A Theory of Stochastic Choice under Uncertainty

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Abstract

In this paper we propose a characterization of stochastic choice under risk and under uncertainty. We presume that decision makers’ actual choices are governed by randomly selected states of mind, and study the representation of decision makers’ perceptions of the stochastic process underlying the selection of their state of mind. The connections of this work to the literatures on random choice, choice behavior when preference are incomplete; choice of menus; and grades of indecisiveness are also discussed.

Keywords: Random choice; decision making under uncertainty; Incomplete preferences; Preferences over menus; Subjective state space

JEL classification numbers: D01, D81

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1 Introduction

In this paper, we develop a theory of random choice under uncertainty and under risk motivated by the recognition that there are situations in which the decision maker’s tastes are subject to random variations. In these situations, a decision maker’s choice behavior displays a stochastic pattern represented by a probability distribution on the set of alternatives.

The idea advanced in this paper is that variability in choice behavior is an expression of internal conflict among distinct inclinations, or distinct “selves,” of the decision maker, whose assessments of the alternatives are different. We refer to these inclinations as ”states of minds” and assume that, analogous to a state of nature, a state of mind resolves the uncertainty surrounding a decision maker’s true subjective beliefs and/or tastes. Our theory presumes that, at a meta level, decision makers entertain beliefs about their likely state of mind when having to choose among uncertain, or risky, prospects; that their actual choice is determined by the state of mind that obtains; and that the observed choice probabilities are consistent with these beliefs. In other words, a decision maker’s state of mind governs his choice behavior in the sense that, when having to choose among acts (or lotteries), a state of mind, encompassing beliefs and risk attitudes, is selected at random and that state of mind decides which alternative is chosen. The focus of our investigation is the representation of the decision maker’s perception of the stochastic process underlying the selection of his state of mind. We presume that this process is accessible by introspection and that it agrees with the empirical distribution characterizing the random choice rule.

The fact that states of mind are preference relations has two crucial aspects. First, it renders the evaluation of the outcomes—acts or lotteries, as the case may be—inherently dependent on the state (of mind). Second, unlike states of nature, which are observable, states of mind are inherently private information. These aspects raise two difficulties: Because the preference relation is state dependent, subjective expected utility theory fails to deliver a unique prior. In the present context, since our main purpose is to obtain a unique prior that characterizes the random choice rule, this shortcoming is fatal. Second, because states of mind are private information, they express themselves, indirectly, through choices among menus rather than directly through the choice of acts. To overcome the first difficulty, we apply a modified version of the model of Karni and Schmeidler (1980). To overcome
the second, building on ideas introduced by Kreps (1979) and developed by Dekel, Lipman, and Rustichini (2001), we derive preferences over acts from those on menus. We assume that a decision maker is characterized by two primitive preference relations: a preference relation on the set of menus of alternatives depicting his actual choice behavior and a preference relation on hypothetical mental state-act lotteries. The preference relation on the set of menus induces preferences on the set of mental acts (that is, mappings from the set of states of mind to the set of uncertain, or risky, prospects). Both the preference relation on the set of mental acts and that on the mental state-act lotteries are assumed to satisfy the von Neumann-Morgenstern axioms; jointly they also satisfy an axiom dubbed consistency. Roughly speaking, consistency requires that when mental state-act lotteries are mapped into mental acts induced by menus, the two aforementioned preference relations be in agreement. This model yields a representation of the preference relations over mental acts induced by menus that takes the form of subjective expected utility with state-dependent utility functions defined on uncertain, or risky, prospects and a unique subjective prior on the set of states of mind. The distribution on the mental state space characterizes the decision maker’s stochastic choice behavior.

More formally, let \( \{ \succ_\omega \mid \omega \in \Omega \} \) be a set of preference relation on the set, \( \mathcal{H} \), of Anscombe-Aumann (1963) acts, and assume that they satisfy the axioms of expected utility theory. A menu, \( \mathcal{M} \), is a non-empty subset of Anscombe-Aumann acts. An act induced by \( \mathcal{M} \), denoted \( f_\mathcal{M} \), is an assignment to each \( \omega \in \Omega \) of an act \( h \) such that \( h \succ_\omega h' \), for all \( h' \in \mathcal{M} \). We denote by \( F \) the set of acts induced by menus. Let \( \succ \) be a preference relation on the set of all menus. Define the induced preference relation on \( F \) as follows: \( f_\mathcal{M} \succ f_{\mathcal{M}'} \) if \( \mathcal{M} \succ \mathcal{M}' \). Broadly speaking, the main result of this paper is identifying necessary and sufficient conditions that yield the following representation: There exist a continuous, non-constant, real-valued function \( u \) on \( \Omega \times \mathcal{H} \) that is affine in its second argument and a probability distribution \( \eta \) on \( \Omega \) such that for all \( f_\mathcal{M}, f_{\mathcal{M}'} \in F \),

\[
f_\mathcal{M} \succ f_{\mathcal{M}'} \iff \sum_{\omega \in \Omega} \eta(\omega) [u(\omega, f_\mathcal{M}(\omega))-u(\omega, f_{\mathcal{M}'}(\omega))] \geq 0.
\]

Moreover, \( u \) is unique up to a cardinal unit-comparable transformation, and \( \eta \) is essentially unique.

In the context of risk, this representation is similar to that of Dekel,
Lipman, and Rustichini (2001). However, the uniqueness of $\eta$ is specific to our model.\footnote{Sadowski (2013) obtained uniqueness of the probabilities in the model of Dekel et. al. (2001) by enriching the model with objective states.}

The theory developed in this paper is related not only to the literature on random choice but also to the literature on choice behavior when preference relations are incomplete, the literature on choice of menus, and the work on grades of indecisiveness.

Applying our model to menus of lotteries, we show that our theory implies the axioms of Gul and Pesendorfer (2006). Hence, the probability measure $\eta$ generates their random utility and random choice model. All our preference relations are defined ex ante, at an earlier stage, before the actual choice among various acts/lotteries. In that stage, the decision maker chooses among menus of alternatives. We do not make explicit the later, ex post, choice, but, as indicated above, we assume that it is consistent with the expectations the decision maker has at the earlier stage. To make the connection between the two stages more explicit, one can follow Ahn and Sarver (2013), who join together the ex ante model of Dekel et. al. (2001) with the ex post random choice of Gul and Pesendorfer (2006).

The representation of incomplete preferences under uncertainty specifies a set of utility-probability pairs and requires that one alternative be strictly preferred over another if and only if the former yields higher subjective expected utility than the latter according to each utility-probability in the set.\footnote{See Galaabaatar and Karni (2013). In the case of incomplete preferences under risk, we identify states of mind with utility function and the analogous results are Dubra, Maccheroni, and and Ok., (2004) and Shapley and Baucells, (2008).} In this context, we identify states of mind with utility-probability pairs. When the alternatives are noncomparable, the choice may be random. Our model implies that the likelihood that one alternative is chosen over another is the measure (according to $\eta$) of the subset of the states of mind that prefer that alternative.

We also show that Minardi and Savochkin’s (2014) notion of grades of indecisiveness between two Anscombe-Aumann acts, say $f$ and $g$, can be represented by the probability $\eta$ of the set $\{\omega \in \Omega \mid f \succ_{\omega} g\}$.

The model developed in this paper is related to the literature on probabilistic choice originated by Luce and Suppes (1965) and later developed by Loomes and Sugden (1995). Recently, Melkonyan and Safra (2014) ax-
iomatized the utility components of two families of such preferences. One family satisfies the independence axiom. This paper complements and extends that model by characterizing the inherent probability distribution over the possible states of mind (possible tastes).

A more detailed discussion of the connections between this paper and these branches of the literature appears in section 3, following the presentation of our theory in the next section. The proofs are relegated to section 4.

