Economics Letters 196 (2020) 109535

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

On the indeterminacy of the representation of beliefs by probabilities

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ARTICLE INFO

Article history:
Received 1 July 2020
Received in revised form 30 August 2020
Accepted 31 August 2020
Available online 9 September 2020

JEL classification:
D8
D81
D83

Keywords:
Subjective probabilities
Prior beliefs
Rank-dependent utility
Probabilistic sophistication
Event exchangeability

ABSTRACT

This paper examines the indeterminacy of the representations of beliefs by subjective probabilities in non-expected utility theories of decision making under uncertainty. These theories include the rank-dependent utility models, probabilistically sophisticated choice and probabilistic sophistication based on event exchangeability.

1. Beliefs and probabilities

Bayesian statistics requires a prior to serve as a basis for the updating the probabilities of events contingent on the acquisition of information. Bayes' rule is a procedure for updating the probabilities, but is silent about the nature of the prior.

Bayesian decision theory, pioneered by Ramsey (1931) and de Finetti (1937) and culminating in the seminal work of Savage (1954), presumes that decision makers entertain beliefs regarding the likelihoods of events and that these beliefs manifest themselves in their choice behavior. Moreover, Bayesian decision theory also presumes that decision makers' beliefs can be quantified by probability measures and that these measures may be inferred from the observed patterns of choice. Consequently, the subjective probability representing a decision maker's belief is a natural Bayesian prior.

According to subjective expected utility (henceforth SEU) theory decision making under uncertainty is a process of integrating two separate cognitive subroutines – the assessment of the likelihoods of events and the evaluation of the possible consequences that may obtain in these events. Furthermore, a decision maker's assessment of the likelihoods of events, or beliefs, may be inferred from her choices and, to the extent that it is quantifiable as a probability measure, constitutes the sought after Bayesian prior. The necessary and sufficient conditions for the existence of a unique probability measure representing a decision maker's beliefs is Savage's (1954) notion of the foundations of statistics.

It is by now generally recognized that the utility function and probability measure that figure in the SEU models of Savage (1954) and of Anscombe and Aumann (1963) are *jointly unique* (that is, the probability is unique given the utility and the utility is unique, up to a positive affine transformation, given the probability). Consequently, the uniqueness of the probabilities in these theories hinges on assigning the consequences *utility values* that are independent of the underlying events. In other words, the probability that figures in SEU theory is unique provided that courses of action that yield the same consequence in every event (that is, constant acts) yield the same utility in every event (that is, constant utility acts). This convention, however, is not implied by the underlying axiomatic structure and, consequently, lacks testable implications in the context of Savage's analytical framework.[2] Put differently, *the subjective probability in the theories of Savage (1954) and of Anscombe and Aumann (1963) are arbitrary theoretical constructs devoid of choice-based meaning and, consequently, do not necessarily represent the decision maker's belief even when such belief exists and is quantified by probability.*[3] From the viewpoint of Bayesian statistics, the fact that the

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1 Excellent suggestions by an anonymous referee are gratefully acknowledge.

2 To be sure, the axiomatic structure of Savage (1954), especially his P3 and P4, and the monotonicity axiom in Anscombe and Aumann (1963) imply that the risk-attitudes are state independent. These assumptions require that shape of the utility function be state-independent, but not its values.

3 For more detailed discussion of this issue and references see Karni (2014).

https://doi.org/10.1016/j.econlet.2020.109535
0165-1765/

subjective probabilities in SEU theory are based on a convention that lacks behavioral foundations suggests that the practice of assuming a uniform prior in Bayesian statistical analysis (with a corresponding unique state-dependent utility function) is at least as compelling as, and more practical than, using the subjective probabilities implied by SEU theory.

