Incomplete Preferences and Random Choice Behavior

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Abstract

This paper includes axiomatic characterizations of random choice behavior that is due to incomplete preferences. It proposes a model of irresolute choice and examines its applications to decision making under certainty, uncertainty, and risk.

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Keywords: Random choice, incomplete preferences, irresolute choice, Knightian uncertainty, multi-prior expected multi-utility representations, multi-utility representations.

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1 Introduction

Situations in which decision makers find feasible alternatives difficult, if not impossible, to compare and choose from are common. As von Neumann and Morgenstern, (1947) admitted, "It is conceivable – and even in a way more realistic – to allow for cases where the individual is neither able to state which of two alternatives he prefers nor that they are equally desirable." Depending on the context, this difficulty may be the result of the complexity of the alternatives or, for lack of experience, the inability to assess their, potentially long-run, consequences. A topical example is the decision whether or not to vaccinate against COVID-19, and whether, when and how often to accept a booster shot.

Leonard Savage broached the appropriateness of the postulate that all alternatives are readily comparable (i.e., that the preference relation is complete) in a letter to Karl Popper dated March 25, 1958, in which he discusses his work on the choice-based foundations of subjective probabilities. "There is, though," Savage wrote "a postulate that insists that economic situations can be ranked in a linear order by the subject, and I freely admit that this seems to me to be a source of much difficulty in my theory. This stringent postulate is in conflict with the common experience of vagueness and indecision, and if I knew a good way to make a mathematical model of those phenomena, I would adopt it, but I despair of finding one."[1]

Aumann (1962) questioned not only the descriptive validity of the completeness postulate but also its normative justification. "Of all the axioms of utility theory," he wrote, "the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from the normative viewpoint."

Danan and Ziegelmeyer (2006), Sautua (2017), and Cettolin and Riedl (2019) provide evidence of the prevalence of incomplete preferences in experimental settings. Yet with few exceptions, the theories of individual decision making – under certainty, risk, or uncertainty – presume that the preference relations depicting individual choice behavior are complete.

When the preference relations are complete, all alternatives are comparable and, in general, decision makers exhibit resolute choice behavior. By contrast, when the preference relations are incomplete, there are alternatives

[1] This correspondence is reproduced by Carlo Zappia (2020).

that are noncomparable, and, facing a choice between such alternatives, decision makers display indecisiveness (e.g., procrastination, hesitation, and irresolute choice). Bewley (2002) suggests that if among the noncomparable alternatives there is one that may be regarded as the status quo, or default, alternative, it is chosen.[2] Danan (2010) analyzes the implications of choice behavior that invokes deliberate randomization.[3] Evren et al. (2019) model choice behavior based on secondary criterion of the top cycle among all undominated alternatives in the feasible set relative to a complete and transitive binary relation.

I address the same issue, proposing a new model, dubbed *irresolute choice model* (henceforth ICM). Taking preference relations on choice sets as a primitive concept and departing form the completeness postulate, the model characterizes random choice behavior between noncomparable alternatives by a collection of nested partial orders each depicting different choice probabilities. The idea that stochastic choice is related to incomplete preferences may be traced to Luce (1959). However, the ICM is very different from, and may be regarded as an alternative to, Luce's model.[4]

The literature offers a variety of axiomatic models characterizing the representations of incomplete preferences under certainty (Ok [2002] Evren and Ok [2011]); risk (Shapley and Baucells [1998] and Dubra et al. [2004]); and uncertainty (Bewley [2002], Seidenfeld et al. [1995], Nau [2006], Ok et al. [2012], Galaabaatar and Karni [2013], and Riella [2015]). Unlike the case of complete preferences, in which representation characterizes choice behavior (i.e., the alternative that commands the highest representation value is chosen), in the case of incomplete preferences, the representations do not, in general, characterize the choice behavior. The main objective of this paper is to propose a model that connects the representations of incomplete preferences to choice behavior.

The underlying premise of this work is that when facing a choice among noncomparable alternatives, decision maker's actions are triggered by impulses, or signals, that are inherently random, or appear to be random to an observer who is not privy to the workings of the decision maker's mind. In either case, insofar as the observer is concerned, the decision maker's

[2] See also Masatlioglu and Ok (2005).

[3] See further discussion of this work in the concluding section.

[4] Further discussion of the relation between the model of this paper and Luce's model and its extensions will be easier to follow after the exposition of the ICM and is therefore discussed in the concluding section.

choices appear to be random. I propose a general framework within which representations of *probabilistic choice behavior* are obtained that depend on the context (i.e., whether the decision problem is under certainty, risk or uncertainty).

The main novelty of this work is the approach to modeling of random choice behavior, which is more conceptual than technical. The incompleteness of the preference relation is modeled as a continuum of strict partial orders on the relevant choice sets depicting the binary relations "one alternative is strictly preferred over another with probability that is at most $\alpha \in [0,1]$." These strict partial orders are linked by a monotonicity requirement. The results are characterizations of probabilistic choice representations.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 applies the model to decision making under certainty. Section 4 applies the model to subjective expected utility theory. Section 5 provides concluding remarks and a brief review of the relevant literature.

2 A Model of Random Choice

2.1 Preliminaries

Let A denote a *choice set.* Elements of A are *alternatives*. Denote by $\succ$ irreflexive and transitive binary relation on A, dubbed *strict preference relation*. For any alternatives $a, a' \in A$, $a \succ a'$ is the proposition that, facing a choice between these two alternatives, a decision maker characterized by $\succ$ chooses the alternative a. This behavior has the usual interpretation that a is *strictly preferred* over a'. I assume throughout that $\succ$ on A is nonempty.

The strict preference relation, $\succ$, induces the following derived binary relations on A. For all $a, a' \in A$,

(a) The *weak preference relation,* $\succcurlyeq$, is defined by: $a \succcurlyeq a'$ if, for all $a'' \in A$, $a'' \succ a$ implies that $a'' \succ a'$.[5]

(b) The *indifference relation*, $\sim$, is defined by $a \sim a'$ if $a \succcurlyeq a'$ and $a' \succcurlyeq a$.

(c) The *noncomparability relation* $\bowtie$, is defined by: $a \bowtie a'$ if $\neg (a \succcurlyeq a')$ and $\neg (a' \succcurlyeq a)$.

(d) The *negation of* $\succ$, denoted $\not\succ$, is defined by $a \not\succ a'$ if $\neg (a' \succ a)$.[6]

[5]Clearly, $a \succ a'$ implies that $a \succcurlyeq a'$.

[6]Note that $\not\succ$ is reflexive but not necessarily transitive. The weak preference relation defined here was introduced in Galaabaatar and Karni (2013). Its significance and impli-

It is natural to suppose that if presented with a choice between two alternatives, a and a', a decision maker would choose the former act if $a \succcurlyeq a'$ and $\neg (a' \succcurlyeq a)$. However, if $a \bowtie a'$ or $a' \sim a$, then the preference relation does not indicate which of the two alternatives will be chosen. Moreover, since $\succcurlyeq \supseteq \bowtie \cup \sim$, $a \succcurlyeq a'$ does not imply that a will be chosen form the subset $\{a, a'\}$.