2 Stochastic Choice Theory

2.1 The analytical framework: Revealed preferences over mental acts induced by menus

2.1.1 Acts and preferences

Let $\mathcal{X}$ be a finite set of outcomes, and denote by $\Delta(\mathcal{X})$ the set of all probability measures on $\mathcal{X}$. For each $p, q \in \Delta(\mathcal{X})$, and $\alpha \in [0, 1]$, define $\alpha p + (1 - \alpha) q \in \Delta(\mathcal{X})$ by $(\alpha p + (1 - \alpha) q)(x) = \alpha p(x) + (1 - \alpha) q(x)$, for all $x \in \mathcal{X}$.

Let $S$ be a finite set of material states (or states of nature), and denote by $H$ the set of all mappings from $S$ to $\Delta(\mathcal{X})$. Elements of $H$ are referred to as acts. $^3$ For all $h, h' \in H$, and $\alpha \in [0, 1]$, define $\alpha h + (1 - \alpha) h' \in H$ by $(\alpha h + (1 - \alpha) h')(s) = \alpha h(s) + (1 - \alpha) h'(s)$, for all $s \in S$, where the convex operation $\alpha h(s) + (1 - \alpha) h'(s)$ is defined as above. Under this definition, $H$ is a convex subset of the linear space $\mathbb{R}^{[X] \rightarrow [S]}$.

Let $\mathcal{P}$ be the set of all preference relations on $H$ whose structure is depicted by the following axioms:

(A.1) (Strict total order) The preference relation $\succ$ is asymmetric and negatively transitive.

(A.2) (Archimedean) For all $h, h', h'' \in H$, if $h \succ h'$ and $h' \succ h''$, then $\beta h + (1 - \beta) h'' \succ h'$ and $h' \succ \alpha h'(1 - \alpha) h''$ for some $\alpha, \beta \in (0, 1)$.

$^3$See Anscombe-Aumann (1963).
(A.3) (Independence) For all \( h, h', h'' \in H \) and \( \alpha \in (0, 1] \), \( h \succ h' \) if and only if \( \alpha h + (1 - \alpha) h'' \succ \alpha h' + (1 - \alpha) h'' \).

(A.4) (Nontriviality) \( \succ \) is not empty.

By the expected utility theorem, a preference relation satisfies (A.1)-(A.4) if and only if there exists a nonconstant real-valued function, \( w(x, s) \), on \( X \times S \), unique up to cardinal unit-comparable transformation,\(^4\) such that, for all \( h, h' \in H \),\(^5\)

\[
h \succ h' \Leftrightarrow \sum_{s \in S} \sum_{x \in X} w(x, s) [h(s)(x) - h'(s)(x)] > 0.
\]

2.1.2 States of mind and mental acts induced by menus

Let \( \Omega \) be a finite, nonempty set and consider the subset of preferences \( P^\Omega = \{ \succ_{\omega} \in P : \omega \in \Omega \} \). We refer to \( \succ_{\omega} \) as a state of mind depicting a possible mood, or persona, of the decision maker. To avoid notational redundancy we assume that \( \succ_{\omega} \neq \succ_{\omega'} \) for all \( \omega \neq \omega' \). To simplify the notation, we also identify \( \succ_{\omega} \) with \( \omega \) and refer to \( \Omega \) as the mental state space. Let \( \succ_{\omega} \) be a binary relation on \( H \) defined by \( h \succ_{\omega} h' \) if \( - (h' \succ_{\omega} h) \). Then, \( \succ_{\omega} \) is complete and transitive.

A menu is a nonempty subset of \( H \). Let \( \mathcal{M} := 2^H \setminus \{ \emptyset \} \) be the set of menus, and denote by \( M \) its generic element. For each \( M \in \mathcal{M} \), define a correspondence \( \varphi_M : \Omega \rightarrow H \) as follows: For every \( \omega \in \Omega \),

\[
\varphi_M (\omega) = \{ h \in M : h \succ_{\omega} h', \forall h' \in M \}.
\]

The correspondence \( \varphi_M \) maps mental states to subsets of \( \omega \)-equivalent acts.

Let \( \hat{F} := \{ f : \Omega \rightarrow H \} \) be the set of all mappings from \( \Omega \) to \( H \). Elements of \( \hat{F} \) are referred to as mental acts. The set \( \hat{F} \) is a convex set (with respect to the operation \( (\alpha f + (1 - \alpha) f')(\omega) = \alpha f(\omega) + (1 - \alpha) f'(\omega) \), for all \( \omega \in \Omega \)). Let \( F := \{ f \in \hat{F} : \exists M \in \mathcal{M} \ : \forall \omega, \ f(\omega) \in \varphi_M (\omega) \} \). An element of \( F \) is dubbed a mental act induced by \( M \). It is an element of \( \hat{F} \) that maximizes all

\(^4\)A function \( \hat{w}(x, s) \) is said to be cardinal unit-comparable transformation of \( w(x, s) \) if there exist a real number \( b > 0 \) and \( a \in \mathbb{R}^S \) such that \( \hat{w}(x, s) = bw(x, s) + a(s) \), for all \( (x, s) \in X \times S \).

\(^5\)See Kreps (1988).
preference relations $\omega \in \Omega$ over $M$. We denote by $f_M \subset F$ the set of mental acts induced by $M$. Let $\succsim$ be a preference relation on $M$ depicting the decision maker’s observable behavior when faced with choice among menus. We assume that $\succsim$ satisfies the analogue of axioms (A.1)-(A.4), where mixtures of menus are defined by mixing all possible pairs of acts in which the acts belong to distinct menus. Define an induced preference relation $\succeq$ on $F$ by $f \succeq f'$ if $f \in f_M$, $f' \in f_{M'}$ and $M \succsim M'$. This definition presumes that, when decision makers compare two menus, $M$ and $M'$, they imagine selecting mental acts from $f_M$ and $f_{M'}$, respectively, and comparing them. There can be two different mental acts $f, f' \in f_M$. Clearly, if $f, f' \in f_M$ then $f \sim f'$. Henceforth, we treat $f_M$ as an element of $F$. Unlike $\succsim$, the induced preference relation $\succeq$ on $F$ is defined on mental constructs which are not directly observable and must be inferred from the decision maker’s choice of menus. Clearly, the induced preference relation $\succeq$ on $F$ satisfies the axioms (A.1)-(A.4). The following lemma ensures that these requirements are nonvacuous.

**Lemma 1:** $F$ is a convex set.

**Remark 1.** The mental state space is analogous to the subjective state space introduced by Kreps (1979). However, unlike Kreps, who derived the existence of such space from preference for flexibility (that is, from preferences over menus), we take the existence of such space as a primitive aspect of the model (which finds its expression in preferences over menus). The subjective state space in Kreps (1979) is non-unique. Dekel, Lipman and Rustichini (2001) axiomatized the existence and uniqueness of Kreps’ subjective state space. In this paper, the uniqueness of the mental state space is assumed. In some applications (e.g., when the preference relation on acts, in the case of uncertainty, or lotteries, in the case of risk, is incomplete) the existence and uniqueness are implied by the representation (see discussion in section 3.1 below).

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6See also Dekel, Lipman, Rustichini and Sarver (2007).

7In principle one can derive the existence and uniqueness of the mental state space by applying the result of Dekel, Lipman and Rustichini (2001). This would require the extension of the menus to allow mixtures of acts and apply their axioms. We do not pursue this here because it is secondary to the main thrust of this work.
2.2 The analytical framework: State-act lotteries and introspective preferences

2.2.1 State-act lotteries and their decomposition

Let \( \hat{\Omega} (\Omega \times H) \) be the set of probability distributions on \( \Omega \times H \) with finite supports. Elements of \( \hat{\Omega} (\Omega \times H) \) are state-act lotteries. A state-act lottery \( \hat{\omega} \in \hat{\Omega} (\Omega \times H) \) is said to be non-degenerate if \( \mu_\ell(\omega) := \Sigma_{h \in H} \ell(\omega, h) > 0 \), for every \( \omega \in \Omega \). Define \( \mu(\ell) = (\mu_\ell(\omega))_{\omega \in \Omega} \), for all \( \ell \in \hat{L}(\Omega \times H) \), then \( \mu(\ell) \in \Delta(\Omega) \).

