Incomplete Preferences and Random Choice

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Abstract

Incomplete preferences admits noncomparable alternatives. This paper advances the proposition that choice among such alternatives is inherently random and proposes a random choice model, to describe random binary choices and random choice functions depicting random choices from sets of alterantives. Representation of the random-choice behavior and are characterized. Experiments designed to test the model in the context of multi-prior expected mult-utility model and the special case of Knightian uncertainty are described and analyized.

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Keywords: Incomplete preferences, Random Choice, Stochastic choice functions, Knightian uncertainty, Multi-prior expected multi-utility representations.

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1 Introduction

In many situations requiring a choice among feasible alternatives, decision makers find the alternatives difficult, if not impossible, to compare. Depending on the context, this difficulty may be due to complexity of the alternatives, or lack of experience that makes it impossible to assess their, possibly long run, consequences. This fact was recognized by von Neumann and Morgenstern, who admitted that "it is conceivable–and even in a way more realistic–to allow for cases where the individual is neither able to state which of two alternatives he prefers nor that they are equally desirable." (von Neumann and Morgenstern [1947]).

The appropriateness of the completeness postulate was broached by Leonard Savage in a letter to Karl Popper, dated March 25, 1958, in which Savage discusses his work on the choice-based foundations of subjective probabilities. Savage wrote: "There is, though, a postulate that insists that economic situations can be ranked in a linear order by the subject, and I freely admit that this seems to me to be a source of much difficulty in my theory. This stringent postulate is in conflict with the common experience of vagueness and indecision, and if I knew a good way to make a mathematical model of those phenomena, I would adopt it, but I despair of finding one."[1]

Robert Aumann questioned not only the descriptive validity of the completeness axiom but also its normative justification. "Of all the axioms of utility theory," he wrote, "the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from the normative viewpoint" (Aumann [1962]).

Evidence of the prevalence of incomplete preferences in experimental settings is provided Danan and Ziegelmeyer (2006), Sautua (2017), Costa-Gomez, et. al (2019), and Cettolin and Riedl (2019).

Yet, with few exceptions, the theories of individual decision making, whether under certainty, risk, or uncertainty, presume that the preference relations describing individual choice behavior are complete. When the preference relation is complete, all alternatives are comparable and decision makers exhibit resolute choice behavior, except when the alternatives belong to the same indifference class. When the preference relation is incomplete, deci-

[1] This correspondence is reproduces by Carlo Zappia form Leonard Jimmy Savage Papers, archived at the Manuscript and Archives Department of Yale University Library as MS 695, Box 25, Folder 622.

sion makers irresolute choice behavior extends to noncomparable alternatives and the existing models provide no guidance insofar as predicting the choice among such alternatives.[2] The purpose of this work is to fill this lacuna by proposing a model that departs form the completeness postulate and depicts choice behavior among noncomparable alternatives.

The main idea is that when having to decide among noncomparable alternatives, a decision maker's choices are triggered by impulses, or signals, that are themselves random or look like such to an observer that is not privy to the working of the decision maker's mind. In either case, insofar as the observer is concerned, the decision maker's choices are inherently random. In the proposed model the impulses, or signals, are "mental decoys," invoked randomly, and aid decision makers "to make up their minds" and resolve their indecisiveness. The model has predictable *probabilistic choice behavior.* In other words, given a menu of noncomparable alternatives, the model predicts the probabilities of the different alternatives being selected.

The behavioral manifestations of incomplete preferences are inertia and unpredictability. Inertia describes the lack of response to exogenous changes in the relevant environment caused by the decision maker's inability to compare the status quo, or default, alternative with new feasible alternatives. Unpredictability expresses the fact that when new feasible alternatives dominate the status quo but are noncomparable among themselves, the decision maker's choice among such alternatives is random.

Despite growing interest, in recent years, in modeling and studying random choice behavior, there is no systematic evidence attributing such behavior specifically to preference incompleteness.[3] This lack of evidence reflects the absence of experimental designs that would allow the identification of random choice behavior restricted to noncomparable alternatives. A contribution of this paper is novel experimental designs that allow the empirical study of random choice behavior in the presence of incomplete preferences. At the heart of these designs are incentive-compatible mechanisms by which the range of incompleteness and decision makers' perceived likelihoods of their eventually choosing the various noncomparable alternatives are elicited. Given these parameters, the model yields predictions that are testable against the decision makers' actual choices.

[2] Ok and Tserenjigmidz (2020) is an exception. Their work is discussed in the next and the concluding sections.

[3] A brief review of the relevant literature appears in the concluding section.

For lack of systematic evidence to provide the scaffolding for the construction of a theory by induction, the proposed model is arrived at based on logical considerations. Methodologically, this approach is consistent with Einstein's view of the role of theory. According to Einstein "on principle, it is quite wrong to try founding a theory on observable magnitudes alone. In reality the very opposite happens. It is the theory which decides what we can observe."[4] Using the proposed model as a guide to what can be observed, and the experimental designs that would allow its testing, this paper advances a meaningful theory of random choice behavior that is attributed to preference incompleteness.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 applies the model to subjective expected utility theory. Section 4 describes the experiments designed to test the model in the context of subjective expected utility theory. Concluding remarks and a brief review of the relevant literature appear in Section 5. The proofs are collected in Section 6.

2 A Model of Random Choice Behavior

The proposed model applies to decision making under certainty, under risk, and under uncertainty.

2.1 The analytical framework

Let the *choice set,* A, be a topological space whose elements are *alternatives.* Denote by $\succ$ a binary relation on A that is a continuous strict partial order dubbed a *preference relation.* Formally, $\succ$ is irreflexive and transitive and, for all $a \in A$, the sets $\{a' \in A \mid a' \succ a\}$ and $\{a' \in A \mid a \succ a'\}$ are open. I assume throughout that $\succ \neq \varnothing$. For any alternatives $a, a' \in A$, $a \succ a'$ has the usual interpretation that a is *strictly preferred* over a', which is taken to mean that a decision maker, whose preference relation is $\succ$, facing a choice between these two alternatives will choose the alternative a. The relation $\neg (a \succ a')$ is reflexive but not necessarily transitive (i.e., it is not necessarily a preorder).

[4]This is a quote from a conversation with Werner Heisenberg. Einstein's General Relativity is a prime example of this deductive methodology.

The strict preference relation, $\succ$, induces three derived relations on A. For all $a, a' \in A$, the induced *weak preference relation,* $\succeq$, is defined by: $a \succeq a'$ if, for all $a'' \in A$, $a'' \succ a$ implies that $a'' \succ a'$; the induced *indifference relation*, $\sim$, is defined by $a \sim a'$ if $a \succeq a'$ and $a' \succeq a$; and the *noncomparability relation* $\bowtie$, defined by: $a \bowtie a'$ if $\neg (a \succeq a')$ and $\neg (a' \succeq a)$.[5]

It is natural to suppose that if presented with a choice between two alternatives, a and a', a decision maker would choose the former act if $a \succeq a'$ and $\neg (a' \succeq a)$. However, if $a \bowtie a'$ or $a' \sim a$, then the preference relation does not single out an alternative that will be chosen.

In what follows I propose to model irresolute choice behavior as a set $\{\trianglerighteq^{\alpha} | \alpha \in [0,1]\}$ of binary relations on A, dubbed *random choice relations.* Given any $a, a' \in A$, the interpretation of $a \trianglerighteq^{\alpha} a'$ is that, facing a choice between a and a', a is chosen with probability α. Clearly, $a \succ a'$ implies $a \trianglerighteq^{1} a'$ and, jointly, $a \succeq a'$ and $\neg (a' \succeq a)$ if and only if $a \trianglerighteq^{1} a'$.