2.2 Irresolute choice model

The basic premise of this work is that, facing a choice between noncomparable or indifferent alternatives, the decision maker behaves *as if* he is awaiting a signal that would resolve his indecision and, thereby, determine his choice. The signal is presumed to be generated by a stochastic process whose nature is not specified extraneously but has the following behavioral expression. Facing a choice between noncomparable or indifferent alternatives, the decision maker may procrastinate while waiting for a signal and then choose in a manner that reflects the underlying randomness of the signal-generating process.[7] Consequently, to the outside observer, the decision maker displays stochastic choice behavior.

To formalize this idea, I model *irresolute choice behavior* as a set $\{\succ^\alpha | \alpha \in [0,1]\}$ of binary relations on A, dubbed *probabilistic choice relations.* For each $\alpha \in [0,1]$, the derived relations $\succcurlyeq^\alpha$, $\sim^\alpha$, $\bowtie^\alpha$ and $\succcurlyeq^\alpha$ are defined follows: $a \succcurlyeq^\alpha a'$ if, for all $a'' \in A$, $a'' \succ^\alpha a$ implies that $a'' \succ^\alpha a'$; $a \sim^\alpha a'$ if $a \succcurlyeq^\alpha a'$ and $a' \succcurlyeq^\alpha a$; $a \bowtie^\alpha a'$ if and only if $\neg (a \succcurlyeq^\alpha a')$ and $\neg (a' \succcurlyeq^\alpha a)$; $a \succcurlyeq^\alpha a'$ if $\neg (a' \succ^\alpha a)$.

Given any $a, a' \in A$, the interpretation of $a \succ^\alpha a'$ is as follows: Facing a choice between the alternatives a and a', alternative a is strictly preferred and, hence, chosen, over a' with probability that is at least α. In other words, for all $\alpha' < \alpha$, $a \succ^\alpha a'$ implies that $a \succ^{\alpha'} a'$. Hence, $\succ^\alpha \subseteq \succ^{\alpha'}$. Moreover, if $a \succcurlyeq^\alpha a'$ then $a \succ^{\alpha'} a'$ for all $\alpha' < \alpha$. Given any $a, a' \in A$, let $\bar{\alpha}(a, a') :=$

cations were investigated and discussed in Karni (2011), who showed that the relations $\succcurlyeq$ and $\succcurlyeq$ agree if and only if $\succ$ is negatively transitive and $\succcurlyeq$ is complete. The relation $\succ$ is not the asymmetric part of $\succcurlyeq$. The indifference relation as is defined here, introduced in Galaabaatar and Karni (2013), is equivalent to that of Eliaz and Ok (2006).

[7]For example, the underlying process may have the structure of the drift-diffusion model, in which procrastination is measured by the response time. See, for example, Ian Krajbich et al. (2014) and Baldassi et al. (2020).

$\sup\{\alpha \in [0,1] \mid a \succ^{\alpha} a'\}$.[8] Then, if $a \sim a'$, which implies that $a \sim^{\alpha} a'$ for all $\alpha \in [0,1]$, $a \succcurlyeq^{\bar{\alpha}(a,a')} a'$ implies that $\bar{\alpha}(a,a')$ is the exact probability that a is chosen from the set $\{a, a'\}$, and a' is chosen with probability $1 - \bar{\alpha}(a,a')$. Clearly, $a \succ a'$ implies that $a \succcurlyeq^{1} a'$. Henceforth, to maintain consistency, I use the symbol $\succ^{1}$ instead of $\succ$ to denote the strict preference relation. Consistently with the interpretation of the probabilistic choice relations, $a \succcurlyeq^{1} a'$ implies that a is chosen from the set $\{a, a'\}$ with probability that is at least, and therefore equal to, one. If $a \sim a'$ then, insofar as the probability of a chosen over a' is concerned, the model is silent.

By definition, $a \succcurlyeq^{\alpha} a'$ if and only if $\neg (a' \succ^{\alpha} a)$. Hence, the irresolute choice behavior may be equivalently modeled as a set of binary relations $\{\succcurlyeq^{\alpha} \mid \alpha \in [0,1]\}$ on A. Note that $\neg (a' \succ^{\alpha} a)$ means the statement "a' is chosen over a with probability at least α" is false. Hence, by the preceding argument, $\alpha \leq 1 - \bar{\alpha}(a,a')$. Let $\bar{\alpha}(a',a) := \sup\{\alpha \in [0,1] \mid \neg (a' \succ^{\alpha} a)\}$ then $\bar{\alpha}(a',a) = 1 - \bar{\alpha}(a,a')$. The interpretation of $a \succcurlyeq^{\bar{\alpha}(a,a')} a'$ is as follows: Facing the choice between alternatives a and a' such that $\neg(a \sim a')$, a is chosen with probability $\bar{\alpha}(a,a')$. Hence, $a \succcurlyeq^{\bar{\alpha}(a,a')} a'$ if and only if $a \succcurlyeq^{\bar{\alpha}(a,a')} a'$ and the two statements of the model are in agreement. Note also that, since $\succ^{\alpha} \subseteq \succ^{\alpha'}$ for all $\alpha' < \alpha$, we have $\succcurlyeq^{\alpha} \subseteq \succcurlyeq^{\alpha'}$.

The proposed ICM is a refinement of decision models that admit incomplete preferences; as such, it is super-imposed on the models that axiomatize decision making under certainty, risk, or uncertainty with incomplete preferences. Therefore, to analyze the behavioral implications of the ICM in these contexts, I superimpose the structure of the ICM on the relevant decision models.

3 Irresolute Choice Behavior under Certainty

3.1 An axiomatic characterization

Let the choice set A be a nonempty topological space, and denote by $\succcurlyeq$ a preorder on A. For every $a \in A$, the upper and lower $\succcurlyeq$-contour sets of a are defined, respectively, by $\mathbb{U}_{\succcurlyeq}(a) = \{a' \in A \mid a' \succcurlyeq a\}$ and $\mathbb{L}_{\succcurlyeq}(a) = \{a' \in A \mid a \succcurlyeq a'\}$. The preorder $\succcurlyeq$ is *continuous* if $\mathbb{U}_{\succcurlyeq}(a)$ and $\mathbb{L}_{\succcurlyeq}(a)$ are closed, for all

[8] That the supremum exists follows from the fact that the set is bounded and that $\neg (a' \sim a)$ implies that there is $\alpha' \in [0,1]$ such that $a \succ^{\alpha'} a'$. Hence, the set is nonempty.

$a \in A$. A nonempty set $\mathcal{U}$ of real-valued functions on A is said to *represent* $\succcurlyeq$ if, for all $a, a' \in A$, $a \succcurlyeq a'$ if and only if $u(a) \geq u(a')$, for all $u \in \mathcal{U}$.

Let $\{\succ^{\alpha} | \ \alpha \in [0,1]\}$ be a set of probabilistic choice relations on A, and $\{\succcurlyeq^{\alpha} | \ \alpha \in [0,1]\}$ the corresponding model expressed in terms of the negations of $\succ^{\alpha}$. For each $\alpha \in [0,1]$ the structure of $\succcurlyeq^{\alpha}$ is depicted, axiomatically, as follows:

(P1) (**Partial preorder**) For each $\alpha \in [0,1]$ $\succcurlyeq^{\alpha}$ is transitive and reflexive.

(P2) (**Continuity**) For every $a \in A$ and $\alpha \in [0,1]$, $\mathbb{U}_{\succcurlyeq^{\alpha}}(a)$ and $\mathbb{L}_{\succcurlyeq^{\alpha}}(a)$ are closed.