The claim that the subjective probabilities are arbitrary, or indeterminate, is not confined to SEU theory. In fact, it applies to all models that invoke Savage's (1954) analytical framework. This claim was recently challenged by Chew and Wang (2020) who argue that "... it appears that the betweenness property may serve as the boundary for robustness of the indeterminacy problem among non-expected utility preferences". In particular, Chew and Wang assert that, unlike the models that exhibit betweenness, the representation of beliefs in the rank-dependent utility model (henceforth, RDU) is implied by the structure of the underlying preference relations. To justify their assertion, Chew and Wang consider the possibility that the utility function is state-dependent and show that, unlike in SEU or non-expected utility models with the betweenness property, in the RDU model this cannot by itself result in indeterminacy of the subjective probabilities. However, the RDU models assume that the utility function is *state and rank independent*. I argue below that this choice of a utility function is as arbitrary as the assumption that the utility function in Savage's model is state-independent. Moreover, if we consider that the utility function may be both state and rank dependent, which is not excluded by the underlying axiomatic structure, then the subjective probabilities in the model are indeterminate. It is worth reiterating that I assume throughout that the *preference relation exhibit state independence* and the issue is the representation.[4]

In addition to the RDU models I consider the probabilistically sophistication choice models of Machina and Schmeidler (1992, 1995) and the event-exchangeability based probabilistic sophistication model of Chew and Sagi (2006), and show that in these, more general, non-expected utility theories, the same indeterminacy of the subjective probability prevails. I conclude with a brief discussion of the alternative analytical frameworks that allow the identification of subjective probability that represent a decision maker's belief.

2. The indeterminacy of subjective probabilities

2.1. Subjective expected utility

Let S be a set of *states* and C an arbitrary set of *consequences.* Subsets of S are *events*.[5] Let $F := \{f : S \to C\}$, the set of mapping from S to C, constitute the choice set. Elements of F are *acts,* representing alternative courses of action. A preference relation, denoted $\succcurlyeq$, is a binary relation on F. The asymmetric and symmetric parts of $\succcurlyeq$, are denoted by $\succ$ and $\sim$, respectively. Axiomatizations of subjective expected utility theory provides necessary and sufficient conditions for the representation of $\succcurlyeq$ by an expected utility functional. Formally, for all $f, g \in F$,

$$f \succcurlyeq g \Leftrightarrow \int_S u\left(f\left(s\right)\right) d\pi\left(s\right) \geq \int_S u\left(g\left(s\right)\right) d\pi\left(s\right),$$

where u is a bounded, real-valued function on C and π is a finitely additive probability measure on the algebra 2^S of all events.

The utility function and the probability measure that figure in this representation are jointly unique in the sense described in the introduction. Specifically, π is unique given the state-independent utility function u. Thus, letting $\gamma : S \to \mathbb{R}_{++}$ such that $\int_S \gamma\left(s\right) d\pi\left(s\right) = 1$, and defining $\hat{u}\left(x, s\right) = u\left(x\right)/\gamma\left(s\right)$ and $\hat{\pi}\left(E\right) = \gamma\left(E\right)\pi\left(E\right)$, for all $x \in C$ and $E \subset S$, we get an alternative representation as follows: For all $f, g \in F$,

$$f \succcurlyeq g \Leftrightarrow \int_S \hat{u}\left(f\left(s\right), s\right) d\hat{\pi}\left(s\right) \geq \int_S \hat{u}\left(g\left(s\right), s\right) d\hat{\pi}\left(s\right).$$

Since there is nothing in the structure of the preference relation that requires the utility function, as opposed to the preference relation, to be state independent, there is no way of deciding, based on choice behavior, whether π or $\hat{\pi}$ or any other probability measure on S is the correct representation of the decision maker's belief.

Chew and Wang (2020) showed that essentially the same argument can be applied to non-expected utility models that display betweenness.[6] Next I examine their assertion that the indeterminacy does not extend to rank-dependent utility theory.

2.2. Rank-dependent utility

The RDU models – Quiggin's (1982) Anticipated Utility and Yaari's (1987) Dual Theory as well as the general RDU model of Chew (1989a) – are theories of choice under risk (that is, the domain of the preference relation is the set, $\Delta\left(C\right)$, of probability distributions on C). According to these models a preference relation, $\succcurlyeq$ on $\Delta\left(C\right)$, is represented by $p \mapsto \int_C u\left(x\right) d\left(g \circ p\right)\left(x\right)$, where u is a real-valued function on C and $g : [0, 1] \to [0, 1]$ is nondecreasing, continuous and onto function, dubbed *probability transformation function*.