2.2 The Random Choice Model

Facing a choice between alternatives that are ranked by the strict preference relation, the decision maker chooses the preferred alternative with probability one. Otherwise, facing a choice between two alternatives, say a and a', a third alternative, $a'' \in A$, dubbed *mental decoy*, is randomly selected. This alternative serves as a reference that the decision maker relies upon to resolve his indecision. If the third alternative is weakly inferior to a and is noncomparable to a' then the decision maker chooses the alternative a, and if it is inferior to a' and noncomparable to a then the decision maker chooses a'. Otherwise, the decision maker procrastinates while waiting for another mental decoy to be randomly selected that would allow him to resolves the indecision along the lines indicated above.[6] The mental decoy formalizes the

[5] It is customary to define weak preference relations as the negation of the strict preference relation $\succ$. Formally, given a binary relation $\succ$ on A, define a binary relation $\succcurlyeq$ on A by $a \succcurlyeq a'$ if $\neg (a' \succ a)$. Eliaz and Ok (2006) study the distinction between noncomparability and indifference. The weak preference relation defined here was introduced in Galaabaatar and Karni (2013). Its significance and implications were investigated and discussed in Karni (2011). In particular, Karni showed that the weak preference relations $\succcurlyeq$ and $\succeq$ agree if and only if $\succ$ is negatively transitive and $\succeq$ is complete. Note that $\succ$ is not the asymmetric part of $\succeq$. The indifference relation defined here, was introduced in Galaabaatar and Karni (2013), is equivalent to that of Eliaz and Ok (2006).

[6] Equivalently, if the third alternative is weakly preferred to a' and is non-comparable to a, then the decision maker chooses the alternative a. More on this in the concluding

idea of a random signal generated by unspecified mental or exogenous process that determined the choice.

Intuitive support to the reasoning underlying the mental decoy idea is provided by the well-known "decoy effect" in consumer decisions. The decoy effect pertains to a pattern of choice behavior according to which, when facing a choice between two products that have multiple attributes, but are noncomparable in the sense that neither product has more of all the desirable attributes than the other, the introduction of a third product that is dominated (in the sense of having less of the desirable attributes) by one of the existing products but not by another, tilts the consumer choice towards the dominating product. A third alternative dominated by both products does not affect the choice behavior and does not produce significant shift in market share.[7] The mental decoy captures the same idea with strict preference instead of attribute-wise domination and random selection of the decoy alternative. Henceforth, I refer to the stochastic process of choosing the decoy alternative as the *mental decoy process.*

To formalize this idea, let $\mathcal{B}_A$ denote the Borel $\sigma-$ algebra on A and P a probability measure on the measurable space $(A, \mathcal{B}_A)$. For any given $a, a' \in A$ define

$$\Psi\left(a \gtrdot a'\right) = \{a'' \in A \mid a \succcurlyeq a'' \text{ and } a' \bowtie a''\}. \tag{1}$$

Obviously, if $a \sim a'$ then $\Psi\left(a \gtrdot a'\right) = \Psi\left(a' \gtrdot a\right) = \varnothing$.

Define a function $p : A \times A \to [0,1]$ by:

$$p\left(a, a'\right) = \frac{P\left(\Psi\left(a \gtrdot a'\right)\right)}{P\left(\Psi\left(a \gtrdot a'\right) \cup \Psi\left(a' \gtrdot a\right)\right)} \text{ if } \neg\left(a \sim a'\right) \tag{2}$$

$$p\left(a, a'\right) \in [0,1] \text{ if } a \sim a'. \tag{3}$$

Given a choice between a and a', is $p\left(a, a'\right)$ is the probability that a third alternative is selected that would lead the decision maker to choose a from the set $\{a, a'\}$, as opposed to procrastination.

The random choice model, or RCM for short, maintains that, facing a choice between a and a', a decision maker characterized by $(\succ, P)$, chooses a with probability $p\left(a, a'\right)$ and a' with probability $1 - p\left(a, a'\right)$. Formally, for all $a, a' \in A$, $a \trianglerighteq^{p(a,a')} a'$. It is easy to verify that if $a \succ a'$ then $p\left(a, a'\right) = 1$

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[7]See Huber et al. (1982). For a more recent discussion in Ok, Ortoleva and Riella (2015).

and $a \succcurlyeq a'$ and $\neg(a' \succcurlyeq a)$ if and only if $p(a,a') = 1$. If $a \sim a'$ then the model is agnostic insofar as the prediction of the probabilistic choice is concerned.

2.3 Representation of the RCM

The RCM is quite general and applies to decision making under certainty, under risk, and under uncertainty. To begin with, I study the representation of model as it applies to general continuous preference relations that admit multi-utility representation.[8] Assume that A is locally compact Hausdorff space that is also σ-compact, and that $\succ$ is a continuous strict partial order on A.[9] Then, it follows from Evren and Ok (2011) Theorem 1, that there is a set, $\mathcal{V}$, of continuous real-valued functions on A such that, for all $a, a' \in A$,

$$a \succcurlyeq a' \Longleftrightarrow v(a) \geq v(a'), \ \forall v \in \mathcal{V}, \tag{4}$$
$$a \sim a' \Longleftrightarrow v(a) = v(a'), \ \forall v \in \mathcal{V}, \tag{5}$$
$$a \bowtie a' \Longleftrightarrow \exists v, \hat{v} \in \mathcal{V} \text{ such that } v(a) > v(a') \text{ and } \hat{v}(a) < \hat{v}(a'). \tag{6}$$

Let $\mathcal{V}$ be endowed with the product topology and denote by $\mathcal{B}_{\mathcal{V}}$ the Borel $\sigma-$ algebra on $\mathcal{V}$. Let $(\mathcal{V}, \mathcal{B}_{\mathcal{V}}, \lambda)$ be a probability space. For every given $a, a' \in A$, let $\mathcal{V}(a,a') := \{v \in \mathcal{V} \mid v(a) \geq v(a')\}$, the set of utility functions according to which a is weakly preferred to a'. Then $\mathcal{V}(a,a') \in \mathcal{B}_{\mathcal{V}}$ and the probability that, facing a choice between a and a', a decision maker characterized by a preference relation $\succ$ on A and the probability measure λ on $\mathcal{V}$ chooses a is $\lambda(\mathcal{V}(a,a'))$. The intuitive idea underlying this assertion is that a utility function is drawn at random from the set $\mathcal{V}$ according to the probability measure λ, and the alternative that is ranked higher according to this utility function is chosen.[10]

Definition: The choice relation $\{\trianglerighteq^{\lambda(\mathcal{V}(a,a'))} \mid a, a' \in A\}$ *represents the RCM* if $p(a,a') = \lambda(\mathcal{V}(a,a'))$, for all $a, a' \in A$.

Given $v \in \mathcal{V}$, the equivalence class of $a \in A$ is defined by $I(a \mid v) = \{a' \in A \mid v(a) = v(a')$. For all $a, a' \in A$ let $\Gamma(a, a' \mid v) := I(a' \mid v) \cap \Psi(a \gtrdot a')$.

[8] See Ok (2002) and Evren and Ok (2011) for axiomatic characterizations of multi-utility representations.

[9] A topological space is locally compact if every point in it has an open neighborhood with compact closure. It is σ-compact if it can be written as a union of countably many of its compact subsets.

[10] In social choice theory the same idea is captured by the random dictator mechanism.

Then, $\mathcal{P}(a,a') := \{\Gamma(a,a' \mid v) \mid v \in \mathcal{V}(a,a')\}$ is a partition of $\Psi(a \gtrdot a')$. Define $\Gamma(a,a' \mid \mathcal{V}(a,a')) = \cup_{v\in\mathcal{V}(a,a')}\Gamma(a,a' \mid v)$, for all $a,a' \in A$.

The following Theorem is a representation of the RCM $\{\trianglerighteq^{p(a,a')} \mid a,a' \in A\}$ in terms of the multi-utility representation of the preference relation, $\succ$.