The representation of irresolute choice behavior requires that the random choice relations in the set $\{\succcurlyeq^{\alpha} | \ \alpha \in [0,1]\}$ be linked. The next axiom provides this link.

(P3) (**Monotonicity**) For all $\alpha, \alpha' \in [0,1]$, $\succcurlyeq^{\alpha} \subseteq \succcurlyeq^{\alpha'}$ if and only if $\alpha' \leq \alpha$.

Lemma 1. The irresolute choice model $\{\succcurlyeq^{\alpha} | \ \alpha \in [0,1]\}$ satisfies monotonicity if and only if, for every $a \in A$, $\mathbb{U}_{\succcurlyeq^{\alpha}}(a) \subseteq \mathbb{U}_{\succcurlyeq^{\alpha'}}(a)$ if and only if $\alpha' \leq \alpha$.

Proof. Monotonicity is equivalent to the proposition, for all $a, a' \in A$, $a' \succcurlyeq^{\alpha} a$ implies that $a' \succcurlyeq^{\alpha'} a$ if and only if $\alpha' \leq \alpha$. The last statement is equivalent to the proposition, for all $a \in A$, $\mathbb{U}_{\succcurlyeq^{\alpha}}(a) \subseteq \mathbb{U}_{\succcurlyeq^{\alpha'}}(a)$ if and only if $\alpha' \leq \alpha$. ▲

The following theorem extends Evren and Ok (2011) Corollary 1, to include irresolute choice behavior.[9]

Theorem 1: *Let A be a locally compact separable metric space and $\{\succcurlyeq^{\alpha} | \ \alpha \in [0,1]\}$ be binary relations on A. Then, the following conditions are equivalent:*

(i) *For every $\alpha \in [0,1]$, $\succcurlyeq^{\alpha}$ satisfies (P1) and (P2) and jointly $\succcurlyeq^{\alpha}$, $\alpha \in [0,1]$, satisfy (P3).*

(ii) *There exists a collection $\{\mathcal{U}^{\alpha} \mid \alpha \in [0,1]\}$ of real-valued, continuous, strictly $\succcurlyeq^{\alpha}$ $-$increasing, functions such that, for every $\alpha \in [0,1]$, $\mathcal{U}^{\alpha}$ represents $\succcurlyeq^{\alpha}$, and $\alpha \geq \alpha'$ if and only if $\mathcal{U}^{\alpha} \supseteq \mathcal{U}^{\alpha'}$.*

Proof. (Sufficiency) Suppose that A is a locally compact separable metric space and $\{\succcurlyeq^{\alpha} | \ \alpha \in [0,1]\}$ be binary relations on A satisfying (P1) and (P2)

[9]Other results of Evren and Ok (2011), including their Theorem 1 and Corollaries 2 and 3, may be extended in the same way.

then, by Evren and Ok (2011) Corollary 1, for each $\alpha \in [0,1]$, there exists a set $\mathcal{U}^{\alpha}$ of real-valued, continuous, functions representing $\succcurlyeq^{\alpha}$and every $u \in \mathcal{U}^{\alpha}$ is strictly $\succcurlyeq^{\alpha}$ $-$increasing. Let $\mathcal{U}^{\alpha}$ be the set of all (continuous) real functions u such that $a \succcurlyeq^{\alpha} a'$ implies $u(a) \geq u(a')$ and $\mathcal{U}^{\alpha'}$ be the set of all continuous real functions u such that $a \succcurlyeq^{\alpha'} a'$ implies $u(a) \geq u(a')$. Then $\succcurlyeq^{\alpha} \subseteq \succcurlyeq^{\alpha'}$ if and only if $u \in \mathcal{U}^{\alpha'}$ then $u \in \mathcal{U}^{\alpha}$. Thus, $\mathcal{U}^{\alpha'} \subseteq \mathcal{U}^{\alpha}$. By the representation, $\mathbb{U}_{\succcurlyeq^{\alpha'}}(a) \supseteq \mathbb{U}_{\succcurlyeq^{\alpha}}(a)$ if and only if $\mathcal{U}^{\alpha} \supseteq \mathcal{U}^{\alpha'}$. By (P3) and Lemma 1, $\alpha \geq \alpha'$ if and only if $\mathbb{U}_{\succcurlyeq^{\alpha'}}(a) \supseteq \mathbb{U}_{\succcurlyeq^{\alpha}}(a)$. Hence, $\alpha \geq \alpha'$ if and only if $\mathcal{U}^{\alpha} \supseteq \mathcal{U}^{\alpha'}$.

(Necessity) Assume that (ii) holds. Corollary 1 of Evren and Ok (2011) implies that, for every $\alpha \in [0,1]$, $\succcurlyeq^{\alpha}$satisfies (P1) and (P2*).* Suppose that $\alpha' \leq \alpha$ if and only if $\mathcal{U}^{\alpha} \supseteq \mathcal{U}^{\alpha'}$. By the representation , $\mathcal{U}^{\alpha} \supseteq \mathcal{U}^{\alpha'}$ if and only if $\mathbb{U}_{\succcurlyeq^{\alpha'}}(a) \supseteq \mathbb{U}_{\succcurlyeq^{\alpha}}(a)$. Hence, $\alpha' \leq \alpha$ if and only if $\mathbb{U}_{\succcurlyeq^{\alpha'}}(a) \supseteq \mathbb{U}_{\succcurlyeq^{\alpha}}(a)$, for all $a \in A$, which, by Lemma 1 is equivalent to (P3). ■

The uniqueness of the representation is as follows: Given any nonempty subset $\mathcal{U}^{\alpha}$ of $\mathbb{R}^{A}$, define the map $\Upsilon_{\mathcal{U}^{\alpha}} : A \to \mathbb{R}^{\mathcal{U}^{\alpha}}$ by $\Upsilon_{\mathcal{U}^{\alpha}}(a)(u) := u(a)$. Two nonempty subsets $\mathcal{U}^{\alpha}$ and $\mathcal{V}^{\alpha}$ of continuous real-valued functions on A represent the same preorder if, and only if, there exists an $f : \Upsilon_{\mathcal{U}^{\alpha}}(A) \to \Upsilon_{\mathcal{V}^{\alpha}}$ such that (i) $\Upsilon_{\mathcal{V}^{\alpha}} = f(\Upsilon_{\mathcal{U}^{\alpha}})$; and (ii) for every $b, c \in \Upsilon_{\mathcal{U}^{\alpha}}(A)$, $b > c$ if and only if $f(b) > f(c)$.[10]

The property $\alpha' > \alpha$ if and only if $\mathcal{U}^{\alpha'} \supset \mathcal{U}^{\alpha}$ is dubbed *nestedness*.

3.2 The indifference relation

The case in which the alternatives under consideration belong to the same indifference class requires special attention. By definition, $a \sim^{1} a'$ if and only if $a \succcurlyeq^{1} a'$ and $a' \succcurlyeq^{1} a$.

Lemma 2: For all $a, a' \in A$, $a \succcurlyeq^{1} a'$ if and only if $u(a) \geq u(a')$, for all $u \in \mathcal{U}^{1}$.