Assume that \( \ell \) is non-degenerate and consider the function \( J : \hat{L}(\Omega \times H) \to \hat{F} \) defined by

\[
J(\ell)(\omega) = \sum_{h \in H} \frac{\ell(\omega, h)}{\mu_\ell(\omega)} h, \text{ for all } \omega \in \Omega.
\]

For the uniform distribution \( \mu(\omega) = \frac{1}{|H|} \) in \( \Delta(\Omega) \), define the function \( K : F \to \hat{L}(\Omega \times H) \) by

\[
K(f)(\omega, h) = \begin{cases} \frac{1}{|H|} & f(\omega) = h \\ 0 & \text{otherwise} \end{cases}
\]

Clearly, \( K(f) \) is non-degenerate and \( J(K(f)) = f \). Henceforth, we focus the attention on \( \ell \in \hat{L}(\Omega \times H) \) such that \( J(\ell) \in F \). Define

\[
L(\Omega \times H) = \{ \ell \in \hat{L}(\Omega \times H) \mid \ell \text{ non-degenerate and } J(\ell) \in F \}.
\]

**Lemma 2:** \( L(\Omega \times H) \) is a convex set.

2.2.2 Introspective preferences and consistency

Let \( \succ^* \) be a binary relation on \( L(\Omega \times H) \) and assume that \( \succ^* \) satisfies the analogue of axioms (A.1) - (A.4). We refer to \( \succ^* \) as introspective preference relation. Define \( \succ^* \) on \( L(\Omega \times H) \), by \( \ell \succ^* \ell' \) if \( \neg(\ell' \succ^* \ell) \). These preference relations express the decision maker’s beliefs about his behavior if he could choose between lotteries \( L(\Omega \times H) \). We presume that the decision maker is conscious that he may experience different moods, and that he is capable of expressing preferences not only among acts given a certain mood (which is captured by the state of mind \( \succ_\omega \)) but also across acts in different moods.
For example, the decision maker is supposed to be able to claim, upon introspection, that he prefers listening to Beethoven 9th symphony when in good mood over listening to Mozart’s requiem when depressed. Because, in general, the decision maker does not get to choose his mood, the introspective preferences are hypothetical and can only be expressed verbally.\footnote{Note that this is analogous to the assertion in Karni and Schmeidler (1980) that the decision maker is capable of comparing watching a football game in an open stadium if it is sunny to watching it on TV at home if it rains.}

Two state-act lotteries \( \ell \) and \( \ell' \) are said to agree outside \( \omega \) if \( \ell (\omega', \cdot) = \ell' (\omega', \cdot) \), for all \( \omega' \in \Omega \setminus \{ \omega \} \). Similarly, two mental acts, \( f \) and \( f' \) are said to agree outside \( \omega \) if \( f (\omega') = f' (\omega') \), for all \( \omega' \in \Omega \setminus \{ \omega \} \). Following Karni and Schmeidler (1980) we make the following definition.

**Definition 1:** A state of mind \( \omega \in \Omega \) is obviously null if, \( f \sim f' \) for all \( f, f' \in F \) that agree outside \( \omega \), and there exist \( \ell, \ell' \in L(\Omega \times H) \) that agree outside \( \omega \), such that \( \ell \succ^* \ell' \); it is obviously nonnull if there are \( f, f' \in F \) that agree outside \( \omega \) and \( f \succ f' \).

The following example demonstrates that the definitions above are not vacuous.

**Example:** Let \( \Omega = \{ \omega, \omega' \} \) and consider \( M = \{ h, h' \} \) and \( M' = \{ h, h'' \} \) such that \( h \succ_w h' \succ_w h'', h' \succ_w h'' \succ_w h \). Then \( f_M (\omega) = h, f_M (\omega') = h' \) and \( f_{M'} (\omega) = h, f_{M'} (\omega') = h'' \) (see Figure 1). Denote \( \ell_M = K (f_M) \) and \( \ell_{M'} = K (f_{M'}) \). Then \( \ell_M (\omega, h) = \frac{1}{2} = \ell_{M'} (\omega, h) \), \( \ell_M (\omega, h) = 0 = \ell_{M'} (\omega, h) \) for all \( h \in H \setminus \{ h \} \), \( \ell_M (\omega', h') = \frac{1}{2} = \ell_{M'} (\omega', h'') \), \( \ell_M (\omega', h') = 0 \) for all \( h' \in H \setminus \{ h'' \} \). Hence, \( \ell_M \) agrees with \( \ell_{M'} \) outside \( \omega' \). Suppose that \( \ell_M \succ^* \ell_{M'} \). If \( J (\ell_M) \sim J (\ell_{M'}) \) then \( \omega' \) is obviously null and if \( J (\ell_M) \succ J (\ell_{M'}) \) then \( \omega' \) is obviously nonnull.

Insert Figure 1

With this in mind we introduce an axiom analogous to strong consistency of Karni and Schmeidler’s (1980).

**(A.5) Consistency I.** For all \( \omega \in \Omega \) and all non-degenerate \( \ell, \ell' \in L(\Omega \times H) \) such that \( \ell \) agrees with \( \ell' \) outside \( \omega \): if \( J (\ell) \succ J (\ell') \) then \( \ell \succ^* \ell' \).

Conversely, for such \( \ell, \ell' \), if \( \omega \) is obviously nonnull then \( \ell \succ^* \ell' \) implies \( J (\ell) \succ J (\ell') \).
2.3 Representation of preferences over menus and introspective preferences: Uncertainty

The following theorem, which is analogous to theorem 3 of Karni and Schmeidler (1980), asserts the existence and describes the uniqueness properties of a subjective expected utility representations of the preference relations $\succsim$ on the set of mental acts induced by menus and $\succsim^*$ on the set of state-act lotteries.

**Theorem 1:** Let $\succsim^*$ on $L(\Omega \times H)$ and $\succsim$ on $F$ be binary relations. The following conditions are equivalent:

1. (a.i) The asymmetric parts of $\succsim^*$ and $\succsim$ satisfy (A.1) - (A.4) and jointly they satisfy (A.5).
2. (a.ii) There exist continuous, non-constant, real-valued function $u$ on $\Omega \times H$ that is affine in its second argument, and a probability distribution $\eta$ on $\Omega$ such that for all $f_M, f_{M'} \in F$,

$$f_M \succsim f_{M'} \iff \sum_{\omega \in \Omega} \eta(\omega) [u(\omega, f_M(\omega)) - u(\omega, f_{M'}(\omega))] \geq 0,$$

and, for all $\ell, \ell' \in L(\Omega \times H)$,

$$\ell \succsim^* \ell' \iff \sum_{\omega \in \Omega} \sum_{h \in H} u(\omega, h) \ell(\omega, h) \geq \sum_{\omega \in \Omega} \sum_{h \in H} u(\omega, h) \ell'(\omega, h).$$

3. (b) $u$ is unique up to cardinal unit-comparable transformation.
4. (c) For obviously null $\omega \in \Omega$, $\eta(\omega) = 0$, and for obviously non-null $\omega \in \Omega$, $\eta(\omega) > 0$. Moreover, if all states of mind are obviously non-null then $\eta$ is unique.

The proof is given in Section 5.

**Remark 2.** Karni and Mongin (2000) showed that, unlike in the work of Karni, Schmeidler and Vind (1983) in which the subjective probabilities are arbitrary, the subjective probabilities in Karni and Schmeidler (1980) are the unique representation of the decision maker’s beliefs about the likely realization of the material states. The same argument applies to this model.