To address the issue of indeterminacy of subjective probability in this context it is necessary, therefore, to convert the RDU models to theories of decision making under uncertainty. Wakker (1990) accomplished this conversion by invoking a capacity of the Choquet expected utility (henceforth, CEU) model of Schmeidler (1987) to induce a probability transformation function in the RDU model. More formally, according to the CEU model a preference relation, $\hat{\succcurlyeq}$ on F, is represented by $f \mapsto \int_S u\left(f\left(s\right)\right) d\varphi\left(s\right)$, where u is a real-valued function on C and $\varphi : 2^S \to [0, 1]$ is a capacity measure. Given a preference relation $\hat{\succcurlyeq}$ on F that has a CEU representation, define a binary relation $\succcurlyeq^*$ on 2^S as follows: $A \succcurlyeq^* B$ if and only if $\varphi\left(A\right) \geq \varphi\left(B\right)$. Suppose that a decision maker whose preference relation $\hat{\succcurlyeq}$ on F has beliefs that are represented by a finitely-additive subjective probability measure, π on 2^S, such that $A \succcurlyeq^* B$ if and only if $\pi\left(A\right) \geq \pi\left(B\right)$. Thus, $\varphi\left(A\right) \geq \varphi\left(B\right)$ if and only if $\pi\left(A\right) \geq \pi\left(B\right)$. Therefore, there exists strictly increasing and onto probability transformation function $g : [0, 1] \to [0, 1]$ defined by $\varphi\left(E\right) = g\left(\pi\left(E\right)\right)$, for all $E \in 2^S$.[7] Then

$$\int_S u\left(f\left(s\right)\right) d\varphi\left(s\right) = \int_S u\left(f\left(s\right)\right) dg\left(\pi\left(s\right)\right).$$

But $\int_S u\left(f\left(s\right)\right) dg\left(\pi\left(s\right)\right) = \int_C u\left(x\right) dg\left(\pi\left(f^{-1}\left(x\right)\right)\right)$, for all $f \in F$. Let $p\left(x\right) := \pi\left(f^{-1}\left(x\right)\right)$, for all $x \in C$, then

$$\int_C u\left(x\right) dg\left(\pi\left(f^{-1}\left(x\right)\right)\right) = \int_C u\left(x\right) d\left(g \circ p\right)\left(x\right).$$

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4 RDU models that admit state-dependent preferences are discussed in Chiu (1996) and Chew and Wakker (1996).

5 The nature of these sets depend on the specific model. In Savage's (1954) theory the set of states is infinite and consequences are arbitrary. In the model of Anscombe and Aumann (1963) the set of states is finite and consequences are lotteries on an arbitrary set of outcomes.

6 If F is a convex set in a linear space then a preference relation $\succcurlyeq$ on F is said to display betweenness if for all $f, g \in F$, $f \succ g$ implies $f \succ \alpha f + (1-\alpha) g \succ g$, for all $\alpha \in (0, 1)$. A review of the models that display betweenness is given in Chew (1989b).

7 Wakker (1990) Lemma 6 gives necessary and sufficient condition for the existence of such probability transformation function.

Therefore, we have a RDU model based on a subjective probability measure π on S.

Consider next the claim of Chew and Wang (2020). Let the act $x_E y \in F$ deliver the monetary payoffs x and y in the events E and it's complement, E^c, respectively, (that is, $(x_E y)(s) = x$ if $s \in E$ and $(x_E y)(s) = y$ if $s \in E^c$) assume that the utility function $u(x) = x$.[8]

If $x < y$ then the RDU representation is:

$$z = xg(\pi(E)) + y(1 - g(\pi(E))).$$

Let $\gamma : 2^S \to \mathbb{R}_{++}$ be a function, then

$$z = \frac{x}{\gamma(E)}\gamma(E)\, g(\pi(E)) + \frac{y}{\gamma(E^c)}\gamma\left(E^c\right)(1 - g(\pi(E))).$$

Choose any probability measure $\hat{\pi}$ on S such that $\hat{\pi}(E) > 0$ if and only if $\pi(E) > 0$. Define $\hat{g}\left(\hat{\pi}(E)\right)$ and γ by the equation

$$\hat{g}\left(\hat{\pi}(E)\right) = \frac{\gamma(E)\, g(\pi(E))}{\gamma(E)\, g(\pi(E)) + \gamma(E^c)(1 - g(\pi(E)))}, \forall E \in 2^S.$$