Theorem 1: *If $\succcurlyeq$ on A is represented by (4) then $\{\trianglerighteq^{\lambda(\mathcal{V}(a,a'))} \mid a,a' \in A\}$ represents the random choice model $\{\trianglerighteq^{p(a,a')} \mid a,a' \in A\}$ if and only if*

$$\frac{P(\Gamma(a,a' \mid \mathcal{V}(a,a')))}{P(\Gamma(a,a' \mid \mathcal{V}(a,a')) \cup \Gamma(a',a \mid \mathcal{V}(a',a)))} = \lambda(\mathcal{V}(a,a')),\ \forall a,a' \in A. \quad (7)$$

2.4 Stochastic choice functions

The RCM is based on pairwise, or binary, choices. Many situations of interest, however, require a choice from menus, or budget sets, that include more than two alternatives. Formally, a *menu* $M \subset A$ is a nonempty compact subset of alternatives. Let $\mathcal{M}$ denote the set of menus. To apply the RCM to these type of decision problems, I extend it to include (stochastic) choices from menus.

Following Ok and Tserenjigmidz (2020), a *stochastic choice function* is a map $\rho : A \times \mathcal{M} \to [0,1]$ such that $\Sigma_{a\in M}\rho(a,M) = 1$ and $\rho(a',M) = 0$, for every $M \in \mathcal{M}$ and $a' \in A\backslash M$. For every $M \in \mathcal{M}$ and $a \in M$, let $D(M,\succ) := \{a' \in M \mid \exists a \in M,\ a \succ a'\}$ be the set of *dominated alternatives in the menu M according to* $\succ$ and $\max(M,\succ) := M\backslash D(M,\succ)$ the subset of M that consists of *undominated alternatives according to* $\succ$. Clearly, $\max(M,\succ)$ is either a singleton or it consists of noncomparable and/or indifferent alternatives.[11]

Ok and Tserenjigmidz (2020) characterized stochastic choice functions induced by lack of strict preference. To depict their result in terms of the notations and the definitions of this paper, a preorder $\succcurlyeq$ on A is said to be *regular* if the symmetric part of $\succcurlyeq$ agrees with the indifference relation induced by $\succcurlyeq$. A stochastic choice function ρ on A is said to be *induced by lack of strict preference* if there exists a regular preorder $\succcurlyeq$ on A such that $\rho(a;M) > 0$ if and only if $a \in \max(M,\succcurlyeq)$ for every $M \in \mathcal{M}$. Ok and Tserenjigmidz showed that a stochastic choice function ρ on $\mathcal{M}$ is induced by lack of strict preference if, and only if, it satisfies the Stochastic Chernoff axiom (i.e., $\forall M, M' \in \mathcal{M}$ with $M \subseteq M'$ and $a \in M$, $\rho(a,M') > 0$ implies

[11] That $\max(M,\succ) \neq \varnothing$ is an implication of the compactness of M and continuity of $\succ$.

$\rho(a, M) > 0$) and Stochastic Condorcet axiom (i.e., $\forall M \in \mathcal{M}$ and $a \in M$, $\rho(a, \{a, a'\}) > 0\ \forall a' \in M$ with $\rho(a', M) > 0$ implies $\rho(a, M) > 0$).

The stochastic choice function induced by lack of strict preference, predicts that only alternatives in the undominated set be chosen with positive probabilities. It is silent insofar as the assignment of probabilities to the different alternatives in the set is concerned. I show next that a stochastic choice function induced by lack of strict preference that is based on the RCM (i.e., induced by the mental decoy process) predicts the probabilities that the different elements of the undominated set be chosen.

Define

$$\Psi(a \gtrdot M) = \{a'' \in A \mid a \succcurlyeq a'' \text{ and } a' \bowtie a'', \forall a' \in \max(M, \succ)\ \}. \tag{8}$$

Clearly, $\Psi(a \gtrdot M) = \cap_{a' \in M} \Psi(a \gtrdot a')$. If $\mid M \mid > 2$ and $\neg(a \sim a', \forall a, a' \in \max(M, \succ))$ then $a \sim a'$ for some $a, a' \in \max(M, \succ)$ implies $\Psi(a \gtrdot M) = \Psi(a' \gtrdot M)$. If $a \sim a'$, for all $a, a' \in \max(M, \succ)$ then $\Psi(a \gtrdot M) = \varnothing$, for all $a \in \max(M, \succ)$.

Define a stochastic choice function ρ as follows: If $a \nsim a'$, for all $a, a' \in \max(M, \succ)$ (i.e., no two undominated alternatives are indifferent to one another) then

$$\rho(a, M) = \frac{P(\Psi(a \gtrdot M))}{P(\cup_{a' \in M} \Psi(a' \gtrdot M))}, \forall a \in \max(M, \succ)\ . \tag{9}$$

If $a \sim a'$, for all $a, a' \in \max(M, \succ)$ then

$$\rho(a, M) \in [0, 1]. \tag{10}$$

If there is a proper subset $\hat{M} \subset \max(M, \succ)$ such that $\hat{a} \sim \hat{a}'$ for all $\hat{a}, \hat{a}' \in \hat{M}$, then

$$\rho\left(a, \hat{M}\right) \in [0, \frac{P\left(\cup_{\hat{a} \in \hat{M}} \Psi(\hat{a} \gtrdot M)\right)}{P(\cup_{a' \in M} \Psi(a' \gtrdot M))}] \text{ and } \rho\left(a, M \backslash \hat{M}\right) = \frac{P(\Psi(a \gtrdot M))}{P(\cup_{a' \in M} \Psi(a' \gtrdot M))}. \tag{11}$$

If $a \notin \max(M, \succ)$ then

$$\rho(a, M) = 0. \tag{12}$$

The interpretation of $\rho(a, M)$ extends that of the pairwise choices and is as follows: Facing a choice from a menu M if it contains a unique undominated alternative, the decision maker chooses it with probability one.

Otherwise, (that is, the undominated set of alternatives in not a singleton) a mental decoy, $a'' \in A$, is drawn at random. If a'' is weakly inferior to some $a \in M$ and is noncomparable to all the other undominated alternatives in M, then the decision maker chooses the alternative a. If there is no alternative in the menu that satisfies this condition, the decision maker procrastinates and waits for another draw from A. The process continues until a decision is made. If all the undominated alternatives in M are indifferent to one another, then the model is agnostic regarding the probability that any particular alternative will be chosen. If a proper subset of the undominated alternatives are indifferent to one another then if the decoy alternative is weakly inferior to the indifferent alternatives and is noncomparable to the other undominated alternatives then the prediction of the model is that one of the indifferent alternatives in the undominated subset be chosen but is agnostic regarding the probability that any particular alternative in the indifferent subset will be chosen. Finally, the probability that a dominated alternative is chosen is zero.

The random choice model maintains that, facing a choice from a menu M, a decision maker characterized by $(\succ, P)$, chooses the alternative a with probability $\rho(a, M)$. Formally, for all $(a, M) \in A \times \mathcal{M}$, $a \trianglerighteq^{\rho(a,M)} M$.

2.5 Representation of random choice functions

Define $\mathcal{V}(a, M) := \cap_{a' \in \max(M,\succ)} \mathcal{V}(a, a')$. For all $a, a' \in A$ and $M \in \mathcal{M}$ let

$$\Gamma(a, a' \mid \mathcal{V}(a, M)) := \cup_{v \in \mathcal{V}(a,M)} \Gamma(a, a' \mid v),$$

and, for all $(a, M) \in A \times \mathcal{M}$ let

$$\Gamma(a, M \mid \mathcal{V}(a, M)) := \cap_{a' \in \max(M,\succ)} \Gamma(a, a' \mid \mathcal{V}(a, M)).$$

Theorem 2: *If $\succcurlyeq$ on A is represented by (4) then $\{\trianglerighteq^{\lambda(\mathcal{V}(a,M))} \mid (a, M) \in A \times \mathcal{M}\}$ represents the random choice model $\{\trianglerighteq^{\rho(a,M)} \mid (a, M) \in A \times \mathcal{M}\}$ if and only if*

$$\frac{P(\Gamma(a, M \mid \mathcal{V}(a, M)))}{P\left(\cup_{a' \in \max(M,\succ)} \Gamma(a', M \mid \mathcal{V}(a', M))\right)} = \lambda(\mathcal{V}(a, M)), \ \forall (a, M) \in A \times \mathcal{M}. \tag{13}$$

3 Random Choice under Uncertainty

For over half a century, subjective expected utility theory has been the dominant model of decision making under uncertainty. Because of it prominent role and rich analytical framework, I explore the application of the RCM to subjective expected utility theory. For the purpose of exposition, I adopt the model of Galaabaatar and Karni (2013). This model admits incomplete beliefs and tastes, and includes Bewley's Knigthian uncertainty model (i.e., complete tastes and incomplete beliefs), and the model of complete beliefs and incomplete tastes as special cases.