Applying the model of Karni and Schmeidler to the preference relations $\succsim_{\omega \in P}$, it can be shown that the utility function in the representations in Theorem 1 takes the linear form $u(\omega, h) = \sum_{s \in S} \pi(s) \sum_{x \in X} u(\omega, x) h(s)(x)$, 

10
where $\pi (\omega, \cdot)$ is a probability measure on the material state space, $S$, representing the beliefs of a decision maker whose mood is $\succ_\omega$.

Finally, note that, by definition and (1), the representation of $\hat{\succ}$ on $\mathcal{M}$ is as follows:

$$M \hat{\succ} M' \iff \sum_{\omega \in \Omega} \eta (\omega) [u (\omega, f_M (\omega)) - u (\omega, f_{M'} (\omega))] > 0.$$  

The importance of the preference relation on menus is that, unlike $\succ$ on $\mathcal{F}$, it is observable in the sense of depicting actual choice behavior.

### 2.4 Representation of preferences over menus and introspective preferences: Risk

#### 2.4.1 The analytical framework: Mental acts induced by menus

The analysis of preferences over menus under risk is the special case of preferences over menus under uncertainty in which the set of material states space is a singleton and can be ignored. Menus are non-empty subsets of $\Delta (X)$. Let $\mathcal{M}_r := 2^{\Delta (X)} \setminus \{\emptyset\}$ be the set of menus. For each $M \in \mathcal{M}_r$, define the correspondence $\varphi^r_M : \Omega \rightarrow \Delta (X)$ as follows: For every $\omega \in \Omega$,

$$\varphi^r_M (\omega) = \{p \in M \mid p \succ_\omega q, \forall q \in M\}.$$  

The correspondence $\varphi^r_M$ maps the set of mental states to subsets of $\omega$–equivalent lotteries. In the present context, one interpretation of mental state is risk attitude.

Let $\hat{G} := \{g : \Omega \rightarrow \Delta (X)\}$ be the set of all mappings from $\Omega$ to $\Delta (X)$. Elements of $\hat{G}$ are dubbed AA mental acts. Clearly, $\hat{G}$ is a convex set under the usual definition. Let $G := \{g \in \hat{G} \mid \exists M \in \mathcal{M}_r : \forall \omega, g (\omega) \in \varphi^r_M (\omega)\}$. We denote by $g_M \subseteq G$ the set of AA mental acts induced by $M$. Let $\succ_r$ be a preference relation on $\mathcal{M}_r$ depicting the decision maker’s observable behavior when faced with choice among menus. As before, we assume that $\succ_r$ satisfies the analogue of axioms (A.1) - (A.4) where mixtures of menus are defined by mixing all possible pairs of acts in which the acts belong to distinct menus.

Define an induced preference relation $\succ_G$ on $G$ by: $g \succ g'$ if $g \in g_M$, $g' \in g_{M'}$ and $M \hat{\succ} M'$, By argument analogous to Lemma 1, $G$ is a convex set. We assume that the induced preference relation $\succ_G$ on $G$ satisfies the analogue of axioms (A.1) - (A.4).

\footnote{AA for Anscombe and Aumann.}
2.4.2 The analytical framework: Mental states-roulette lotteries

Let \( \hat{\mathcal{L}}(\Omega \times \Delta(X)) \) be the set of simple probability distributions over \( \Omega \times \Delta(X) \). As before, let \( \mu_\ell(\omega) := \sum_{p \in \Delta(X)} \ell(\omega, p) \), for every \( \omega \in \Omega \).

Assume that \( \ell \) is non-degenerate and consider the function \( I : \hat{\mathcal{L}}(\Omega \times \Delta(X)) \rightarrow \hat{G} \) defined by

\[
I(\ell)(\omega) = \sum_{p \in \Delta(X)} \frac{\ell(\omega, p)}{\mu_\ell(\omega)} p, \text{ for all } \omega \in \Omega.
\]

For the uniform distribution \( \mu(\omega) = \frac{1}{|\Omega|} \) in \( \Delta(\Omega) \), define the function \( T : G \rightarrow \hat{\mathcal{L}}(\Omega \times \Delta(X)) \) by

\[
T(g)(\omega, p) = \begin{cases} \frac{1}{|\Omega|} & g(\omega) = p \\ 0 & \text{otherwise} \end{cases}
\]

Clearly, \( T(g) \) is non-degenerate and \( I(T(g)) = g \). Define

\[
\mathcal{L}(\Omega \times \Delta(X)) = \{ \ell \in \hat{\mathcal{L}}(\Omega \times \Delta(X)) \mid \ell \text{ non-degenerate and } I(\ell) \in G \}.
\]

By the same argument as in the proof of Lemma 2, \( \mathcal{L}(\Omega \times \Delta(X)) \) is a convex set.

2.4.3 Consistency and representation

Let \( \succ_L \) be a preference relations on \( \mathcal{L}(\Omega \times \Delta(X)) \) and assume that it satisfies the analogue of (A.1)-(A.4). Analogously to Definition 1, a state of mind \( \omega \in \Omega \), is said to be obviously null if, for all \( g, g' \in G \) such that \( g \) agrees with \( g' \) outside \( \omega \), \( g \sim_G g \) and there exist that \( \ell, \ell' \in \mathcal{L}(\Omega \times \Delta(X)) \) such that \( \ell \) agrees with \( \ell' \) outside \( \omega \), and \( \ell \succ_L \ell' \). It is obviously nonnull if \( g \succ_G g \), for some \( g, g' \in G \) such that \( g \) agrees with \( g' \) outside \( \omega \).

The next axiom is analogous to (A.5).

**(A.6) Consistency II.** For all \( \omega \in \Omega \), and all non-degenerate \( \ell, \ell' \in \mathcal{L}(\Omega \times \Delta(X)) \) such that \( \ell \) agrees with \( \ell \) outside \( \omega \), \( I(\ell) \succ_G I(\ell') \) implies \( \ell \succ_L \ell' \). Conversely, if \( \omega \) is obviously nonnull then \( \ell \succ_L \ell' \) implies \( I(\ell) \succ_G I(\ell') \).
Corollary: Let $\succeq_L$ on $L(\Omega \times \Delta(X))$ and $\succeq_G$ on $G$ be binary relations then the following conditions are equivalent:

(a.i) The asymmetric parts of $\succeq_L$ and $\succeq_G$ satisfy (A.1) - (A.4) and jointly they satisfy (A.6).

(a.ii) There exist a non-constant, real-valued function $u$ on $\Omega \times \Delta(X)$ affine in its second argument, and a probability distributions $\lambda$ on $\Omega$ such that for $g^*_M, g'^*_M \in G$,

$$g^*_M \succ_G g'^*_M \iff \sum_{\omega \in \Omega} \lambda(\omega) [u(\omega, g^*_M(\omega)) - u(\omega, g'^*_M(\omega))] \geq 0, \quad (3)$$

and, for all $\ell, \ell' \in L(\Omega \times \Delta(X))$,

$$\ell \succ_L \ell' \iff \sum_{\omega \in \Omega} \left[ \sum_{p \in \Delta(X)} u(\omega, p) \ell(\omega, p) - \sum_{p \in \Delta(X)} u(\omega, p) \ell'(\omega, p) \right] \geq 0. \quad (4)$$

(b) $u$ is unique up to cardinal unit-comparable transformation.

(c) If $\omega \in \Omega$ is obviously null then $\lambda(\omega) = 0$, and if it is obviously non-null then $\lambda(\omega) > 0$. Moreover, if all states of mind are obviously non-null then $\lambda$ is unique.

The proof is similar to that of Theorem 1 and is not given here.