Proof. By definition $a \succcurlyeq^{1} a'$ if $\hat{a} \succ^{1} a$ then $\hat{a} \succ^{1} a'$, for all $\hat{a} \in A$. Hence, by definition, $a' \succcurlyeq^{1} a''$ implies that $a \succcurlyeq^{1} a''$. By Theorem 1, this is equivalent to $u(a') \geq u(a'')$ implying that $u(a) \geq u(a'')$, for all $u \in \mathcal{U}^{1}$. Consider a sequence $(a''_n) \subset A$ such that $a' \succcurlyeq^{1} a''_n$ for $n = 1, 2, ...$ and $a' = \lim_{n\to\infty} a''_n$. This is equivalent to $u(a') \geq u(a''_n)$ for $n = 1, 2, ...$ and, by the continuity of u, $u(a') = \lim_{n\to\infty} u(a''_n)$, for all $u \in \mathcal{U}^{1}$. Moreover, $a \succcurlyeq^{1} a''_n$, $n = 1, 2, ..$, which is

[10]See Evren and Ok (2011). Note that, in general, for arbitrary multi-utility representations, $\mathcal{V}^{\alpha}$ and $\mathcal{V}^{\alpha'}$, of two preorders, $\succcurlyeq^{\alpha}$ and $\succcurlyeq^{\alpha'}$, such that $\succcurlyeq^{\alpha} \subset \succcurlyeq^{\alpha'}$ does not imply that $\mathcal{V}^{\alpha} \subset \mathcal{V}^{\alpha'}$.

equivalent to $u(a) \geq u(a''_n)$, $n = 1, 2, ...$ and $u(a) \geq \lim_{n\to\infty} u(a''_n) = u(a')$, for all $u \in \mathcal{U}^1$. Hence, by Theorem 1, $a \succcurlyeq^1 a'$ if and only if $u(a) \geq u(a')$, for all $u \in \mathcal{U}^1$. ▲

By definition of $\sim^1$ and Lemma 2, $a \sim^1 a'$ if and only if $u(a) = u(a')$, for all $u \in \mathcal{U}^1$. By Theorem 1, $\mathcal{U}^\alpha \subseteq \mathcal{U}^1$ for all $\alpha \in [0,1]$. Thus, $a \sim^1 a'$ implies that $a \sim^\alpha a'$, $a \sim^{\alpha'} a'$, for all $\alpha, \alpha' \in [0,1]$. Consequently, the irresolute choice model is silent with regard to the probability of selection of any alternatives belonging to the same indifference class.

3.3 Canonical signal space and random choice

The premise underlying the stochastic choice behavior depicted by the ICM is that choices between noncomparable or indifferent alternatives are governed by unspecified of random signals-generating process. Consider the choice between two alternatives, a and a' such that $\neg(a \sim a')$, then the probability of a signal that would resolve the indecision in favor of a is $\bar{\alpha}(a, a')$. Let $p(a, \{a, a'\})$ denote the probability of choosing the alternative a from the set $\{a, a'\}$. Then, $p(a, \{a, a'\}) = \bar{\alpha}(a, a')$, for all $a, a' \in A$. By the representation of the ICM this is the case if and only if $u(a) \geq u(a')$, for all $u \in \mathcal{U}^{\bar{\alpha}(a,a')}$.

Given an ICM $\{\succcurlyeq^\alpha | \ \alpha \in [0,1]\}$, define a function $F : 2^{\mathcal{U}} \backslash \varnothing \to [0,1]$ as follows: For $\alpha \in [0,1]$, $F(\mathcal{U}^\alpha) = \alpha$. By definition, $F(\mathcal{U}^0) = 0$, $F(\mathcal{U}^1) = 1$, and $F(\mathcal{U}^\alpha) \geq F\left(\mathcal{U}^{\alpha'}\right)$, for all $\alpha \geq a'$. Then, for all $a, a' \in A$, $p(a, \{a, a'\}) = F\left(\mathcal{U}^{\bar{\alpha}(a,a')}\right)$. In other words, if $\neg(a \sim a')$, the decision maker behaves *as if* a function u is selected from $\mathcal{U}^1$ according to a probability distribution F and a is chosen if $u \in \mathcal{U}^{\bar{\alpha}(a,a')}$ and a' is chosen if $u \in \mathcal{U}^1 \backslash \mathcal{U}^{\bar{\alpha}(a,a')}$. Therefore, the set $\mathcal{U}^1$ may be taken to be the *canonical signal space*.

It is worth underscoring that if $\bar{\alpha}(a', a) = 0$ then there is no $u \in \mathcal{U}^0$ such that $u(a') > u(a)$. To grasp this, consider two alternatives, $a, a' \in A$ such that $\bar{\alpha}(a', a) = 0$. Since $\bar{\alpha}(a', a) = 0$ if and only if $\bar{\alpha}(a, a') = 1$, by Theorem 1, $u(a) \geq u(a')$, for all $u \in \mathcal{U}^1$. But $\mathcal{U}^0 \subseteq \mathcal{U}^1$ implies that for no $u \in \mathcal{U}^0$ it holds that $u(a') > u(a)$.

3.4 Probabilistic choice

Many decision problems require the decision maker to choose an alternative from a finite set of feasible alternatives that include more than two elements. To see how the ICM may be applied to choice from such sets, consider the following adaptation of the model.

Let $M \subset A$ be a feasible set of alternatives and, to simplify the exposition, suppose that no two alternatives in M belong to the same indifference class. An alternative $a \in M$ is said to be *dominated* if for no $\alpha \in [0,1]$ it holds that $a \succcurlyeq^{\alpha} a'$, $\forall a' \in M \backslash \{a\}$. Let $D(M)$ denote the subset of dominated alternatives in M and let $UD(M) = M \backslash D(M)$ denote the subset of *undominated* alternatives in M.[11] Note that $UD(M)$ is nonempty.

Let $UD(M) = \{a_1, ..., a_m\}$. For each $a_i \in M$ define $\Lambda_i(M) = \{\alpha \in [0,1] \mid a_i \succcurlyeq^{\alpha} a_j, \forall a_j \in M \backslash \{a_i\}\}$. In words, $\Lambda_i(M)$ is the set of indices designating the random choice relations that rank the alternative a_i (weakly) higher than any other alternative in the menu M. Define $\underline{\alpha}(a_i; M) = \inf \Lambda_i(M)$ and $\bar{\alpha}(a_i; M) = \sup \Lambda_i(M)$.[12] By definition, $\underline{\alpha}(a_i; M)$ and $\bar{\alpha}(a_i; M)$ are the indices of the probabilistic choice relations such that $\succcurlyeq^{\bar{\alpha}(a_i;M)} \subseteq \succcurlyeq^{\alpha} \subseteq \succcurlyeq^{\underline{\alpha}(a_i;M)}$, for all $\alpha \in \Lambda_i(M)$. Without loss of generality assume that the elements of $UD(M)$ are rearranged in a ascending order of set inclusion (i.e., $\succcurlyeq^{\underline{\alpha}(a_1;M)} \subseteq \succcurlyeq^{\underline{\alpha}(a_2;M)} \subseteq, ..., \subseteq \succcurlyeq^{\underline{\alpha}(a_m;M)}$). If $\underline{\alpha}(a_1; M) = 1$ then, by Theorem 1, a_1 is the only element of the undominated set and $D(M) = \{a_2, a_3, ..., a_m\}$. In general, we have $1 > \underline{\alpha}(a_1; M) > \underline{\alpha}(a_2; M) >, ..., > \underline{\alpha}(a_{m-1}; M) > \underline{\alpha}(a_m; M) = 0$.[13]