3.1 The analytical framework and the RCM

Let S be a finite set of *states.* Subsets of S are *events.* Let X be a finite set of outcomes and denote by $\Delta(X)$ the set of all probability distributions on X. For each $\ell, \ell' \in \Delta(X)$ and $\alpha \in [0,1]$ define $\alpha\ell + (1-\alpha)\ell' \in \Delta(X)$ by $(\alpha\ell + (1-\alpha)\ell')(x) = \alpha\ell(x) + (1-\alpha)\ell'(x)$, for all $x \in X$.

Let $H := \Delta(X)^S$ be the choice set. Elements of H are referred to as *acts*. For all $h, h' \in H$ and $\alpha \in [0,1]$, define $\alpha h + (1-\alpha) h' \in H$ by $(\alpha h + (1-\alpha) h')(s) = \alpha h(s) + (1-\alpha) h'(s)$. Under this definition H is a convex subset of the linear space $\mathbb{R}_+^{|X|\times|S|}$. Denote by $\mathcal{B}_H$ the trace on H of the the Borel $\sigma-$ algebra on $\mathbb{R}_+^{|X|\times|S|}$, and let P be a probability measure on the measurable space $(H, \mathcal{B}_H)$.

Let $\succ$ a strict partial order on H and, for all $f, g \in H,$ define $\Psi(g \gtrdot f) := \{h \in H \mid g \succcurlyeq h \text{ and } f \bowtie h\}$. Then, $\Psi(g \gtrdot f)$,$\Psi(f \gtrdot g) \in \mathcal{B}_H$ and, according to the RCM, $f \trianglerighteq^{p(f,g)} g$, where $p(f,g) = P(\Psi(f \gtrdot g)) / (P(\Psi(f \gtrdot g) \cup \Psi(f \lessdot g)))$.

3.2 Representation of random choice

The literature on decision making under uncertainty with incomplete preferences deals with axiomatic characterizations of multi-prior expected multi-utility representation.[12] For the present purpose I adopt the product repre-

[12]These contributions include Seidenfeld, Schervish, and Kadane. (1995), Nau (2006), Ok, Ortoleva and Riella (2012), and Galaabaatar and Karni (2013). Bewley (2002) dealt with the special case of multi-prior representation of incomplete beliefs. Expected multi-utility representations were axiomatized by Shapley and Baucells (1998) and Dubra, Maccheroni and Ok (2004). The former is a special case of Galaabaatar and Karni (2013). The latter is a special case of Ok et. al (2012).

sentation of the weak preference relation of Galaabaatar and Karni (2013).

Let $\Delta(S)$ be a set of probability distributions (priors) on S and $\Pi \subset \Delta(S)$. Let $\mathcal{U}$ be a set of real-valued functions on X and let $\Phi := \Pi \times \mathcal{U}$. Since $\pi \in \mathbb{R}_+^{|S|}$ and $u \in \mathbb{R}^{|X|}$, (π, u) is a vector in $\mathbb{R}_+^{|S| \times |X|}$ and $\Phi \subset \mathbb{R}_+^{|S| \times |X|}$. Denote by $\langle \widehat{\Phi} \rangle$ the closure of the convex cone in $\mathbb{R}^{|X| \cdot |S|}$ generated by Φ. Then, for all $f, g \in H$,

$$f \succcurlyeq g \Leftrightarrow \sum_{s \in S} \pi(s) \left(\sum_{x \in X} f(x,s) u(x) \right) \geq \sum_{s \in S} \pi(s) \left(\sum_{x \in X} g(x,s) u(x) \right), \ \forall (\pi, u) \in \Phi. \tag{14}$$

Moreover, if $\Phi' = \Pi' \times \mathcal{U}'$ represents $\succ$ in the sense of (14), then $\langle \widehat{\Phi'} \rangle = \langle \widehat{\Phi} \rangle$ and $\pi(s) > 0$, for all $s \in S$.

To simplify the notations, when there is no risk of confusion, I denote $U(f(s); u) = \Sigma_{x \in X} f(x,s) u(x)$ and $U(f : (\pi, u)) = \Sigma_{s \in S} \pi(s) U(f(s); u)$. Let

$$\Phi(f,g) := \{(\pi, u) \in \Phi \mid U(f : (\pi, u)) \geq U(g : (\pi, u))\}. \tag{15}$$

Suppose that, when facing a choice between f and g, a probability-utility pair, $(\widetilde{\pi}, \widetilde{u})$, is randomly drawn from Φ, and f is chosen over g if and only if $(\widetilde{\pi}, \widetilde{u}) \in \Phi(f,g)$. Denote by $\mathcal{B}_\Phi$ the trace on Φ of the Borel $\sigma-$algebra on $\mathbb{R}_+^{|S| \times |X|}$ and let λ be a probability measure on the measurable set $(\Phi, \mathcal{B}_\Phi)$. Then $\Phi(f,g) \in \mathcal{B}_\Phi$ and the probability that, facing a choice between f and g, a decision maker characterized by a preference relation $\succ$ on H and the probability measure λ on Φ, chooses f is $\lambda(\Phi(f,g))$.

The multi-prior expected multi-utility representation is a special case of the representation (4) in which preferences are represented by multiple linear utility functions over acts. Formally, substituting H for A, Φ for $\mathcal{V}$ and $\Phi(f,g)$ for $\mathcal{V}(f,g)$, and applying Theorem 1 we get:

Corollary 1: *If $\succcurlyeq$ on H is represented by (14) then $\{\trianglerighteq^{\lambda(\Phi(f,g))} \mid f, g \in H\}$ represents the random choice model $\{\trianglerighteq^{p(f,g)} \mid f, g \in H\}$ if and only if*

$$\frac{P(\Gamma(f, g \mid \Phi(f,g)))}{P(\Gamma(f, g \mid \Phi(f,g)) \cup \Gamma(g, f \mid \Phi(g,f)))} = \lambda(\Phi(f,g)), \ \forall f, g \in H. \tag{16}$$

3.3 Special cases

Knightian uncertainty (Bewley 2002) pertains to the case of complete tastes (i.e., on the subset of constant acts the preference relation $\succ$ on H is transitive, irreflexive and negatively transitive) and attributes the incompleteness of the preference relation solely to incomplete beliefs. This is a special case in which, in the representation (14), $\Phi = \{u\} \times \Pi$. Applying our definition to this case, let $\Pi(f,g) := \{\pi \in \Pi \mid U(f:(\pi,u)) \geq U(g:(\pi,u))\}$,then $\Phi(f,g) = \{u\} \times \Pi(f,g)$ and $\lambda(\Phi(f,g)) = \lambda(\Pi(f,g))$, for all $f,g \in H$. Denote by $f \trianglerighteq^{\lambda(\Pi(f,g))} g$ the choice relation that selects f form the set $\{f,g\}$ with probability $\lambda(\Pi(f,g))$. Then, an immediate implication of Corollary 1 is the following, (where $\Pi(f,g)$ stands for $\{u\} \times \Pi(f,g)$):

Corollary 2: *If $\succcurlyeq$ on H displays Knightian uncertainty and is represented by (14) then $\trianglerighteq^{\lambda(\Pi(f,g))}$ represents the random choice model if and only if, for all $f,g \in H$,*

$$\frac{P(\Gamma(g,f \mid \Pi(f,g)))}{P(\Gamma(g,f \mid \Pi(f,g)) \cup \Gamma(f,g \mid \Pi(f,g)))} = \lambda(\Pi(f,g)). \tag{17}$$