Remark 3. For every $\omega \in \Omega$, and $\ell \in L(\Omega \times \Delta(X))$, $\ell(\omega, \cdot)$ is a compound lottery. Hence, by the reduction of compound lottery axiom,

$$\ell(\omega, p)(x) = \sum_{p \in \Delta(X)} \ell(\omega, p)p(x), \quad \forall x \in X.$$ 

Since $u(\omega, \cdot)$ in (4) is affine, $u(\omega, p) = \sum_{x \in X} u(\omega, x)p(x)$. Hence, for every $\omega \in \Omega$, and $\ell \in L(\Omega \times \Delta(X))$,

$$\sum_{p \in \Delta(X)} u(\omega, p) \ell(\omega, p) = \sum_{x \in X} u(\omega, x) \sum_{p \in \Delta(X)} \ell(\omega, p)p(x).$$

3 Relation to the Literature

3.1 Choice Behavior when Preferences are Incomplete

When the preference relation is complete, there is no distinction between preference and choice behavior. The representation of the preferences is the
choice criterion. By contrast, when the preference relation is incomplete, and the choice is between non-comparable alternatives, the representation does not indicate which of the alternatives will be selected. In particular, the choice between alternatives that are non-comparable may be random. This, however, does not mean that non-comparable alternatives are equally likely to be selected. If one alternative is “almost better” then the other, then it stands to reason that it is more likely to be chosen. To lend this idea concrete meaning we note that, in general, in subjective expected utility theory with incomplete preferences one alternative is strictly preferred over another if its subjective expected utility is greater according to a set of pairs of utilities and subjective probabilities.\textsuperscript{10} Special cases include complete tastes, in which the one alternative is strictly preferred over another if its subjective expected utility is greater according to a set of subjective probabilities, and complete beliefs, in which one alternative is strictly preferred over another if its subjective expected utility is greater according to a set utilities functions. Similarly, in expected utility theory under risk, one alternative is strictly preferred over another if its expected utility is greater according to a set of utilities functions.\textsuperscript{11}

Note that if the representation involves a set, $\Psi$, of probability-utility pairs, as in the case of subjective expected utility theory with incomplete preferences, or a set of utility functions, $\mathcal{U}$, as in the case of expected utility theory under risk with incomplete preferences, then the set of states of mind is uniquely defined. More specifically, each $(\pi, U) \in \Psi$ defines a state of mind $\succsim_{\omega}$ which is the preference relation on $\mathcal{F}$ induced by the functional $\sum_{s \in S} \pi(s) \sum_{x \in X} U(x) f(x, s)$. Similarly, in the case of risk, each $u \in \mathcal{U}$ defines a state of mind $\succsim_{\omega}$ which is the preference relation on $\mathcal{G}$ induced by the functional $\sum_{x \in X} u(x)p(x)$. The uniqueness is an implication of the uniqueness of the corresponding representations. To state the uniqueness result, consider first the case of incomplete preferences under risk. Following Dubra et. al. (2004) denote by $\langle \mathcal{U} \rangle$ the closure of the convex cone generated by all the functions in $\mathcal{U}$ and all the constant function on $\Delta(X)$. If $\mathcal{V}$ is another set of utility functions representing the same incomplete preference relation under risk, then $\langle \mathcal{V} \rangle = \langle \mathcal{U} \rangle$.\textsuperscript{12} Galaabaatar and Karni (2013) obtained analogous

\textsuperscript{10}See Galaabaatar and Karni (2013).
\textsuperscript{12}Dubra et. al. (2004) includes that case in which $X$ is a compact set in a metric space.
uniqueness result in the case of incomplete preferences under uncertainty.

In all of these instances, two alternatives are non-comparable if one is preferred over the other according to some elements in the corresponding set (e.g., some utility-probability pairs) and the second alternative is preferred over the first according to the rest of the elements in the corresponding set. It seems natural to suppose that the likelihood that the first alternative is chosen depends on the measure of the set of utilities and/or probabilities, as the case may be, according to which the expected utility of the first is larger than that of the second. This presumption expresses the idea that one probability-utility pair is selected at random and the corresponding state of mind that governs the particular choice. The question is what is the appropriate measure on the mental state space that describes this random selection process?

The theory developed in this paper suggests that the set of utility probability pairs correspond to the set, $\Omega$, of states of mind; that the decision maker can assess the likelihoods of distinct states of mind by introspection; and that the likelihood of a particular state of mind (or utility probability pair) is selected to decide between the alternatives is given by $\eta$.

Suppose that, when facing a choice among acts, the decision maker behaves as if a state of mind from $\Omega$ is drawn according to the distribution $\eta$, and that this state of mind determines his choice. Specifically, given a menu $M = \{f_1, \ldots, f_n\}$, and assuming that all elements of $\varphi_M(\omega)$ are selected with equal probabilities, the probability that $f_i$ is chosen is $\alpha_i, i = 1, \ldots, n$, is as follows: Let $\Psi_M(f_i) := \{\omega \in \Omega \mid f_i \in \varphi_M(\omega)\}$

$$
\alpha_i = \sum_{\omega \in \Psi_M(f_i)} \eta(\omega) \frac{1}{|\varphi_M(\omega)|}.
$$

(5)

A special case concerns doubleton menus. Let $M = \{f, g\}$ and denote by $\zeta(f, g)$ the probability that $f$ is chosen from the menu $M$. According to our approach,

$$
\zeta(f, g) = \sum_{\omega \in \Psi_M(f)} \eta(\omega) \frac{1}{|\varphi_M(\omega)|}.
$$

(6)

Consider next the case of subjective expected utility theory with incomplete preferences. Galaabaatar and Karni (2013) define a weak preference

The result for the case in which $X$ is finite appears in by Galaabaatar and Karni (2012).
relation $\succ_{GK}$ on $H$ as follows: For all $f, g \in H$, $f \succ_{GK} g$ if $h \succ f$ implies $h \succ g$ for all $h \in H$. Thus, $f \succ_{GK} g$ if and only if there exists a set, $\Psi$, of affine utility functions on $\Delta(X)$ and probability measures on $S$ such that $\sum_{s \in S} \hat{\pi}(s) \sum_{x \in X} \hat{U}(x) [f(x, s) - g(x, s)] = 0$ for some $(\hat{\pi}, \hat{U}) \in \Psi$ and $\sum_{s \in S} \pi(s) \sum_{x \in X} U(x) [f(x, s) - g(x, s)] > 0$ holds for all $(\pi, U) \in \Psi \setminus \{(\hat{\pi}, \hat{U})\}$.

Consider the case in which $\Psi$ is finite. Applying the theory of this paper we obtain the following implications: (a) $f \succ g$ if and only if $\zeta(f, g) = 1$ (that is, $f \succ_\omega g$ for every $\omega \in \Omega$). (b) $f \succ_{GK} g$ if and only if $f \succ_\omega g$ for every $\omega \in \Omega$, with indifference for a subset $\hat{\Omega}$ of $\Omega$. The probability that $f$ is selected is $\zeta(f, g) = 1 - \eta\left(\hat{\Omega}\right)/2$. (c) If $f$ and $g$ are non-comparable (that is, $\neg(f \succ_{GK} g)$ and $\neg(g \succ_{GK} f)$) then the probability that $f$ is selected over $g$ is $\zeta(f, g) \in (0, 1)$.

### 3.2 Grades of indecisiveness

In a recent paper, Minardi and Savochkin (2014) address the issue of choice in the context of incomplete beliefs, represented by a set of priors. They model a decision maker’s inclination to choose one Anscombe-Aumann act over another when he is indecisive. This inclination finds it expression in the decision maker’s reported predisposition to choose one alternative over another. Minardi and Savochkin formalized this idea using, as primitive, a binary relation, $\succsim$ on the set of ordered pairs of acts. They interpret the relation $(f, g) \succsim (f', g')$ as indicating that the decision maker is more confident that $f$ is at least a good as $g$ than that $f'$ is at least as good as $g'$. Minardi and Savochkin give necessary and sufficient conditions on $\succsim$ for the existence of a function, $\mu$, assigning to every pair of Anscombe-Aumann acts, $f$ and $g$, a real number, $\mu(f, g) \in [0, 1]$ such that $(f, g) \succsim (f', g')$ if and only if $\mu(f, g) \geq (f', g')$. Moreover, under these conditions the function $\mu$ is a capacity of the subset of the set of priors according to which the expected utility of $f$ is greater or equal to that of $g$.