Define $J_1 = [1, \underline{\alpha}(a_1; M)]$ and $J_i = (\underline{\alpha}(a_i; M) - \underline{\alpha}(a_{i+1}; M)]$, $i = 1, ..., m-1$. Then, $\mathcal{J} := \{J_1, ..., J_{m-1}\}$ is a partition of the unit interval. Corresponding to $\mathcal{J}$ define a partition of $\mathcal{U}^1$ as follows: Let $P_1(M) := \{u \in \mathcal{U}^1 \mid u \in \mathcal{U}^{\underline{\alpha}(a_1;M)}\}$, $P_i(M) := \{u \in \mathcal{U}^1 \mid u \in \mathcal{U}^{\underline{\alpha}(a_{i+1};M)} / \mathcal{U}^{\underline{\alpha}(a_i;M)}\}$, $i = m-1, ..., 2$, and $P_m(M) := \{u \in \mathcal{U}^1 \mid u \in \mathcal{U}^1 \backslash \mathcal{U}^{\underline{\alpha}(a_{m-1};M)}\}$.[14] Then, $\alpha \in \Lambda_i(M)$ if and only if, for all $u \in P_i$, $u(a_i) \geq u(a_j)$, for all $\forall a_j \in M \backslash \{a_i\}$.

A *stochastic choice function* is a function p that, for every nonempty subset M of A, return a probability distribution $p(M)$ over M. The probability of choosing a from the set M is denoted $p(a, M)$. The stochastic choice function is said to be *induced by the ICM* if there is a function $c : 2^A \backslash \varnothing \rightarrow 2^A \backslash \varnothing$ given by $c(M) = UD(M)$ and

$$p(a_i, M) = \begin{bmatrix} \bar{\alpha}(a_i; M) - \bar{\alpha}(a_{i+1}; M) & \text{if } a_i \in UD(M) \\ 0 & \text{if } a_i \notin UD(M) \end{bmatrix}.$$

[11] Formally, an alternative, $a \in M$ is undominated if, for some $\alpha \in [0,1]$, $a \succcurlyeq^{\alpha} a'$, for all $a' \in M \backslash \{a\}$.

[12] That the infimum and supremum exist follows from the facts that the set $\Lambda_i(M)$ is bounded and, because a_i is undominated, $\Lambda_i(M)$ nonempty.

[13] By definition, $\bar{\alpha}(a_1; M) = 1$, and $\bar{\alpha}(a_i; M) = \underline{\alpha}(a_{i-1}; M)$, for all $i = 2, ..., m$.

[14] Since indifference is not allowed, there is no ambiguity with regard to which element of the partition each utility function belongs to.

for all $M \subseteq A$.

Since $\mathcal{U}^1$ is the canonical signal space, the probability of receiving a signal $u \in P_i$ is:

$$F\left(\mathcal{U}^{\bar{\alpha}(a_i;M)}\right) - F\left(\mathcal{U}^{\bar{\alpha}(a_{i+1};M)}\right) = \bar{\alpha}\left(a_i; M\right) - \bar{\alpha}\left(a_{i+1}; M\right), \ i = 1, ..., m.$$

Hence,

$$p\left(a_i, M\right) = \begin{bmatrix} F\left(\mathcal{U}^{\bar{\alpha}(a_i;M)}\right) - F\left(\mathcal{U}^{\bar{\alpha}(a_{i+1};M)}\right) & \text{if } a_i \in UD\left(M\right) \\ 0 & \text{if } a_i \notin UD\left(M\right) \end{bmatrix}$$

Thus, when facing a choice form M, the decision maker behaves *as if* a utility function $u \in \mathcal{U}^1$ is selected according to the distribution F and the *undominated alternative*, a_i is chosen if $u \in \mathcal{U}^{\underline{\alpha}(a_{i+1};M)} / \mathcal{U}^{\underline{\alpha}(a_i;M)}$, $i = 1, ..., m-1$.

4 Irresolute Choice Behavior under Uncertainty

4.1 The analytical framework

For over half a century, subjective expected utility theory has been the dominant model of decision making under uncertainty. Because of its prominent role and rich analytical framework, I explore the application of the ICM to subjective expected utility theory, invoking the model of Galaabaatar and Karni (2013). This model admits incomplete beliefs and tastes. It includes Bewley's Knigthian uncertainty model (i.e., complete tastes and incomplete beliefs) and the subjective expected multi-utility model (i.e., complete beliefs and incomplete tastes) as special cases.

The analytical framework is that of Anscombe and Aumann (1963). Let S be a finite set of *states.* Subsets of S are *events.* Let X be a finite set of *outcomes* and denote by $\Delta(X)$ the set of all probability distributions on X. For each $q, q' \in \Delta(X)$ and $\gamma \in [0,1]$ define $\gamma q + (1-\gamma) q' \in \Delta(X)$ by $(\gamma q + (1-\gamma) q')(x) = \gamma q(x) + (1-\gamma) q'(x)$, for all $x \in X$.

The choice set is $H := \Delta(X)^S$ (i.e., the set of mapping from S to $\Delta(X)$). The elements of H are *acts*. For all $h, h' \in H$ and $\gamma \in [0,1]$, define $\gamma h + (1-\gamma) h' \in H$ by $(\gamma h + (1-\gamma) h')(s) = \gamma h(s) + (1-\gamma) h'(s)$. Under this definition H is a convex subset of the linear space $\mathbb{R}_+^{|X| \times |S|}$. A constant act, $h \in$

H, is an act such that $h(s) = q$ for all $s \in S$, where $q \in \Delta(X)$. Henceforth, I identify the subset of constant acts with $\Delta(X)$. Hence, $\Delta(X) \subset H$.

Let $\{\succ^\alpha \mid \alpha \in [0,1]\}$ be random choice relations on H depicting irresolute choice behavior. The choice set H is said to be *bounded* if there exist $\bar{h}$ and $\underline{h}$ in H such that $\bar{h} \succ^1 h \succ^1 \underline{h}$, for all $h \in H - \{\bar{h}, \underline{h}\}$.

For each $\alpha \in [0,1]$, let $\mathcal{U}^\alpha$ be a nonempty closed set off real-valued functions on X and, for every $u \in \mathcal{U}^\alpha$, let $\Pi^\alpha(u)$ be a nonempty closed set of probability measures on S. Define $\Phi^\alpha = \{(\pi, u) \mid u \in \mathcal{U}^\alpha,\ \pi \in \Pi^\alpha(u)\}$. Then $\{\Phi^\alpha \mid \alpha \in [0,1]\}$ is said to *represent* the ICM $\{\succ^\alpha \mid \alpha \in [0,1]\}$ if the following conditions hold:

(a) For all $h \in H$ and $(\pi, u) \in \Phi^1$,

$$\sum_{s\in S} \pi(s) \sum_{x\in X} \bar{h}(x,s)u(x) > \sum_{s\in S} \pi(s) \sum_{x\in X} h(x,s)u(x) > \sum_{s\in S} \pi(s) \sum_{x\in X} \underline{h}(x,s)u(x), \tag{1}$$

(b) For all $h, h' \in H$,

$$h \succ^\alpha h' \Leftrightarrow \sum_{s\in S} \pi(s) \sum_{x\in X} h(x,s)u(x) > \sum_{s\in S} \pi(s) \sum_{x\in X} h'(x,s)u(x),\ \forall (\pi, u) \in \Phi^\alpha. \tag{2}$$

4.2 Axiomatic characterization

Following Galaabaatar and Karni (2013), I assume that the random choice relations $\succ^\alpha$, $\alpha \in [0,1]$ have a structure depicted by the following axioms. The first three axioms are well known and require no elaboration.

