the horse race. For example, the conceivable act that pays off $x \in M$ if o_1 obtains, $y \in M$ if o_2 obtains, and 0 otherwise, where $x > 0 > y$ has a concrete meaning – namely, a bet on Incumbent winning the race. More generally, the set of bets is given by $B := \{b : S \to O \cup M \mid b(s) \in M, \forall s \in S\}$. Finally, there are conceivable acts whose payoffs involve mixture of consequences in O and M (e.g., an act that pays off x in s_1, o_3 in state s_2 and y in state s_3) that are absurd.

The decision maker can contemplate choosing simultaneously a conceivable act from A' and a bet from B. Hence, the *conceivable choice set*, $\mathbb{C} := \{(a, b) : S \to O \times M \mid (a, b) \in A' \times B\}$, is the set of conceivable act-bet pairs that map the state space S to the product set $O \times M$.

3.3 The separation of states and consequences

A crucial aspect of Savage's (1954) model is the separation of tastes and beliefs. The valuation of the consequences (i.e., tastes) is independent of the events in which they are affected, and the assessment of the likelihoods of the events (i.e., beliefs) is independent of valuation of the consequences assigned to them. This separation is not always natural, however, and in some important situations the notions of states and consequences are confounded and the preference relation does not satisfy state-independence.

Consider, for example, the following situation described by Aumann in a letter to Savage dated January 1971.[13] A man's love for his wife makes his life without her "less 'worth living.'" The wife falls ill. To survive, she must undergo a routine but dangerous operation. The husband is offered a choice between betting $100 on his wife's survival or on the outcome of a coin flip. Even supposing that the husband believes that his wife has an even chance of surviving the operation, he may still rather bet on her survival, because winning $100 if she does not survive is "somehow worthless." Betting on the outcome of a coin flip, the husband might win but not be able to enjoy his winnings because his wife dies. In this situation, argues Aumann, Savage's notion of states (that is, whether the wife is dead or alive) and consequences are confounded to the point that there is nothing that one may call a consequence (that is, something whose value is state independent).

In his response, Savage admits that the difficulty Aumann identifies is indeed serious. In defense of his model, Savage writes, "The theory of personal probability and utility is, as I see it, a sort of framework into which I hope to fit a large class of decision problems. In this process, a certain amount of pushing, pulling, and departure from common sense may be acceptable and even advisable.... To some - perhaps to you - it will seem grotesque if I say that I should not mind being hung so long as it be done without damage to my health or reputation, but I think it desirable to adopt such language so that the danger of being hung can be contemplated in this framework" (Drèze 1987, p. 78). To the specific example of Aumann, Savage responds "In particular, I can contemplate the possibility that the lady dies medically and yet is restored in good health to her husband" (Drèze 1987, p. 80).The presumption that decision makers engage in such mental exercises when making

[13] The correspondence is reproduced in Drèze (1987) and in the collected works of Aumann (2000).

decisions seems farfetched.

The source of the problem is the formulation of the state space. Both Aumann and Savage take for granted that there are two states, the wife lives and the wife dies. However, because the wife's death affects the husband's well-being, it is also a consequence. It is the double role of the wife's health that confounds states and consequences. This problem can be avoided if the states space is defined using the approach outlined above.

In the scenario described by Aumann there are two basic acts – undergo surgery, a_1, and avoid surgery, a_2, – and two feasible consequences – the wife lives, c_1, and the wife dies, c_2. (Note that $C(a_1) = C(a_2) = \{c_1, c_2\}$). These generate four states as follows:

$A \backslash S$	s_1	s_2	s_3	s_4
a_1	c_1	c_1	c_2	c_2
a_2	c_1	c_2	c_1	c_2

If the husband believes that if his wife is to survive she must undergo the operation (that is, the husband believes that a_2 must necessarily result in c_2) then, for him, the event $\{s_1, s_3\}$ is null.