To connect the model of this paper to the work of Minardi and Savochkin (2014), consider the special case of doubleton menus. Let $M = \{f, g\}$ and denote by $\zeta(f, g)$ the probability that $f$ is chosen from the menu $M$. According to our approach, $\zeta(f, g)$ is given in (6).
If we assume that the ordinal ranking of the elements of $X$ is independent of the decision maker’s state of mind (e.g., if $X$ are monetary payoffs), it is easy to verify that $\zeta(f, g)$ satisfies the properties of the function $\mu$ of Minardi and Savochkin (2014). Thus, the application of our approach to doubleton menus of Anscombe-Aumann acts whose payoffs are roulette lotteries over monetary outcomes yields a result which is analogous to that of Minardi and Savochkin.

Unlike Minardi and Savochkin (2014), whose concern is incomplete beliefs, in our model, there is a single prior on the mental state space, a set of material state-dependent utility functions on outcomes and probability measures on the material state space representing the decision maker’s states of the mind. In our model, distinct states of mind may represent distinct tastes (e.g., risk attitudes) and/or beliefs on the material state space, corresponding to different moods of the decision maker.

### 3.3 Random choice behavior

A random choice rule is an assignment of a probability distribution to every feasible set of alternatives, depicting the relative frequencies according to which a decision maker chooses these alternatives. A random utility function is a (finitely additive) probability measure on a set of utility functions. Gul and Pesendorfer (2006) gave necessary and sufficient conditions for a random choice rule to maximize a random utility function when the set of outcomes is finite and the set of utility functions is the von Neumann-Morgenstern utilities over distributions on the set of outcomes.

The model in this paper is close in spirit to the Gul-Pesendorfer representation of random choice rules. What Gul and Pesendorfer call a decision

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13The properties are: Reflexivity (i.e., $\zeta(f, f) = 1$). Weak transitivity (i.e., for all $f, g, h \in H$, $\zeta(f, g) = 1$ implies $\zeta(f, h) \geq \zeta(g, h)$). Monotonicity (i.e., $f_\omega(\omega) = f$ for all $\omega \in \Omega$ implies $\zeta(f, g) = 1$). Independence (i.e., for all $f, g, h \in H$ and $\alpha \in (0, 1]$, $\zeta(f, g) = \zeta(\alpha f + (1 - \alpha) h, \alpha g + (1 - \alpha) h)$). Reciprocity (i.e., for all $f, g \in H$, $\zeta(f, g) \in (0, 1]$ implies $\zeta(g, f) = 1 - \zeta(f, g)$). Continuity (i.e., for all $f, g, h \in H$ and $\gamma \in [0, 1]$, the sets $\{\alpha \in [0, 1] : \eta(\alpha f + (1 - \alpha) g, h) \geq \gamma\}$ and $\{\alpha \in [0, 1] : \eta(h, \alpha f + (1 - \alpha) g) \geq \gamma\}$ are closed). Non-degeneracy (i.e., $\zeta(f, g) = 0$, for some $f, g \in H$). If, in addition, we assume that the ordinal ranking of the elements of $X$ is independent of the decision maker’s state of mind, and consider doubleton menus $M = \{\delta_x, \delta_y\}$, then our model implies C-Completeness (i.e., either $\zeta(\delta_x, \delta_y) = 1$ or $\zeta(\delta_y, \delta_x) = 1$).

14Since Gul and Pesendorfer (2006) and Ahn and Sarver (2013) deal with menu choice
problem is a menu $M = \{p_1, \ldots, p_n\} \subset \Delta(X)$, and what they refer to as a regular random utility function may be restated in terms of our model as follows: Define

$$N^+(M, p) = \{\omega \in \Omega \mid p \succ \omega \ p', \forall p' \in M, p' \neq p\}.$$ 

Then $\eta$ is regular if,

$$\eta \left( \bigcup_{p \in M} N^+(M, p) \right) = 1, \ \forall M \in \mathcal{M}.$$ 

If $\eta$ is regular, then the random choice rule implied by our model assigns $p_i$, $i = 1, ..., n$, the probability

$$\alpha_M(p_i) = \sum_{\omega \in \Psi_M(p_i)} \eta(\omega). \quad (7)$$

By definition, $\alpha$ is a random choice rule represented by the random utility model depicted in section 2. It can be shown that the random choice rule $\alpha$ satisfies the axioms of Gul and Pesendorfer (2006).\textsuperscript{15}

Ahn and Sarver (2013) synthesized the random choice model of Gul and Pesendorfer (2006) and the menu choice model of Dekel, et. al. (2001) to obtain a representation of a two-stage decision process in which, in the first stage, decision makers choose among menus and their preferences have a representation à la Dekel, et. al., and in the second stage, they make a stochastic choice from the menu selected in the first stage according to a distribution function (and a tie breaking rule) that has a Gul-Pesendorfer representation. Ahn and Sarver identify the necessary and sufficient conditions for

\textsuperscript{15}If $M$ is such that $\varphi_M(\omega)$ is a singleton for all $\omega \in \Omega$, then, for all $p \in M \subset M'$, $\Psi_M(p) \supset \Psi_{M'}(p)$. Hence, $\alpha^M(p) \geq \alpha^{M'}(p)$. Hence, $\alpha$ is monotone. For all $q \in \Delta(X)$ and $\lambda \in (0, 1]$ let $M = \lambda M + (1 - \lambda) \{q\} := \{\lambda p_i + (1 - \lambda) q \mid p_i \in M\}$. Then, by independence, for all $p \in M$, $\Psi_M(p) = \Psi_{M'}(\lambda p + (1 - \lambda) q)$. Hence, $\alpha^M(p) = \alpha^{M'}(\lambda p + (1 - \lambda) q)$. Thus, $\alpha$ is linear. Also, since in our model, decision makers are expected utility maximizers, restricting choice to extreme points entails no essential loss. So $\alpha$ is extreme. Finally, for all $M, M' \in \mathcal{M}$, and $\lambda \in [0, 1]$ let $\lambda M + (1 - \lambda) M' = \{\lambda p + (1 - \lambda) p' \mid p \in M, p' \in M'\}$. Then, $\Phi_{\lambda M + (1 - \lambda) M'}(\lambda p + (1 - \lambda) p') = \{\omega \in \Omega \mid \lambda p + (1 - \lambda) p' \in \varphi_{\lambda M + (1 - \lambda) M'}(\omega)\}$. But $\lambda p + (1 - \lambda) p' \in \varphi_{\lambda M + (1 - \lambda) M'}(\omega)$ implies $p \in \varphi_M(\omega)$ and $p' \in \varphi_{M'}(\omega)$. Hence, variations in $\lambda$ will not change $\Phi_{\lambda M + (1 - \lambda) M'}(\lambda p + (1 - \lambda) p')$. Thus, $\alpha$ is mixture continuous.
the representation of Dekel, et. al. (2001) and that of Gul and Pesendorfer (2006) to be consistent, in the sense that the decision maker’s prior on the subjective state space and state-dependent utility functions agree with the distribution depicting his stochastic choice behavior and the corresponding state-dependent utility functions of Gul and Pesendorfer.

As explained in the Introduction, the model presented in this paper assume that this synthesis exists. If $M$ is a menu that was selected in the first stage then, as we assumed in section (3.1), when facing a choice among lotteries, the decision maker anticipates that a state of mind from $\Omega$ is drawn according to the distribution $\eta$, and that this state of mind determines his choice. If this anticipation is correct then the probability that $p_i$ is chosen is given equation (7).

Despite the similarity, the random choice behavior in this paper is fundamentally different from that of Gul and Pesendorfer (2006) and Ahn and Sarver (2013). First, and foremost, in these models the function that associates each menu with a probability distribution over its elements, the random choice rule, is a primitive concept. By contrast, in the model of section 2, it is a derived concept. Second, the essence of the model of Ahn and Sarver (2013) is consistency between the (ex-ante) anticipated choice and (ex-post) actual stochastic choice, which is exogenously given. The essence of the present model is consistency between the introspective beliefs and the actual beliefs, represented by the probabilities on the mental state space.