(A.1) (**Strict partial order)** For every $\alpha \in [0,1]$, the $\succ^\alpha$ is transitive and irreflexive.

(A.2) (**Archimedean)** For all $f, g, h \in H$, if $f \succ^\alpha g$ and $g \succ^\alpha h$ then there exist $\beta, \gamma \in (0,1)$ such that $\beta f + (1-\beta) h \succ^\alpha g$ and $g \succ^\alpha \gamma f + (1-\gamma) h$.

(A.3) (**Independence**) For all $f, g, h \in H$ and $\alpha \in (0,1]$, $f \succ^\alpha g$ if and only if $\alpha f + (1-\alpha) h \succ^\alpha \alpha g + (1-\alpha) h$.

For each $h \in H$ and every $s \in S$, denote by h^s the constant act that pays off $h(s)$ in every state. The next axiom asserts that if every possible consequence of h, taken as a constant act, is an element of the lower contour set of g according to irresolute choice relation $\succ^{\alpha}$, then the convexity of the lower contour sets implies that any convex combination of the consequences of h is dominated by g. Think of h as representing a subset of the simplex in $\mathbb{R}^{|S|}$ whose elements correspond to subjective probabilities on S that the decision maker may entertain. Since any such combination is $\succ^{\alpha}$ −dominated by g, so is h. Formally,

(A.4) (**Dominance**) For all $h, g \in H$, and $\alpha \in [0,1]$, if $g \succ^{\alpha} h^{s}$ for every $s \in S$, then $g \succ^{\alpha} h$.

The next axiom restates (P3) in terms of the present model.

(A.5) (**Monotonicity**) For all $\alpha, \alpha' \in [0,1]$, $\succ^{\alpha} \subseteq \succ^{\alpha'}$ if and only if $\alpha' \leq \alpha$.

The following theorem characterizes irresolute choice behavior:

Theorem 2: *Let* $\{\succ^{\alpha} | \ \alpha \in [0,1]\}$ *be a set of binary relations on* H. *Then the following conditions are equivalent:*

(*i*) *H is* $\succ^{\alpha}$*-bounded and for each* $\alpha \in [0,1]$, $\succ^{\alpha}$ *satisfies (A.1)–(A.4) and jointly* $\succ^{\alpha}$, $\alpha \in [0,1]$, *satisfy (A.5).*

(*ii*) *For each* $\alpha \in [0,1]$ $\succ^{\alpha}$ *is represented by (1) and (2) and* $\alpha \geq \alpha'$ *if and only if* $\Phi^{\alpha} \supseteq \Phi^{\alpha'}$.

The proof that $\succ^{\alpha}$ satisfies (A.1)–(A.4) if and only if $\succ^{\alpha}$ is represented by (1) and (2) is an immediate implications of Theorem 1 of Galaabaatar and Karni (2013). The proof that (A.5) holds if and only if $\alpha \geq \alpha'$ if and only if $\Phi^{\alpha} \supseteq \Phi^{\alpha'}$ is by the same argument as in Theorem 1 above. The uniqueness of the representation is described in Galaabaatar and Karni (2013) and is not replicated here.

4.3 Special cases

The theory of subjective expected utility with incomplete preferences includes two special cases: the case in which the incompleteness is due solely to incomplete beliefs and the case in which it is due solely to incomplete tastes.

The case of incomplete beliefs was axiomatized by Bewley (2002), who dubbed it "Knightian uncertainty." Tastes completeness, or unambiguous risk attitudes, requires that the restriction of the preference relation to constant acts exhibits negative transitivity. Let $p \in \Delta(X)$ denote the constant act that pays off p in every state. Then tastes completeness is captured by the following:

(A.6) (**Unambiguous risk attitudes**) For all constant acts $p, q, r \in \Delta(X)$, $\neg(p \succ^1 q)$ and $\neg(q \succ^1 r)$ imply $\neg(p \succ^1 r)$.

The corollary below is implied by Theorem 2.

Corollary 1: *Let* $\{\succ^\alpha | \alpha \in [0,1]\}$ *be a set of binary relations on* H. *Then* H *is bounded and, for each* $\alpha \in [0,1]$, $\succ^\alpha$*satisfies (A.1)–(A.4), jointly* $\succ^\alpha$, $\alpha \in [0,1]$, *satisfy (A.5), and* $\succ^1$*satisfies (A.6) if and only if* $\succ^\alpha$*is represented by (1) and (2) with* $\Theta^\alpha = \{u\} \times \Pi^\alpha$ *and* $\alpha \geq \alpha'$ *if and only if* $\Pi^\alpha \supseteq \Pi^{\alpha'}$. *Moreover,* u *is unique up to positive affine transformation, the closed convex hull of* Π^α *is unique and, for each* $\pi \in \Pi^\alpha$, $\pi(s) > 0$ *for all* $s \in S$.

Consider next the case of complete beliefs and ambiguous risk attitudes. For each event E, denote by rEq the act whose payoff is r for all $s \in E$ and q for all $s \in S - E$. Denote by $r\gamma q \in \Delta(X)$ the constant act whose payoff in every state is $\gamma r + (1-\gamma) q$. A *bet* on an event E is the act rEq, whose payoffs satisfy $r \succ^1 q$, where $r, q \in \Delta(X)$.[15]

Suppose that the decision maker considers the constant act $r\gamma q$ preferable to the bet rEq. Because the payoffs are the same, this preference indicates that he believes that γ exceeds the likelihood of E. This belief is said to be *coherent* if it holds that $r'\gamma q'$ is preferable to the bet $r'Eq'$ for all constant acts r' and q' such that $r' \succ^1 q'$. By the same logic a preference of a bet rEq over the constant act $r\gamma q$ means that the decision maker believes the probability of E to exceed γ. A binary relation $\succ^1$ on H is said to exhibit *coherent beliefs* if, for all events E and $r, q, r', q' \in \Delta(X)$ such that $r \succ^1 q$ and $r' \succ^1 q'$, $r\gamma q \succ^1 rEq$ if and only if $r'\alpha q' \succ^1 r'Eq'$, and $rEq \succ^1 r\gamma q$ if and only if $r'Eq' \succ^1 r'\gamma q'$. Note that the structure of a binary relation $\succ^1$ depicted by

(A.1)–(A.4) implies that the decision maker's beliefs are coherent.

The idea of complete beliefs is captured by the following axiom, which is due to Galaabaatar and Karni (2013).

(A.7) (**Complete beliefs**) For all events E and $\gamma \in [0,1]$, and constant acts r and q such that $r \succ^1 q$, either $r\gamma q \succ^1 rEq$ or $rEq \succ^1 r\gamma' q$, for every $\gamma > \gamma'$.

If the decision maker's beliefs are complete, then the incompleteness of the random choice relations $\succ^\alpha$, $\alpha \in [0,1]$, on H is due entirely to the

[15] By monotonicity, $r \succ^1 q$ implies that $r \succ^\alpha q$, for all $\alpha \in [0,1]$.

incompleteness of his tastes. To state the next result I introduce the following additional notations. Let $\langle \mathcal{U}^\alpha \rangle := cl\{con\,(\mathcal{U}^\alpha) + \{\theta \mathbf{1}_X\}_{\theta \in \mathbb{R}}$ (i.e., $\langle \mathcal{U}^\alpha \rangle$ denotes the closure, with respect to the sup-norm topology, of the cone generated by $\mathcal{U}^\alpha$ and the constant real-valued functions on X). The next Corollary is an implication of Theorem 2.