In this context, the constant act that yields the outcome the wife lives, c_1, in every state amounts to conceiving a medical procedure, not currently available, that is guaranteed to save the wife's life. Denote this conceivable treatment by a_3 and and suppose that $C(a_3) = \{c_1\}$. Augmenting the depiction of the states by adding the consequence of this conceivable constant act we get $s_2 = (c_1, c_2, c_1)$ and $s_4 = (c_2, c_2, c_1)$. The state s_4 is a description of a situation in which the wife would die under all currently available treatments but not under treatment a_3. Savage's statement "I can contemplate the possibility that the lady dies medically and yet is restored in good health to her husband" is problematic, because if the outcome "the wife dies during the operation" is a state of nature, then the constant act that delivers the consequence c_1 (the wife lives) must be possible in the state in which she is dead, which is absurd. However, according the approach advanced here, the same statement translates into "I can contemplate the possibility of a treatment that would restore the wife in good health to her husband in circumstance in which she would have died under the currently available treatments." This statement, far from being absurd, it is quite conceivable.

Consider next the husband's betting decision. The availability of a coin flip, f, introduces another basic act and two new feasible consequences, "heads up" (denoted H) and "tails up" (denoted T). Betting also requires monetary payoffs. For simplicity assume that the set of monetary payoffs is a doubleton $M = \{\$0, \$100\}$. This modification requires the expansion of the state space. Since $C(f) = \{H, T\}$, the relevant set of consequences is $C = C(a_1) \cup C(a_2) \cup C(f) \cup M$. Since M and $C(a_1) \cup C(a_2) \cup C(f)$ are disjoint, the parsimonious state space consists of four states: $s_{2H} = (c_1, c_2, H)$, $s_{2T} = (c_1, c_2, T)$, $s_{4H} = (c_2, c_2, H)$ and $s_{4T} = (c_2, c_2, T)$.

Bets are conceivable acts: "bet on heads" (denoted b_1) and "bet on tails" (denoted b_2). The bet b_1 pays off \$100 in the event $H := \{s_{2H}, s_{4H}\}$ and \$0 in the event $T := \{s_{2T}, s_{4T}\}$. A bet on the survival of the wife, b_3, pays off \$100 in the event $E_2 := \{s_{2H}, s_{2T}\}$ and \$0 in the event $E_4 := \{s_{4H}, s_{4T}\}$. In other words, betting on the wife's survival is betting that the operation succeeds. Since the husband's evaluation of the monetary payoff is not independent of whether the wife is dead or alive, according to the traditional approach, the preference relation does not satisfy state-independence. By contrast, under the approach advanced here the husband can contemplate such a bet even if he does not choose a_1. For example, the husband can conceive of choosing simultaneously the imaginary treatment a_3 and the aforementioned bet. If the husband imagines choosing $(a_3, b_1) \in A' \times B$ then the consequences that would have followed are the payoff $(c_1, \$100)$ in the event H and the payoff $(c_1, \$0)$ in the event T.[14] Similarly, imagining choosing (a_3, b_3) would pay off $(c_1, \$100)$ in the event E_2 and $(c_1, \$0)$ in the event E_4. Thus, the consequences of both bets are identical. Therefore, if the husband is an expected utility maximizer and believes that an operation has an equal chance of succeeding or failing, he is indifferent between betting on the outcome of the coin flip and betting on the success of the operation. Thus, the approach to the construction of the state space advanced here disentangles states and consequences, and lends credence to the supposition that the preferences display state independence.

However, the elicitation of the husbands beliefs is a thought experiment whose outcome hinges on a bet the payoff of which depends on the outcome of the act a_1 when a_3 is supposed to be implemented. If the implementation of the imaginary medical treatment a_3 precludes the implementation of a_1 then the scenario described above is inherently hypothetical. Hence, such a bet cannot possibly be settled in practice. The conclusion requires the ad-

[14]See discussion in section 3.2.

mission of preferences over hypothetical choices expressed verbally admitting preferences over counterfactuals. This departure from the revealed preference methodology raises methodological issues that cannot be addressed here. I conclude, therefore, by restating Savage's position on this issue as summarized in the following quote: "There is a mode of interrogation between what I called the behavioral and the direct. One can, namely, ask the person, not how he feels but what he would do in such and such situation. In so far as the theory of decision under development is regarded as an empirical one, the intermediate mode is a compromise between economy and rigor. But in the theory's more normative interpretation as a set of criteria of consistency for us to apply to our decisions, the intermediate mode is just the right one" (Savage, [1954], p. 28).