Lu (2014) and Dillenberger, Lleras, Sadowski and Takeoka (2014) address the issue of identifying the distribution of private information signals from choice behavior. Both invoke preference relation on the nonempty subsets of Anscombe-Aumann (1963) acts but take different approaches. Lu (2014) extends the random choice model of Gul and Pesendorfer (2006) to include decision problems that consist of Anscombe-Aumann acts. In the individual interpretation of Lu’s model, a decision maker receives a signal that affects his choice behavior. The signal is a draw from a distribution on a canonical signal space of beliefs (that is, prior distribution of the material state space) and tastes (that is, a set of utility functions) and is private information. Lu provides an axiomatic characterization of the random choice rule that is necessary and sufficient for it to have a information representation.\footnote{Lu (2014) contains additional results and analysis that are not directly related to this work.}
Dillenberger et al. (2014) propose a theory of subjective learning according to which the preference relations on menus of Anscombe-Aumann (1963) acts reflect decision makers’ anticipated acquisition of private information before a choice of an act from the menu must be made. They analyze the axiomatic structure that allows an uninformed observer to infer from decision makers’ choice behavior, the distribution of the signals (that is, underlying information structure) that govern their ex post choices.\textsuperscript{17}

Despite sharing some features with the theory advanced here, the works of Lu (2014) and Dillenberger et al. (2014) are different from the one of this paper conceptually, methodologically, and structurally. To begin with, their objective is a representation of an analyst’s inference of the decision maker’s private information from his choice behavior. By contrast, in this paper it is the decision maker who is unsure about his own preferences, and the main trust of our analysis is the representation of the decision maker’s beliefs about the evolution of his own preferences and choice behavior. The analysis of Lu and Dillenberger et al. is based on preference relation among menus and are anchored in the revealed preference methodology. By contrast, the model of this paper requires that, in addition to preference relation on menus, the decision maker express his preferences on a set of hypothetical lotteries. This departure from the revealed preference methodology has its benefits in terms of its greater generality. Specifically, this work is concerned with the decision maker’s uncertainty about his preferences which include beliefs as well as his tastes. Moreover, the information structure, which is focus of the analysis of Dillenberger et al. (2014), corresponds to the beliefs in the model of this paper. Their model and analysis neither intended nor can it address the issue of uncertain tastes which is at the core of the present analysis. Finally, analytical framework, the axiomatic structures and the representations of the preferences in the works of Lu and Dillenberger et al. models are different from those presented here.\textsuperscript{17}

\textsuperscript{17}Dillenberger et. al. (2014) also apply their model to the analysis of dynamic decision making tracing the effect of anticipated arrival of information. These aspects of their paper are not directly related to this work.
4 Proofs

4.1 Proof of Lemma 1

Let \( \phi \in \Phi \) and \( \phi' \in \Phi \), \( \phi \geq h, \forall h \in M \) and \( \phi' \geq h, \forall h \in M' \). We need to show that there exist \( \hat{M} \in \mathcal{M} \) such that \( \hat{f}_{\hat{M}} = \alpha f_M + (1 - \alpha) f_{M'} \). Consider the menu

\[
\hat{M} = \{ \alpha f_M (\omega) + (1 - \alpha) f_{M'} (\omega) \mid \omega \in \Omega \}.
\]

Then, by two applications of (A.3), for all \( \omega \in \Omega \)

\[
\alpha f_M (\omega) + (1 - \alpha) f_{M'} (\omega) \geq \omega \alpha f_M (\omega') + (1 - \alpha) f_{M'} (\omega') .
\]

Hence \( \alpha f_M + (1 - \alpha) f_{M'} = \hat{f}_{\hat{M}} \) and is, by definition, an element of \( F \). Thus, \( F \) is a convex set. \( \square \)

4.2 Proof of Lemma 2

Let \( \ell, \ell' \in L (\Omega, H) \) and \( \alpha \in [0, 1] \). To show that \( \alpha \ell + (1 - \alpha) \ell' \in L (\Omega, H) \), we need to show that \( J (\alpha \ell + (1 - \alpha) \ell') \in F \). By definition, for all \( \omega \in \Omega \)

\[
J (\alpha \ell + (1 - \alpha) \ell') (\omega) = \sum_{h \in H} \left[ \frac{\ell (\omega, h)}{\mu_\ell (\omega)} \right] \frac{\alpha \mu_\ell (\omega)}{\alpha \mu_\ell (\omega) + (1 - \alpha) \mu_{\ell'} (\omega)} h + \sum_{h \in H} \left[ \frac{\ell' (\omega, h)}{\mu_{\ell'} (\omega)} \right] \frac{(1 - \alpha) \mu_{\ell'} (\omega)}{\alpha \mu_\ell (\omega) + (1 - \alpha) \mu_{\ell'} (\omega)} h.
\]

Let \( \frac{\alpha \mu_\ell (\omega)}{\alpha \mu_\ell (\omega) + (1 - \alpha) \mu_{\ell'} (\omega)} = \beta \), then \( J (\alpha \ell + (1 - \alpha) \ell') (\omega) = \beta J (\ell) (\omega) + (1 - \beta) J (\ell') (\omega) \), for all \( \omega \in \Omega \). Hence, by the convexity of \( F \), \( J (\alpha \ell + (1 - \alpha) \ell') = \beta J (\ell) + (1 - \beta) J (\ell') \in F \). \( \square \)

4.3 Proof of Theorem 1

The proof is modiﬁes that of the theorem of Karni and Schmeidler (1980), taking into considerations the differences in the domains of the preference relations.

(\( a \)) (Sufﬁciency) By Lemma 2, \( L (\Omega, H) \) is a convex set. Since \( \succ^* \) satisfies (A.1) - (A.4), by the expected utility theorem, there exist continuous,
non-constant, real-valued function \( u \) on \( \Omega \times H \) which is affine in its second argument and unique up to positive linear transformation, such that, for all \( \ell, \ell' \in L(\Omega, H) \)

\[
\ell \succ^* \ell' \iff \sum_{\omega \in \Omega} \sum_{h \in H} u(\omega, h) \ell(\omega, h) \geq \sum_{\omega \in \Omega} \sum_{h \in H} u(\omega, h) \ell'(\omega, h). \quad (8)
\]

Hence, (2) holds. Note that when \( \ell \) is non-degenerate, (8) can be rewritten as

\[
\ell \succ^* \ell' \iff \sum_{\omega \in \Omega} \mu_\ell(\omega) u(\omega, J(\ell)(\omega)) \geq \sum_{\omega \in \Omega} \mu_{\ell'}(\omega) u(\omega, J(\ell')(\omega)).
\]

Since \( F \) is a convex set and \( \succ \) satisfies (A.1) - (A.4), there exist continuous, non-constant, real-valued function \( v \) on \( \Omega \times H \), affine in its second argument and unique up to cardinal unit-comparable transformation, such that for all \( f_M, f_M' \in F \),

\[
f_M \succ f_M' \iff \sum_{\omega \in \Omega} v(\omega, f_M(\omega)) \geq \sum_{\omega \in \Omega} v(\omega, f_M'(\omega)). \quad (9)
\]