Note that few restrictions are imposed on γ. In particular, it is easy to verify that if there are events E_1 and E_2 such that $\hat{\pi}(E_1) = \hat{\pi}(E_2)$ and $\pi(E_1) \neq \pi(E_2)$ then γ must satisfy

$$\frac{\gamma\left(E_1^c\right)}{\gamma(E_1)}\left(1 - \frac{1}{g(\pi(E_1))}\right) = \frac{\gamma\left(E_2^c\right)}{\gamma(E_2)}\left(1 - \frac{1}{g(\pi(E_2))}\right).$$

Let

$$K\left(\pi(E), \gamma(E), \gamma\left(E^c\right)\right) := \gamma(E)\, g(\pi(E)) + \gamma\left(E^c\right)(1 - g(\pi(E)))$$

and define

$$u(x, E) := \frac{K_E(\pi(E), \gamma(E), \gamma(E^c))}{\gamma(E)} x \text{ and}$$

$$u\left(y, E^c\right) := \frac{K_E(\pi(E), \gamma(E), \gamma(E^c))}{\gamma(E^c)} y.$$

Then the representation may be written as

$$z = u(x, E)\, \hat{g}\left(\hat{\pi}(E)\right) + u\left(y, E^c\right)\left(1 - \hat{g}\left(\hat{\pi}(E)\right)\right),$$

is a state and rank dependent utility function that is linear in the monetary payoff.

Consider next the case $x > y$. Let

$$z := x(1 - g(1 - \pi(E))) + yg(1 - \pi(E)).$$

define $\hat{g}\left(1 - \hat{\pi}(E)\right)$ by

$$\hat{g}\left(1 - \hat{\pi}(E)\right) = \frac{\gamma(E^c)\, g(1 - \pi(E))}{\gamma(E)(1 - g(1 - \pi(E))) + \gamma(E^c)\, g(1 - (1 - \pi(E)))}.$$

Let

$$\hat{K}\left(\pi(E), \gamma(E), \gamma\left(E^c\right)\right) := \gamma(E)(1 - g(1 - \pi(E))) + \gamma\left(E^c\right) g(1 - \pi(E)),$$

and

$$\hat{u}(x, E) := \frac{\hat{K}(\pi(E), \gamma(E), \gamma(E^c))}{\gamma(E)} x \text{ and}$$

$$\hat{u}\left(y, E^c\right) := \frac{\hat{K}(\pi(E), \gamma(E), \gamma(E^c))}{\gamma(E^c)} y.$$

Then the representation may be written as

$$z = \hat{u}(x, E)\left(1 - \hat{g}\left(1 - \hat{\pi}(E)\right)\right) + \hat{u}\left(y, E^c\right)\hat{g}\left(1 - \hat{\pi}(E)\right).$$

[8] While this assumption entails no loss of generality, strictly speaking, this is Yaari's (1987) Dual Theory.

Thus, if the utility is state and rank dependent then the π prior is arbitrary. Chew and Wang's (2020) assertion of determinacy is based on the implicit assumption that the utility function may be state-dependent but not rank-dependent. In this case, the probability transformation function must satisfy $g(r) + g(1 - r) = 1$, for all $r \in [0, 1]$.[9] However, nothing in the structure of the RDU preferences requires that the utility function be rank-independent. In other words, the choice of a representation in which the utility function is state and rank independent is a convention that is not implied by the underlying preference relation. Consequently, the same indeterminacy that inflicts the SEU model is present in RDU model. Baccelli (2019) made a similar claim that applies to situations in which the probabilities may be act-dependent. He showed that in such situations, if state-dependent utility functions are admissible but act-dependent utilities are not then the indeterminacy issue is resolved. However, in view of the fact that the restriction of the utility function to be act-independent is not implied by the underlying axiomatic structure, the issue of identifiably of the subjective probability representation of the decision maker's beliefs remains unresolved.

More generally, let $S = \{s_1, \ldots, s_n\}$ and C be an interval in $\mathbb{R}$ representing monetary payoffs. Consider $f \in F$, such that, without loss of generality, $f(s_1) = x_1 < \ldots < f(s_n) = x_n$. Given a probability measure π on S, define the cumulative distribution function $H(\cdot \mid f)$ on C corresponding to f as follows: $H(x_i \mid f) = \pi\{s \mid f(s) \leq x_i\}$. Then, the RDU representation of a preference relation $\succcurlyeq$ on F is give by: $f \to \Sigma_{i=1}^n u(x_i)\, g(H(x_i \mid f) - H(x_{i-1} \mid f))$. In this representation, the utility function is state and rank independent.