References

[1] Ahn, David, S. and Haluk Ergin (2010) "Framing Contingencies," *Econometrica*, 78, 655-695.

[2] Anscombe, Francis J. and Robert J. Aumann (1963) "A Definition of Subjective Probability," *Annals of Mathematical Statistics* 43, 199–205.

[3] Arrow, Kenneth (1965) *Aspects of the Theory of Risk-Bearing*, Helsinki: Yrjo Jahnsson Foundation.

[4] Aumann, Robert J. (2000) *Collected Papers*, MIT Press, Cambridge MA.

[5] de Finetti, B. (1937), "La Prevision: Ses Lois Logiques, ses Sources Subjectives." *Annales de l'Institut Henri Poincare*, 7: 1-68. (Translated in Kyburg, H.E. and H.E. Smokler, (eds.) *Studies in Subjective Probability*, New York: John Wiley and Sons, 1963).

[6] Drèze, J. H. (1987) *Essays on Economic Decisions under Uncertainty.* Cambridge: Cambridge University Press.

[7] Ellsberg, Daniel (1961) "Risk, Ambiguity and the Savage Axioms," *Quarterly Journal of Economics* 75, 643-669.

[8] Epstein, Larry, Marinacci, Massimo and Kyoungwon Seo (2007) "Coarse Contingencies and Ambiguity," *Theoretical Economics* 2, 355-394.

[9] Gilboa, Itzhak, Pestlewaite, Andrew and David Schmeidler (2009) "Is It Always Rational to Satisfy Savage's Axioms?" *Economic and Philosophy* 25, 285-296.

[10] Fishburn, Peter (1970) *Utility Theory for Decision Making.* John Wiley and Sons, New York.

[11] Karni, Edi and David Schmeidler (1991) "Utility Theory with Uncertainty," in Werner Hildenbrand and Hugo Sonnenschein, eds., *Handbook of Mathematical Economics* vol. IV. Amsterdam: Elsevier Science Publishers B.V.

[12] Karni, Edi and Marie-Louise Vierø (2013) ""Reverse Bayesianism": A Choice-Based Theory of Growing Awareness," *American Economic Review*, 103, 2790–2810.

[13] Karni, Edi and Marie-Louise Vierø (2015a) "Probabilistic Sophistication and Reverse Bayesianism," *Journal of Risk and Uncertainty,* 50, 189-208.

[14] Karni, Edi and Marie-Louise Vierø (2015b) "Awareness of Unawareness: A Theory of Decision Making in the Face of Ignorance," unpublished manuscript.

[15] Kreps, David M. (1979) "A Representation Theorem for 'Preference for Flexibility'," *Econometrica* 47, 565-576.

[16] Kreps, David M. (1988) *Notes on the Theory of Choice.* Westview Press, Boulder.

[17] Kreps, David M. (2013) *Microeconomic Foundations I.* Princeton Univerity Press, Princeton

[18] Machina, Mark J. (2003) "States of the World and the State of Decision Theory," In *The Economics of Risk*. Donald Meyer (ed.) W.E. Upjohn Institute for Employment Research.

[19] Ramsey, Frank P. (1926), "Truth and Probability." (Published in Braithwaite, R. B. and F. Plumpton, *The Foundation of Mathematics and Other Logical Essays*. London: Routledge and Kegan, 1931).

[20] Savage, Leonard J. (1954) *The Foundations of Statistics.* New York: Wiley. Second revised edition 1972.

[21] Schmeidler, David and Peter Wakker (1987) "Expected Utility and Mathematical Expectation," in John Eatwell, Murray Milgate, and Peter Newman, eds., *The New Palgrave: A Dictionary of Economics.* London: Macmillan Press.