Fix \( \bar{h} \), an interior point of \( H \), let \( B^{\bar{h}} \subset H \) be a closed ball centered at \( \bar{h} \), denote by \( \bar{h}(\omega) \) the unique maximizer of \( \succ^\omega \) over \( B^{\bar{h}} \) and let \( M = \{ \bar{h}(\omega) \mid \omega \in \Omega \} \) be a menu consisting of these maximal points. Since \( \omega \neq \omega' \) implies \( \succ^\omega \neq \succ^\omega' \), \( \bar{h}(\omega) \neq \bar{h}(\omega') \) and each \( \bar{h}(\omega) \) has a neighborhood \( N^h_\omega \) such that, for all \( h \in N^h_\omega \) and \( \omega \neq \omega', h \succ^\omega \bar{h}(\omega') \) and \( \bar{h}(\omega') \succ^\omega h \). Hence, for all \( h \in N^h_\omega \) and \( M_\omega = \{ h \} \cup \{ h(\omega') \mid \omega' \neq \omega, \omega' \in \Omega \} \),

\[
f_{M_\omega}(\omega') = \begin{cases} h & \omega' = \omega \\ \bar{h}(\omega') & \omega' \neq \omega \end{cases}
\]

Define \( \ell_M = K(f_M) \) and consider an obviously nonnull \( \omega \in \Omega \). Let \( L_\omega \) denote the subset of lotteries in \( L(\Omega, H) \) that agree with \( \ell_M \) outside \( \omega \) and denote \( F_\omega = J(L_\omega) \). By (A.5), \( \succ^* \) restricted lotteries in \( L_\omega \) and \( \succ \) restricted to \( F_\omega \) agree (that is, for all \( \ell, \ell' \in L_\omega \), and \( J(\ell), J(\ell') \in F_\omega \)) then \( \ell \succ^* \ell' \) if and only if \( J(\ell) \succ J(\ell') \). For the restricted relations \( \succ^* \) and \( \succ \) the functions \( u(\omega, \cdot) \) and \( v(\omega, \cdot) \) constitute, respectively, von Neumann-Morgenstern utility functions on the subset of acts \( H_\omega = \{ h \in H \mid h = f_M(\omega), f_M \in F_\omega \} \). By the preceding argument, \( H_\omega \) contains the open neighborhood \( N^h_\omega \). Hence,
by the affinity and uniqueness of the von Neumann–Morgenstern utility functions \( u(\omega, \cdot) \) and \( v(\omega, \cdot) \), \( v(\omega, \cdot) = b(\omega) u(\omega, \cdot) + a(\omega) \), where \( b(\omega) > 0 \). As each \( v(\omega, \cdot) \) can be rescaled by subtracting \( a(\omega) \), we assume (without loss of generality) that \( v(\omega, \cdot) = b(\omega) u(\omega, \cdot) \). For \( \omega \) obviously null let \( b(\omega) = 0 \). Thus, \( v(\omega, h) = b(\omega) u(\omega, h) \), for all \( \omega \in \Omega \) and \( h \in H \).

By (A.4), there exist an obviously nonnull state of mind, thus, \( b(\omega) > 0 \) for some \( \omega \in \Omega \). Define \( \eta(\omega) = b(\omega) / \Sigma_{\omega' \in \Omega} b(\omega') \) and observe that, by (9),

\[
f_M \succeq f_{M'} \iff \sum_{\omega \in \Omega} \eta(\omega) [u(\omega, f_M(\omega)) - u(\omega, f_{M'}(\omega))] \geq 0.
\]

Hence, (1) holds.

(Necessity) The proof is immediate and is omitted.

(b) This is an immediate implication of the uniqueness of \( v \) in (9).

(c) Assume that all \( \omega \) are obviously nonnull and consider the menu \( \tilde{M} \) defined in part (a). Since \( \omega \) is obviously nonnull then there are \( f, f' \in F \), such that \( f \) agrees with \( f' \) outside \( \omega \), and \( f \succeq f' \). Hence, (1) implies that

\[
\eta(\omega) [u(\omega, f(\omega)) - u(\omega, f'(\omega))] > 0.
\]

Thus, \( \eta(\omega) > 0 \) for all \( \omega \).

Suppose that there exists \( \eta' \neq \eta \) that, in conjunction with \( u \), satisfy the representations in (1) and (2). We may write (1) as

\[
J(\ell) \succeq J(\ell') \iff \sum_{\omega \in \Omega} \eta(\omega) [u(\omega, J(\ell)(\omega)) - u(\omega, J(\ell')(\omega))] \geq 0
\]

\[
\iff \sum_{\omega \in \Omega} \eta'(\omega) [u(\omega, J(\ell)(\omega)) - u(\omega, J(\ell')(\omega))] \geq 0.
\]

Since \( \eta \neq \eta' \), there are \( \omega, \omega' \in \Omega \), such that \( \eta(\omega) > \eta'(\omega) \) and \( \eta(\omega') < \eta'(\omega') \). For \( p \in [0, 1] \) define

\[
\ell_p(\omega, \bar{h}(\omega)) = \eta(\omega) p, \quad \ell'_p(\omega', \bar{h}(\omega')) = \eta'(\omega') (1 - p)
\]

\[
\ell_p(\omega, \bar{h}(\omega')) = \eta(\omega) (1 - p), \quad \ell'_p(\omega', \bar{h}(\omega)) = \eta'(\omega') p
\]

\[
\ell_p(\omega', \bar{h}(\omega)) = \eta(\omega') \text{ and } \ell'_p(\omega, \bar{h}(\omega')) = \eta(\omega)
\]

and, for all \( \omega'' \in \Omega \setminus \{\omega, \omega'\} \), \( \ell_p(\omega'', \bar{h}(\omega'')) = \ell'_p(\omega'', \bar{h}(\omega'')) \). Then

\[
J(\ell_p)(\omega) = ph(\omega) + (1 - p) \bar{h}(\omega'), \quad J(\ell'_p)(\omega') = \bar{h}(\omega),
\]

23
and
\[ J (\ell_p') (\omega') = (1 - p) \tilde{h} (\omega') + p\tilde{h} (\omega), \]
and \( J (\ell_p) (\omega'') = J (\ell_p') (\omega'') \), for all \( \omega'' \in \Omega \setminus \{\omega, \omega'\} \).

By definition, \( \tilde{h} (\omega) \succ_{\omega} \tilde{h} (\omega') \) and \( \tilde{h} (\omega') \succ_{\omega} \tilde{h} (\omega) \). Hence, \( J (\ell_p) \succ J (\ell_p') \)
if and only if
\[ p\eta (\omega) \left[ u (\omega, \tilde{h} (\omega)) - u (\omega, \tilde{h} (\omega')) \right] + (1 - p) \eta (\omega') \left[ u (\omega', \tilde{h} (\omega)) - u (\omega', \tilde{h} (\omega')) \right] \geq 0 \]
if and only if
\[ p\eta' (\omega) \left[ u (\omega, \tilde{h} (\omega)) - u (\omega, \tilde{h} (\omega')) \right] + (1 - p) \eta' (\omega') \left[ u (\omega', \tilde{h} (\omega)) - u (\omega', \tilde{h} (\omega')) \right] \geq 0. \]
But \( u (\omega, \tilde{h} (\omega)) - u (\omega, \tilde{h} (\omega')) > 0 \) and \( u (\omega', \tilde{h} (\omega)) - u (\omega', \tilde{h} (\omega')) < 0 \).
Moreover, \( \eta (\omega) > 0 \) and \( \eta' (\omega') > 0 \). Thus, there exists \( \bar{p} \) such that
\[ \bar{p}\eta (\omega) \left[ u (\omega, \tilde{h} (\omega)) - u (\omega, \tilde{h} (\omega')) \right] + (1 - \bar{p}) \eta (\omega') \left[ u (\omega', \tilde{h} (\omega)) - u (\omega', \tilde{h} (\omega')) \right] = 0 \]
but then, since \( \eta (\omega) > \eta' (\omega) \) and \( \eta (\omega') < \eta' (\omega') \)
\[ \bar{p}\eta' (\omega) \left[ u (\omega, \tilde{h} (\omega)) - u (\omega, \tilde{h} (\omega')) \right] + (1 - \bar{p}) \eta' (\omega') \left[ u (\omega', \tilde{h} (\omega)) - u (\omega', \tilde{h} (\omega')) \right] < 0. \]
This is the required contradiction. \( \square \)
References


Games and Economic Behavior, 62, 329-347.