Corollary 2: *Let $\{\succ^\alpha \mid \alpha \in [0,1]\}$ be binary relations on H. Then H is bounded and, for each $\alpha \in [0,1]$, $\succ^\alpha$ satisfies (A.1)–(A.4), jointly $\succ^\alpha$, $\alpha \in [0,1]$, satisfy (A.5) and $\succ^1$ satisfies (A.7), if and only if $\succ^\alpha$ is represented by (1) and (2) with $\Theta^\alpha = \mathcal{U}^\alpha \times \{\pi\}$ and $\alpha \geq \alpha'$ if and only if $\mathcal{U}^\alpha \supseteq \mathcal{U}^{\alpha'}$. Moreover, the probability measure, π, is unique and $\pi(s) > 0$, for all $s \in S$, and if $\mathcal{V}^\alpha$ is another set of real-valued functions on X that represent $\succ^\alpha$ in the sense of (2) then $\langle \mathcal{V}^\alpha \rangle = \langle \mathcal{U}^\alpha \rangle$.*

5 Behavioral Implications

Any theory that purports to describe natural or social phenomena must have clear testable predictions and implications. To render the proposed ICM meaningful, I describe briefly some of its behavioral implications in the context of a simple portfolio problem. I also describe experiments designed to test qualitative and quantitative properties of the model, pointing out the kind of observations that would contradict the model.

5.1 A simple portfolio problem

Let there be two financial assets: a risk-free asset, whose rate of return is zero, and a risky asset whose rates of return are r_1 or r_2, in the states s_1 and s_2, respectively, where $r_1 > 0 > r_2$. Consider a risk-averse decision maker displaying Knigthian uncertainty, and let the set of his subjective probabilities of state s_1 be $[\underline{\pi}, \bar{\pi}]$. Suppose that the decision maker's initial wealth is w_0, which he must allocate between the two assets. Denote by B the investment in the risky asset, which may be positive or negative depending on whether the decision maker buys or sells the risky asset. The decision-maker's problem is to choose B.

According to the ICM, the decision is triggered by a signal $\pi \in [\underline{\pi}, \bar{\pi}]$ that induces a choice of B that maximizes $\pi u(w_0 + Br_1) + (1-\pi)\, u(w_0 + Br_2)$. By risk aversion, there is a unique solution, denoted $B^*(\pi; r_1, r_2)$, given by the necessary and sufficient condition:

$$\pi u'(w_0 + B^*(\pi; r_1, r_2) r_1) r_1 + (1-\pi) u'(w_0 + B^*(\pi; r_1, r_2) r_2) r_2 = 0.$$

Clearly, $B^*(\cdot; r_1, r_2)$ is a monotonic increasing function.

The prediction of the ICM is that the choice of B is random and is depicted by a cumulative distribution function $G(B^*(\pi; r_1, r_2)) = F(\pi)$, for all r_1, r_2. Moreover, a change in the rates of returns may induce a random change in the portfolio position triggered by a new signal $\pi' \in [\underline{\pi}, \bar{\pi}]$. Specifically, consider a decrease of the positive return from r_1 to r_1'. Define $\hat{\pi}$ by the equation

$$\hat{\pi} u'(w_0 + B^*(\hat{\pi}; r_1', r_2) r_1') r_1' + (1-\hat{\pi}) u'(w_0 + B^*(\hat{\pi}; r_1', r_2) r_2) r_2 = 0.$$

Then decision maker chooses *to increase or decrease* the investment in the risky asset depending on whether π' is larger or small than $\hat{\pi}$. Specifically, $B^*(\pi'; r_1', r_2) > (\leq) B^*(\pi; r_1, r_2))$ if and only if $\pi'(>) \leq \hat{\pi}$.

The random choice behavior described above is different from Bewley's dictum "if in doubt do nothing." Applied to the initial portfolio choice, Bewley's dictum predicts that the decision maker will chose to stay put, not investing in the risky asset unless $\underline{\pi} r_1 + (1-\underline{\pi}) r_2 > 0$. Suppose that the decision maker invested in the risky asset (i.e., $B^*(\pi; r_1, r_2) > 0$). Then, unlike the prediction of the ICM, Bewley's dictum predicts that the decision maker displays inertia by not adjusting his portfolio position if the variations in the rate of return r_1' are in the range $\underline{r}_1 < r_1' < \bar{r}_1$, where $\bar{r}_1$ and $\underline{r}_1$ are defined, by $\underline{\pi}\bar{r}_1 + (1-\underline{\pi}) r_2 = 0$ and $\bar{\pi}\underline{r}_1 + (1-\bar{\pi}) r_2 = 0$, respectively.

5.2 Experiments

Generally speaking, testing the proposed ICM requires that alternatives the decision maker considers to be noncomparable be identified and the agreement between the observed choices among such alternatives and the probabilistic choices predicted by the model evaluated.

In the cases of decision making under risk and under uncertainty, the monotonicity of the preference relations with respect to first-order stochastic dominance is a property that transcends individual idiosyncratic risk attitudes. Consequently, the multi-prior expected multi-utility model with incomplete preferences displays *probabilistic choice monotonicity with respect* *to first-order stochastic dominance*. Formally, if an act h first-order stochastically dominates an act g and f is noncomparable to either h or g, then the probability that f is selected from the pair $\{f, g\}$ is greater than the probability that it is selected from the pair $\{f, h\}$.

Similar reasoning applies in the case of decision making under certainty in which the alternatives are multi-attribute goods and the incompleteness is due to the inability of the decision maker to compare alternatives that have different attributes. Formally, if an alternative a dominates an alternative a' in the sense that it has more of the positive attributes and/or less of the negative ones, and a'' is an alternative that is noncomparable to either a or a', then the probability that a'' is selected from the pair $\{a'', a'\}$ is greater than the probability that it is selected from the pair $\{a'', a\}$.

The degree of incompleteness of a decision maker's preference relation is a personal characteristic. Therefore, to obtain testable implications of the ICM, one needs formal measures of the degree of incompleteness and an elicitation scheme by which it is possible to determine individual degrees of incompleteness. In the context of decision making under uncertainty, Karni and Vierø (2021) introduced such measures as well as incentive compatible mechanisms by which the incompleteness displayed by a preference relation may be elicited.