The other special case is that of complete beliefs (see Galaabaatar and Karni [2013]). In this case, we have expected multi-utility representation according to which (14) holds with $\Phi = \{\pi\} \times \mathcal{U}$. Let $\mathcal{U}(f,g) := \{u \in \mathcal{U} \mid U(f:(\pi,u)) \geq U(g:(\pi,u))\}$, then, $\Phi(f,g) = \{\pi\} \times \mathcal{U}(f,g)$ and $\lambda(\Phi(f,g)) = \lambda(\mathcal{U}(f,g))$, for all $f,g \in H$. Denote by $f \trianglerighteq^{\lambda(\mathcal{U}(f,g))} g$ the choice relation that selects f form the set $\{f,g\}$ with probability $\lambda(\mathcal{U}(f,g))$. Then, an immediate implication of Corollary 1 is the following (where $\mathcal{U}(f,g)$ stands for $\{\pi\} \times \mathcal{U}(f,g)$):

Corollary 3: *If $\succcurlyeq$ on H displays Knightian uncertainty and is represented by (14) then $\trianglerighteq^{\lambda(\mathcal{U}(f,g))}$represents the random choice model if and only if, for all $f,g \in H$,*

$$\frac{P(\Gamma(g,f \mid \mathcal{U}(f,g)))}{P(\Gamma(g,f \mid \mathcal{U}(f,g)) \cup \Gamma(f,g \mid \mathcal{U}(f,g)))} = \lambda(\mathcal{U}(f,g)). \tag{18}$$

All these results can be extended to stochastic choice functions in a straightforward manner.

4 Experimental Tests

Any meaningful theory that purports to describe natural or social phenomena must be accompanied by clear testable implications. A theory of choice behavior is no exception. To render the proposed RCM meaningful, I discuss next experiments designed to test it in the context of decision making under uncertainty and under risk. Generally speaking, testing the proposed RCM requires that acts, in the case of uncertainty, and lotteries, in the case of risk, that the decision maker considers to be noncomparable be identified, and the agreement between the observed choices among such acts and the random choices predicted by the model evaluated.

There is an analogy between the theory of risk-aversion and the RCM that is useful to keep in mind when considering their behavioral implications. In both instances the structure of the underlying preference relations imposes some strictures on the predictable choice behavior and, at the same time, leaves room for personal variations that are due to distinct individual risk attitudes in the former case, and to idiosyncratic incompleteness and the mental decoy process in the latter case. Monotonicity with respect to first-order stochastic dominance, implied by the preference structure, is a property that transcends individual risk attitudes and idiosyncratic mental decoy processes. Consequently, the multi-prior expected multi-utility model with incomplete preferences displays *probabilistic choice monotonicity with respect to first-order stochastic dominance*. Formally, if an act h first-order stochastically dominates an act g, and f in noncomparable to either h or g, then the probability that f is selected from the pair (f, g) is greater than the probability that it is selected from the pair (f, h).

I describe below experiments designed to test the RCM. These, include the probabilistic choice monotonicity and the hypothesis that decision makers entertain probabilistic beliefs about the stochastic selection of the mental decoys which are manifested in their random choice behavior.

4.1 Probabilistic choice monotonicity

The degree of incompleteness, of a decision maker's preference relation is a personal characteristic. Therefore, to obtain testable implications of the RCM, we need formal measures of the degree of incompleteness and an elicitation scheme by which it is possible to determine the individual degree of incompleteness. Karni and Vierø (2021) introduced such measures and also an incentive compatible mechanisms by which the incompleteness displayed by a preference relation may be elicited. Specifically, let $E \subseteq S$ be an event and denote by $x_E y$ the act that pays off \$$x$ if $s \in E$ and \$$y$ if $s \notin S \backslash E$. If $x > y$ I refer to the act $x_E y$ as *bet on* E. Consider a subject characterized by a preference relation $\succ$ on H. Given a bet $x_E y$ let $\bar{c}(x_E y; \succ) := \inf\{c \in \mathbb{R} \mid c \succ x_E y\}$

and $\underline{c}(x_E y;\succ) := \sup\{c \in \mathbb{R} \mid x_E y \succ c\}$, where c denotes the constant act that pays off \$$c$. Then the range of sure payoffs to which $x_E y$ is noncomparable is $[\underline{c}(x_E y;\succ), \bar{c}(x_E y;\succ)]$. Let $\Phi(\succ)$ be the set of probability-utility pairs in (14) that represent $\succ$. Given $(\pi, u) \in \Phi(\succ)$, define the certainty equivalent, $c((\pi, u); x_E y)$, of $x_E y$ corresponding to (π, u) by:

$$u(c((\pi, u); x_E y)) = \pi(E) u(x) + (1 - \pi(E)) u(y).$$

Then $\bar{c}(x_E y;\succ) = \sup_{(\pi,u)\in\Phi}\{c((\pi, u); x_E y)\}$ and $\underline{c}(x_E y;\succ) = \inf_{(\pi,u)\in\Phi}\{c((\pi, u); x_E y)\}$.

The experimental test of the probabilistic choice monotonicity hypothesis consists of two parts. In the first part the experimenter elicits the subjects degrees of incompleteness. In the second part the subjects are asked to choose among noncomparable bets. Specifically,

Part I - The elicitation of the degree of incompleteness:[13] Fix a bet $x_E y$ on E and $\theta > 0$. At time $t = 0$, the subject is asked to report numbers, $\underline{z}(x_E y;\theta), \bar{z}(x_E y;\theta) \in [y, x]$ such that $\underline{z}(x_E y;\theta) \leq \bar{z}(x_E y;\theta)$. Then a random number, z, is drawn from a uniform distribution on $[y, x]$. In the interim period, $t = 1$, the subject is awarded the bet $x_E y$ if $z < \underline{z}(x_E y;\theta)$ and the outcome z if $z \geq \bar{z}(x_E y;\theta)$. If $\underline{z}(x_E y;\theta) < \bar{z}(x_E y;\theta)$ and $z \in [\underline{z}(x_E y;\theta), \bar{z}(x_E y;\theta))$, then the subject is allowed to choose between the bet $(x-\theta)_E(y-\theta)$ and the outcome $z - \theta$. In the last period, $t = 2$, after it is verified whether or not the event E obtained, all payments are made. Denote this mechanism Λ.

Theorem (Karni and Vierø [2021]) *Given the mechanism* Λ, *there is* $\varepsilon > 0$ *such that, for all* $\theta \in [0, \varepsilon)$, *the subject's unique dominant strategy is to report* $\underline{z}(x_E y;\theta) = \underline{c}(x_E y;\succ)$ *and* $\bar{z}(x_E y;\theta) = \bar{c}(x_E y;\succ)$.

Given a bet $x_E y$, the set of constant acts that are noncomparable to $x_E y$ are: $c \in [\underline{z}(x_E y;\succ), \bar{z}(x_E y;\succ))$.

Part II - Observations of choice behavior- Two methods are possible for generating observations by which the model may be tested: Repeated choices by the same subject and single choices by a group of subjects. In both instances, a set $J = \{1, ..., n\}$ of subjects is recruited for the experiment and the aforementioned scheme is implemented to elicit the range of incompleteness at E, $\underline{z}_j(x_E y;\theta), \bar{z}_j(x_E y;\theta)$, $j \in J$.

In the repeated choices experiment each subject, $j \in J$, is presented with repeated choices between the bet $x_E y$ and a sure payoffs $z_i \in [\underline{z}_j(x_E y;\theta), \bar{z}_j(x_E y;\theta))$,

[13] This elicitation scheme was introduced in Karni and Vierø (2021).

$i = 1, ...m$. The prediction of the RCM is that the frequency of choosing the bet decreases monotonically with the values of z_i.[14]

In the single choice experiment the subjects are divided, randomly, into two groups $J_1 = \{1, ..., k\}$ and $J_2 = \{k+1, ..., n\}$. Subjects belonging to J_1 are asked to choose between the bet $x_E y$ and a sure outcome z_1 and subjects belonging to J_2 are asked to choose between the bet $x_E y$ and a sure outcome z_2, where $z_1, z_2 \in \cap_{j \in J}[\underline{z}_j(x_E y; \theta), \bar{z}_j(x_E y; \theta))$, $z_2 > z_1$. The prediction of the RCM, based on the assumption that the measures of incompleteness, $[\underline{z}_j(x_E y; \theta), \bar{z}_j(x_E y; \theta))$, are equally distributed in the two groups, is that the proportion of subjects from J_1 that choose the bet is smaller than that in J_2.

These predictions of the RCM are derive from the probabilistic choice monotonicity with respect to first-order stochastic dominance.

Knightian uncertainty: In the case of Knightian uncertainty, $\Phi := \{u\} \times \Pi$ in the representation (14) attributes the incompleteness of the preference relation entirely to the incompleteness of the subject's beliefs, represented by a set, Π. Since we are interested in the probabilities of the event E and its complement, Π is depicted by an interval, $[\underline{\pi}(E), \bar{\pi}(E)] \subseteq [0, 1]$, where $\pi \in [\underline{\pi}(E), \bar{\pi}(E)]$ denotes the subjective probability of the event E. In this case, Karni's (2020) modified proper scoring rule can be invoked for the elicitation of the range, $[\underline{\pi}(E), \bar{\pi}(E)]$, of the probabilities of an event E. According to Karni's scheme, at time $t = 0$ the subject is asked to report numbers, $\underline{r}(E, \theta), \bar{r}(E, \theta) \in [0, 1]$ with $\underline{r}(E, \theta) < \bar{r}(E, \theta)$. Then a random number, r, is drawn from a uniform distribution on $[0, 1]$. In the interim period, $t = 1$, the subject is awarded the bet $x_E y$ if $r < \underline{r}(E, \theta)$ and if $r \geq \bar{r}(E, \theta)$, the subject is awarded the right for a lottery $\ell(r; x, y)$ that pays the $\$x$ with probability r and $\$y$ with probability $(1 - r)$. If $r \in [\underline{r}(E, \theta), \bar{r}(E, \theta))$ the subject is allowed to choose between the bet $(x - \theta)_E(y - \theta)$ and the lottery $\ell(r; x - \theta, y - \theta)$, where $\theta > 0$. In the last period, $t = 2$, after it is verified whether or not the event E obtained and the outcome of the lottery is revealed, all payments are made. Denote this mechanism Λ_b.

Karni (2020) proves an elicitation theorem that implies the following result:

Theorem (Karni [2020]) *Given the mechanism* Λ_b, *there is* $\varepsilon > 0$ *such*

[14] This method is discussed in Loomes and Sugden (1998) and was implemented in a study by Loomes, Moffatt, and Robert Sugden (2002). To incetivize the subject to consider the choice seriously, one of the subject's choices is selected, at random, and the subject is rewarded according to the outcome of the selecte bet.

that, for all $\theta \in [0, \varepsilon)$, the subject's unique dominant strategy is to report $\underline{r}(E;\theta) = \underline{\pi}(E)$ and $\bar{r}(E;\theta) = \bar{\pi}(E)$.

The experiment consists of the elicitation of the range of incomplete beliefs $[\underline{\pi}(E), \bar{\pi}(E)]$. In the repeated choices experiment, the subject is presented, repeatedly, with choices between the bet $x_E y$ and lotteries $\ell(r_i; x, y)$, $r_i \in [\underline{r}_j(E;\theta), \bar{r}_j(E;\theta))$, $i = 1, ..., m$. The prediction of the RCM is that the frequency of choosing the bet decreases monotonically with the values of r_i.

In the single choice experiment the subject are divided, randomly. into two groups $J_1 = \{1, ..., k\}$ and $J_2 = \{k+1, ..., n\}$. Subjects belonging to J_1 are asked to choose between the bet $x_E y$ and the lottery $\ell(r_1; x, y)$ and subjects belonging to J_2 are asked to choose between the bet $x_E y$ and the lottery $\ell(r_2; x, y)$, where $r_1, r_2 \in \cap_{j \in J}[\underline{r}_j(E;,\theta), \bar{r}_j(E;\theta)]$ and $r_2 > r_1$. Assuming that the ranges of incompleteness are equally distributed in the two groups, the prediction of the RCM is that the proportion of subjects from J_1 that choose the bet is smaller than that in J_2.

Complete beliefs: In the case of complete beliefs, $\Phi = \mathcal{U} \times \{\pi\}$. Let $p = (x_1, p_1; ..., x_n, p_n) \in \Delta(X)$, where $0 < x_i < x_{i+1}$, $i = 1, ..., n-1$, and define $\bar{c}(p) = \inf\{c \in \mathbb{R} \mid \delta_c \succ p\}$ and $\underline{c}(p) = \sup\{c \in \mathbb{R} \mid p \succ \delta_c\}$, where $\delta_c \in \Delta(X)$ assigns c the unit probability mass. The elicitation scheme is designed to identify the the range $[\underline{c}(p), \bar{c}(p)]$ of sure outcomes that are noncomparable to p. It requires the subject to report, at time $t = 0$, numbers, $\underline{z}(p;\theta), \bar{z}(p;\theta) \in [x_1, x_n]$ such that $\underline{z}(p;\theta) \leq \bar{z}(p;\theta)$. A random number, z, is drawn from a uniform distribution on $[x_1, x_n]$. In the interim period, $t = 1$, the subject is awarded the lottery p if $z < \underline{z}(p;\theta)$ and the outcome z if $z \geq \bar{z}(p;\theta)$. If $z \in [\underline{z}(p;\theta), \bar{z}(p;\theta))$, then the subject is allowed to choose between the lottery $p' = (x_1 - \theta, p_1; ..., x_n - \theta, p_n)$ and the outcome $z - \theta$, where $\theta \in (0, x_1)$. In the last period, the outcome of the lottery is revealed, and all payments are made. Denote this mechanism Λ_t.

Theorem (Karni and Vierø [2021]). *Given Λ_t, there is $\varepsilon > 0$ such that, for all $\theta \in [0, \varepsilon)$, the subject's unique dominant strategy is to report $\underline{z}(p;\theta) = \underline{c}(p)$ and $\bar{z}(p;\theta) = \bar{c}(p)$.*

The experiment consists of the elicitation of the range of incomplete risk attitudes $[\underline{z}(p;\theta), \bar{z}(p;\theta)]$. In the repeated choices experiment the subject is presented, repeatedly, with choices between the lottery p and sure payoffs $c_i \in [\underline{z}(p;\theta), \bar{z}(p;\theta))$, $c_i = 1, ..., m$. The prediction of the RCM is that the frequency of choosing the lottery decreases monotonically with the values of c_i.

In the single choice experiment the subject are randomly divided into two groups $J_1 = \{1, ..., k\}$ and $J_2 = \{k+1, ..., n\}$. Subjects belonging to J_1 are asked to choose between the lottery p and an outcome c_1 and subjects belonging to J_2 are asked to choose between p and the outcome c_2, where $c_1, c_2 \in \cap_{j \in J}[\underline{z}_j(p;\theta), \bar{z}_j(p;\theta)]$ and $c_2 > c_1$. The prediction of the RCM is that the proportion of subjects from J_1 that choose the lottery is smaller than that in J_2.

4.2 Probabilistic choice hypotheses

The experiments described above are designed to test a qualitative property of the RCM, namely, probabilistic choice monotonicity that transcends the idiosyncratic variations of individual stochastic decoy processes. They are not designed to quantify the change in the probabilistic choice behavior in response to variations in the alternatives. Quantifying these responses requires knowledge of the probability measure λ. For example, hypothesizing that λ is the Labesgue measure on $\mathbb{R}^{|X| \times |S|}$ the RCM predicts that, facing a choice between $x_E y$ and a sure outcome, $z \in [\underline{z}(x_E y;\theta), \bar{z}(x_E y;\theta))$, the probability that a decision maker chooses $x_E y$ is $(z - \underline{z}(x_E y;\theta)) / (\bar{z}(x_E y;\succ) - \underline{z}(x_E y;\theta))$. Similarly, under Knightian uncertainty, hypothesizing that λ is the Labesgue measure on $[0,1]$, the RCM predicts that the probability that $x_E y$ is chosen is $(r - \underline{r}(E;\theta)) / (\bar{r}(E;\theta) - \underline{r}(E;\theta))$. Thus, the proposed experiments may be used to test alternative hypotheses regarding the probability measure λ.