An experimental test of the probabilistic choice monotonicity hypothesis in this context is based on observing choices among bets on an event. Formally, a *bet on an event* E is an act that pays off x dollars if E obtains and y dollars otherwise, where $x > y$. The experiment consists of two parts: In the first part, a set $J = \{1, ..., n\}$ of subjects is recruited and the ranges of incompleteness of bets on an event, E, are elicited using the scheme of Karni and Vierø (2021). In the second part, the subjects are asked to choose, repeatedly, between a bet on E and sure payoffs that are noncomparable to the bet. The prediction of the ICM is that the relative frequency of choosing the bet decreases monotonically with the values of the sure payoffs.[16]

The experiments described above are designed to test a qualitative property of the ICM, namely, probabilistic choice monotonicity that transcends the idiosyncratic variations of individual stochastic signal-generating processes. They are not designed to quantify the change in the probabilistic choice be-

[16]This method is discussed in Loomes and Sugden (1998) and was implemented in a study by Loomes, Moffatt, and Sugden (2002). To provide the subjects with an incentive to consider the choice seriously, one of each subject's choices is randomly selected, and the subject is rewarded according to the outcome of the selected alternative.

havior in response to variations in the sets alternatives. To grasp the nature of qualitative constrains imposed by the ICM model on subjects' choice behavior, consider the following experiment. Let $b = x_E y$, $b' = x'_E y'$, and $b'' = x''_E y''$ be three bets on E, where $y'' < y' < y < x < x' < x''$, and suppose that no two of these bets are comparable.[17] The subjects are asked to choose, repeatedly, from the binary set $\{b, b'\}$, and $\{b, b''\}$. Let $\alpha' = p(b', \{b, b'\})$ and $\alpha'' = p(b'', \{b, b''\})$ denote the relative frequency of choosing the b' from the set $\{b, b'\}$ and b'' from the set $\{b, b''\}$. Then the ICM model predicts that: (a) If $\alpha'' \geq \alpha'$, then facing a choice among the three bets, the subject chooses b'' with probability $p(b'', \{b, b', b''\}) = \alpha''$, b with probability $p(b, \{b, b', b''\}) = (1 - \alpha'')$ and $p(b', \{b, b', b''\}) = 0$ (i.e., b' is a dominated bet in the set $\{b, b', b''\}$). (b) If $\alpha'' < \alpha'$, then facing the choice among the three bets, the subject chooses b'' with probability, α'', b with probability $(1 - \alpha')$, and b' with probability $\alpha' - \alpha''$ (i.e., $p(b'', \{b, b'b''\}) = \alpha''$, $p(b, \{b, b'b''\}) = (1 - \alpha')$ and $p(b', \{b, b'b''\}) = \alpha' - \alpha''$).

6 Concluding Remarks

This paper proposes a novel approach to modeling decision making under certainty, risk, and uncertainty in situations in which the preference relations are incomplete. The indecisiveness, due to the noncomparability of the alternatives under consideration, is captured by a set of partial strict orders on the corresponding choice sets. The implied probabilistic choice behavior is characterized. Using the same approach, the ICM can be applied to non-expected utility theories with incomplete preferences (e.g., the dual theory (Maccheroni [2004]), probabilstically sophisticated choice (Karni [2020]) and weighted utility theory (Karni and Zhou [2021])).

6.1 Interpersonal comparisons

Different decision makers may exhibit distinct random choice behaviors because of different attributes of the ICMs that depict their decision-making processes. Specifically, the preference relations may not agree on the sets of alternatives that are noncomparable. For example, one decision maker may strictly prefer an alternative a over a', displaying resolute choice, while

[17] The bets are chosen after the range of incompleteness at E is elicited, using the scheme described in Karni and Vierø (2021).

another decision maker may find the same alternatives noncomparable and display irresolute choice behavior. Even if the decision makers are indecisive with regard to the two alternatives, they may still exhibit distinct random choice patterns, due to distinct underlying signal-generating processes. To grasp this, let the ICM model of the one decision maker be $\{\succ^{\alpha} \mid \alpha \in [0,1]\}$ and that of another be $\{\hat{\succ}^{\alpha} \mid \alpha \in [0,1]\}$. Suppose that both models agree that a and a' are noncomparable. It may still be that $a \succcurlyeq^{\bar{\alpha}(a,a')} a'$ and $a \hat{\succcurlyeq}^{\bar{\alpha}'(a,a')} a'$, for $\bar{\alpha}(a,a') \neq \bar{\alpha}'(a,a')$. According to the ICM, the former decision maker chooses a with probability $p(a,\{a,a'\}) = \bar{\alpha}(a,a')$, and the latter with probability $p'(a,\{a,a'\}) = \bar{\alpha}'(a,a')$.

6.2 Related literature

The recognition that, in many settings, choices are observed to display stochastic patterns led, in recent years, to a revival of interest in modeling and testing stochastic choice behavior.[18] Much of this work - including recent contributions by Echenique and Saito (2019), Ahumada and Ulku (2018), and Horan (2021) - builds on, and extends, the seminal model proposed by Luce (1959). A common feature of these models is a primitive stochastic choice function that is assumed to be the observable object being studied and characterized. The stochastic choice relations are the analogue concept in this paper are . Like the stochastic choice function, these relations are a primitive concept. In every other respect, however, (i.e., the axiomatic structure and the representations), the ICM is different from Luce's original model and its extensions. In particular, Luce's (1959) model requires that the choice probabilities must always be strictly positive. In many instances the empirical choice probability is zero. Consequently, this requirement is regarded as a weakness of the model. Indeed, the aforementioned extensions of Luce's model are intended to address and overcome this weakness by admitting "editing" processes that qualify the supports of the images of the choice functions by eliminating dominated alternatives. By contrast, the ICM naturally admits zero choice probabilities, and dominated alternatives are assigned zero probability, the probabilistic choice relations that are at the heart of the model.

Danan (2010) modeled a two-stage decision-making process according to which, in the first stage, any two alternatives are either ranked in the strict

[18]See Gul et al. (2014), Fudenberg et al. (2015), and Frick et al. (2019).

sense or judged as being equally valuable. If no judgment is rendered comparing their values, the two alternatives are determined to be noncomparable. In the second stage, the alternative that is ranked higher, if such an alternative exists, is selected. Otherwise, one of the alternatives is chosen either by deliberate randomization or selectively. Danan's analysis addresses the vulnerability of the decision process to being manipulated to produce sure losses through a process known as a money pump. In the case of deliberate randomization, choice behavior is based on a signal produced be a randomization device. In terms of the ICM model proposed here, the signal space of the randomizing device is mapped onto the canonical signal space by ascribing to the sets of utility functions that rank one alternative over the other the probability that the first alternative is selected by the randomization device.

Ok and Tserenjigmid (2020) model random choice behavior as random choice functions, which they define and characterize for stochastic choices induced by indifference, indecisiveness, and experimentation. The first two are closely related to the phenomena modeled in this paper. Ok and Tserenjigmid merely assert that the maximal elements of the menu will be chosen with positive probability.[19]

Karni and Safra (2016) study stochastic choice under risk and under uncertainty based on the notion that decision makers' actual choices are governed by randomly selected *states of mind*. They provide axiomatic characterization of the representation of decision makers' perceptions of the stochastic process underlying the selection of their state of mind. In the context of decision making under uncertainty with incomplete preferences, the states of mind are probability-utility pairs in the set Φ.[20] The stochastic choice process corresponds to a subjective probability measure, λ, of the sets Φ^{λ}. Thus, the work of Karni and Safra (2016) may be regarded as providing axiomatic foundations of a subjective version of the ICM.

Finally, although not involving random choice behavior, the idea of nested family of preorders, was explored by Hill (2016). In the context of Knightian uncertainty, Hill proposed that the larger the stake involved, the more confidence the decision maker must have in his judgment before making a decision. Formally, this takes the form of nested preorders with the set of

[19] Ok and Tserenjigmid (2021) propose making rationality comparisons between stochastic choice rules by means of a partial ordering method. According to their method, the stochastic choice model of this paper is maximally rational.

[20] In the special cases of Knightian uncertainty and complete beliefs, the sets of states of mind are Π and $\mathcal{U}^1$, respectively.

prior corresponding to a higher stake decision being contained in that of the lower stake one.

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