Another possible interpretation of the probability measure λ in RCM model is that it quantifies the decision makers' subjective beliefs about the likely realizations of mental decoys that triggers their choices. To test this hypothesis, it is necessary to elicit λ jointly with the range of incompleteness that determines its support.[15] I describe below an experiment designed to test this hypothesis in the case of Knightian uncertainty. Again, the experiment consists of two parts. In the first part the experimenter elicits the probability measure λ on Π, dubbed *second-order beliefs*. In the second part, the experimenter presents the subjects with choices between bets and compares their responses to the predictions of the RCM parametrized by the information generated in the first part.

Part I - The elicitation of the range of second-order beliefs : I

[15] Karni and Safra (2016) characterized decision makers' perceptions underlying their subjective probability measure λ.

describe next an incentive compatible mechanism for the elicitation of the subject's probability measure of his priors regarding the likelihood of an event E. In the case of Knightian uncertainty, the probabilities depicting the subject's beliefs of the event E are represented by an interval, $[\underline{r}(E), r(E)] \subseteq [0,1]$. Denote by $\mathcal{B}_\Pi$ the Borel $\sigma-$ algebra on $[\underline{r}(E), r(E)]$.

In the first part of the experiment, the mechanism of Karni's (2020) is implemented to elicit of the subject's second-order beliefs, λ on of his first-order subjective probabilities of the event E. In the second part, to generate the observations, a bet $x_E y$ is fixed the subjects are offered a choice between the bet and a lottery $\ell(p) = (x, p; y, (1-p)$, $p \in [\underline{r}(E), r(E)]$. The prediction of the RCM is that the bet is selected with probability $\lambda[p - \underline{r}(E)]$.

Part II - The generate observations - Implementing the repeated individual choices method, each subject $j \in J$, is offered choices between the bet $x_E y$ and lotteries form the set, $L := \{\ell(p_i) := (x, p_i; y, (1-p_i) \mid p_i \in [\underline{r}_j(E), \bar{r}_j(E)], i = 1, ..., \tau\}$. These lotteries that are noncomparable to $x_E y$ according to subject j whose elicited second-order belief is λ_j. Denote by n_i, the number of times the subject is offered the choice between the bet and the lottery $\ell(p_i)$ and let $k(\ell(p_i))$ denote the number of times the bet is chosen.The hypothesis to be statistically tested is that $k(\ell(p_i))/n_i = \lambda_j[p_i - \underline{\pi}]$. The agreement between the prediction and the empirical observations is measured by the Euclidean metric $\|\lambda_j[p_i - \underline{\pi}]n_i - k(\ell(p_i))\|$.

The implementation of the single choice method requires that $\cap_{j\in J}[\underline{r}_j(E), r_j(E)] \neq \varnothing$. Fix $p \in \cap_{j\in J}[\underline{r}_j(E), r_j(E)]$ and present each subject with a single choice between $\ell(p)$ and $x_E y$. Let $J(k) \subseteq J$, $k = 1, ..., n$, denote the subsets on J that contain k subjects that chose the bet and, for each $t = 1, ..., n!/k!(n-k)!$, denote by $J(k_t)$ the different compositions of the subjects in $J(k)$. The model predicts that the expected number of subjects that choose the bet is: $\Sigma_{k=1}^n k\xi(k)$, where $\xi(k) = \Sigma_{t=1}^{n!/k!(n-k)!}\left(\Pi_{j\in J(k_t)}\lambda_j[p - \underline{r}_j(E)]\right) \times \left(\Pi_{j\in J\setminus J(k_t)}\lambda_j[\bar{r}(E) - p]\right)$. Let $\gamma(p)$ denoted the number of subjects that chose the bet. The hypothesis to be tested is that $\gamma(p) = \Sigma_{k=1}^n k\xi(k)$.

5 Concluding Remarks

I conclude by pointing out two equivalent formulations of the RCM and a brief review of the related literature.

5.1 Equivalent formulations of the random choice model

Two equivalent formulations of the RCM extends the set of alternatives that may play the role of mental decoy. As in the case of the RCM of Section 2, facing a choice between noncomparable alternatives, a and a', the decision maker receives a signal in the form of a third alternative, a''.

According to the first alternative formulation, the decision maker chooses the alternative a if the third alternative, a'', is weakly preferred to a' and is noncomparable to a. In that case,

$$\Psi'(a \gtrdot a') := \{a'' \in A \mid a'' \succcurlyeq a' \text{ and } a \bowtie a''\}. \tag{19}$$

The second alternative formulation combines the two possibilities. More explicitly, the decision maker chooses the alternative a if the third alternative is either weakly inferior to a and is noncomparable to a', or if it is weakly preferred to a' and is noncomparable to a. Hence, for any $a, a' \in A$,

$$\hat{\Psi}(a \gtrdot a') := \{a'' \in A \mid a \succcurlyeq a'' \text{ and } a' \bowtie a''\} \cup \{a'' \in A \mid a'' \succcurlyeq a' \text{ and } a \bowtie a''\}. \tag{20}$$

In either case, $p(a, a') = P\left(\hat{\Psi}(a \gtrdot a')\right) / P\left(\hat{\Psi}(a \gtrdot a') \cup \hat{\Psi}(a' \gtrdot a)\right)$.

The equivalence among the three formulations follows from Theorem 1 and the fact that in all of these cases, for all $a, a' \in A$, the probability that a is chosen from that set $\{a, a'\}$ is $\lambda(\mathcal{V}(a, a'))$.

The alternative formulations yield the same probabilistic choice behavior but have distinct implications insofar as the procrastination is concerned. In particular, the second alternative formulation admit larger set of decoy alternatives that resolve the indecision. Therefore, the probability of making a decision at each stage of the mental decoy process is larger (i.e., $P\left(\hat{\Psi}(a \gtrdot a') \cup \hat{\Psi}(a' \gtrdot a)\right) > P(\Psi(a \gtrdot a') \cup \Psi(a' \gtrdot a))$. Supposing that the time elapsed between consecutive draws of the mental decoys is the same, the expected delay due to procrastination is shorter according to this formulation.

5.2 Related literature

The recognition that, in many settings, observed choices display stochastic behavior lead to increase interest, in recent years, in modeling and testing stochastic choice behavior.[16] However, as was mentioned before, except of Ok and Tserenjigmidz (2020), these studies do not attribute this behavior specifically to preference incompleteness.

Ok and Tserenjigmidz (2020) model random choice behavior as random choice functions, which they define and characterize for stochastic choices induced by indifference, indecisiveness, and experimentation. The former two are closely related to the present work. In particular, as was shown in Section 2, the random choice function induced by RCM, is consistent with Ok and Tserenjigmidz's axiomatic characterization of stochastic choice functions induced by lack of strict preference. However, unlike Ok and Tserenjigmidz, the random choice function induced by RCM advances a testable hypothesis that quantify probabilistic choice behavior rather than merely asserting that the maximal elements of the menu will be chosen with positive probability.

Karni and Safra (2016) study stochastic choice under risk and under uncertainty based on the notion that decision makers' actual choices are governed by randomly selected states of mind. They provide axiomatic characterization of the representation of decision makers' perceptions of the stochastic process underlying the selection of their state of mind. In the context of decision making under uncertainty with incomplete preferences, the states of mind are probability-utility pairs in the set Φ and the stochastic choice process correspond the subjective measure of the sets $\Phi(f,g)$, $f,g \in H$.[17] Hence, their work can be regarded as providing axiomatic foundations of representation of the the states of mind by the probability measure, λ, and the hypothesis that the probability of choosing f out of the set $\{f,g\}$ is $\lambda(\Phi(f,g))$ as depicted by the RCM.

6 Proofs

6.1 Proof of Theorem 1

We need to show that $p(a,a') = \lambda(\mathcal{V}(a,a'))$ or, equivalently, that $P(\Psi(a \gtrdot a')) = P(\Gamma(a,a' \mid \mathcal{V}(a,a')))$, for all $a,a' \in A$. Given $a,a' \in A$ such that $a \bowtie a'$, define

$$\mathcal{V}(a \gtrdot a') = \{v \in \mathcal{V} \mid \exists a'' \in A \text{ s.t. } v(a) \geq v(a'') \text{ and } v(a') = v(a'')\}. \quad (21)$$

[16]See Luce (1959), Gul et. al (2014), Fudenberg, et. al (2015).

[17]In the special cases of multi-prior expected multi-utility, Knightian uncertainty and complete beliefs, the sets of states of mind are Φ, Π and $\mathcal{U}$, respectively.

Given $a, a' \in A$, define a mapping $\Upsilon : \mathcal{V}(a \gtrdot a') \rightarrow \mathcal{P}(a, a')$ by: $\Upsilon(v) = \Gamma(a, a' \mid v)$. By the uniqueness of $\mathcal{V}$, Υ is well-defined. Let $\Gamma(a, a' \mid v) \in \mathcal{P}(a, a')$ then, by definition, $v(a) \geq v(a'')$ and $v(a') = v(a'')$, for all $a'' \in I(a' \mid v)$. Hence, $I(a \mid v) \cap \Psi(a \gtrdot a') = \Upsilon^{-1}(v)$. Thus, Υ is a bijection.

Since $\mathcal{P}(a, a')$ is a partition of $\Psi(a \gtrdot a')$, we have:

$$\Upsilon(\mathcal{V}(a \gtrdot a')) := \cup_{v \in \mathcal{V}(a \gtrdot a')} \Upsilon(v) = \cup_{v \in \mathcal{V}(a \gtrdot a')} \Gamma(a, a' \mid v) := \Gamma(a, a' \mid \mathcal{V}(a \gtrdot a')) = \Psi(a \gtrdot a'). \tag{22}$$

Hence, $\Upsilon(\mathcal{V}(a \gtrdot a')) = \Psi(a \gtrdot a')$. Note that $\mathcal{V}(a \gtrdot a') \cup \mathcal{V}(a' \gtrdot a) = \mathcal{V}$. Moreover, by the representation (4), the bijection Υ is continuous. Since $\Psi(a \gtrdot a') \in \mathcal{B}_A$, we have that $\Upsilon^{-1}(\Psi(a \gtrdot a')) \in \mathcal{B}_{\mathcal{V}}$, $\forall a, a' \in A$.

I show next that $\mathcal{V}(a \gtrdot a') = \mathcal{V}(a, a')$. Let $v \in \mathcal{V}(a \gtrdot a')$. Then, by definition, there is $a'' \in A$ such that $v(a) \geq v(a'')$ and $v(a') = v(a'')$. Hence, $v(a) \geq v(a')$. Thus, $v \in \mathcal{V}(a, a')$. If $v \in \mathcal{V}(a, a')$ then $v(a) \geq v(a')$. For all $a'' \in I(a' \mid v)$, $v(a') = v(a'')$. Hence, $v(a) \geq v(a'')$. Consequently, $v \in \mathcal{V}(a \gtrdot a')$. Thus, $\mathcal{V}(a \gtrdot a') = \mathcal{V}(a, a')$.

Substituting $\mathcal{V}(a, a')$ for $\mathcal{V}(a \gtrdot a')$ in (22) we get that

$$\Gamma(a, a' \mid \mathcal{V}(a, a')) = \Gamma(a, a' \mid \mathcal{V}(a \gtrdot a')) = \Psi(a \gtrdot a'). \tag{23}$$

Thus,

$$\frac{P(\Gamma(a, a' \mid \mathcal{V}(a, a')))}{P(\Gamma(a, a' \mid \mathcal{V}(a, a')) \cup \Gamma(a', a \mid \mathcal{V}(a', a)))} = \frac{P(\Psi(a \gtrdot a'))}{P(\Psi(a \gtrdot a') \cup \Psi(a \gtrdot a'))}. \tag{24}$$

Since $\lambda(\mathcal{V}) = 1$, we have $\lambda(\mathcal{V}(a, a')) = \lambda(\mathcal{V}(a \gtrdot a')) / \lambda(\mathcal{V})$. Hence, by (24), for all $a, a' \in A$,

$$\lambda(\mathcal{V}(a, a')) = \frac{P(\Gamma(a, a' \mid \mathcal{V}(a, a')))}{P(\Gamma(a, a' \mid \mathcal{V}(a, a')) \cup \Gamma(a', a \mid \mathcal{V}(a', a)))} \tag{25}$$

if and only if

$$\lambda(\mathcal{V}(a, a')) = \frac{P(\Psi(a \gtrdot a'))}{P(\Psi(a \gtrdot a') \cup \Psi(a' \gtrdot a))} = p(a, a'). \tag{26}$$

Thus, $\trianglerighteq^{p(a,a')} = \trianglerighteq^{\lambda(\mathcal{V}(a,a'))}$ for all $a, a' \in A$ if and only if (7). ■

6.2 Proof of Theorem 2

By the proof of Theorem 1, $\Gamma(a,a' \mid \mathcal{V}(a,a')) = \Psi(a \gtrdot a')$. Thus, for all $(a,M) \in A \times \mathcal{M}$,

$$\Gamma(a,M \mid \mathcal{V}(a,M)) = \cap_{a' \in \max(M,\succ)} \Gamma(a,a' \mid \mathcal{V}(a,a')) = \cap_{a' \in \max(M,\succ)} \Psi(a \gtrdot a').$$

Moreover, for all $a, a' \in \max(M,\succ)$,

$$\lambda(\mathcal{V}(a,a')) = \frac{P(\Gamma(a,a' \mid \mathcal{V}(a,a')))}{P(\Gamma(a,a' \mid \mathcal{V}(a,a')) \cup \Gamma(a',a \mid \mathcal{V}(a',a)))}.$$

But $\mathcal{V}(a,M) = \cap_{a' \in \max(M,\succ)} \mathcal{V}(a,a')$. Hence,

$$\lambda(\mathcal{V}(a,M)) = \lambda\left(\cap_{a' \in \max(M,\succ)} \mathcal{V}(a,a')\right) =$$

$$\frac{P\left(\cap_{a' \in \max(M,\succ)} \Gamma(a,a' \mid \mathcal{V}(a,a'))\right)}{P\left(\cup_{a' \in \max(M,\succ)} [\cap_{a'' \in \max(M,\succ)} \Gamma(a',a'' \mid \mathcal{V}(a',a''))]\right)} = \frac{P(\Gamma(a,M \mid \mathcal{V}(a,M)))}{P\left(\cup_{a' \in \max(M,\succ)} \Gamma(a',M \mid \mathcal{V}(a',M))\right)}$$

if and only if

$$\lambda(\mathcal{V}(a,M)) = \frac{P\left(\cap_{a' \in \max(M,\succ)} \Psi(a \gtrdot a')\right)}{P\left(\cup_{a' \in \max(M,\succ)} \left[\cap_{a'' \in \max(M,\succ)} \Psi(a' \gtrdot a'')\right]\right)},$$

for all $(a,M) \in A \times \mathcal{M}$. But $\Psi(a \gtrdot M) = \cap_{a' \in \max(M,\succ)} \Psi(a \gtrdot a')$. Hence,

$$\lambda(\mathcal{V}(a,M)) = \frac{P(\Psi(a \gtrdot M))}{P(\cup_{a' \in M} \Psi(a' \gtrdot M))} = \rho(a,M)$$

for all $M \in \mathcal{M}$ and $a \in \max(M,\succ)$. Thus, $\trianglerighteq^{\rho(a,M)} = \trianglerighteq^{\lambda(\mathcal{V}(a,M))}$, for all $(a,M) \in A \times M$, if and only if (13). ■

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