Consider next another probability measure, $\hat{\pi}$ on S, such that $\hat{\pi}(s) > 0$ if and only if $\pi(s) > 0$. Let $\hat{H}(x_i \mid f) = \hat{\pi}\{s \mid f(s) \leq x_i\}$. Let $\gamma : S \to \mathbb{R}_{++}$ and define

$$\hat{g}\left(\hat{H}(x_i \mid f) - \hat{H}(x_{i-1} \mid f)\right) = \frac{\gamma(s_i)}{r(x_i)} g(H(x_i \mid f) - H(x_{i-1} \mid f)),$$

where $\gamma(s_i) := \gamma\left(H^{-1}(x_i \mid f) - H^{-1}(x_{i-1} \mid f)\right)$ and

$$r(x_i) = \Sigma_{i=1}^n \gamma\left(H^{-1}(x_i \mid f) - H^{-1}(x_{i-1} \mid f)\, g(H(x_i \mid f) - H(x_{i-1} \mid f))\right).$$

Define state and rank dependent utility function by: $\hat{u}(x_i, s_i, r(x_i)) = r(x_i)\, u(x_i) / \gamma(s_i)$. Then the following is an equivalent RDU representation of $\succcurlyeq$ on F,

$$f \to \Sigma_{i=1}^n \hat{u}(x_i, s_i, r(x_i))\, \hat{g}\left(\hat{H}(x_i \mid f) - \hat{H}(x_{i-1} \mid f)\right).$$

Since $\hat{\pi} \neq \pi$, the underlying subjective probability is indeterminate.[10]

The indeterminacy of the subjective probability concerns a formal, choice-based, distinction between probabilities and utilities. As such it does not concern the interpretation and relevance of the alternative equivalent representations and, in particular, the meaning of state and rank dependent utility functions. However, state-dependent utility functions are natural and easy to justify and interpret. For example, in decisions involving health insurance it is intuitively obvious that the utility of the indemnities depend on the underlying state of the decision maker's health. By contrast, the meaning of rank-dependent utility in not obvious. The following example illustrates its relevance. Consider a

[9] Since this implies that $g(0.5) = 0.5$. Thus, the indeterminacy induced by state-dependent rank-independent utility is confined to the Anticipated Utility model of Quiggin (1982).

[10] Because the ranks of the consequences is determined by the underlying preference relation, they are preserved under the rank and state dependent utility functions.

decision maker who places a bet on the outcome of a baseball game. If the decision maker is a fan of one of the teams his utility of the winning prize may depend on who the winner is (e.g., he may enjoy the winning when "his" team wins). This is a case of state-dependent utility. Suppose next that the decision maker is indifferent with regard to which team wins and all he cares about is winning. In this instance, the utility of the winning prize, while it obviously depends on the size of the prize, may also reflect the "pride of winning" (i.e., the extra pleasure of predicted the outcome correctly). In this instance the utility is rank-dependent but not state dependent. In either case, the interpretation of the utility of the prize captures, in addition to the prize itself, the emotional response to winning.

2.3. Probabilistic sophistication

Probabilistic sophistication describes models in which subjective probabilities are defined independently of the functional forms of the representations of the preference relations. The literature offers two alternative formulations of the idea of probabilistic sophistication. The first, due to Machina and Schmeidler (1992, 1995), who coined the term, and Grant (1995), is an axiomatic modeling of choice under uncertainty. The second, due to Chew and Sagi (2006), is founded on the idea of event exchangeability. This approach dispenses of some of the restrictions the models of Machina and Schmeidler and Grant.

The question that I address next is whether the subjective probabilities defined in the theories of probabilistic sophistication necessarily represent decision makers' beliefs.

2.3.1. Probabilistically sophisticated choice

Consider the definition of subjective probability in the probabilistically sophisticated choice (henceforth PSC) theory. Let $x_E y \in F$ and suppose that $x > y$. According to Machina and Schmeidler (1995) the probabilities of the events E and E^c, $\pi(E)$ and $1-\pi(E)$, respectively, are defined by the equivalence $x_E y \sim \pi(E)\delta_x + (1-\pi(E))\delta_y$, where $\delta_x \in \Delta C$ assigns the outcome x the unit probability mass. Assuming that the preference relation on the subset of constant acts, ΔC, is monotonic with respect to first-order stochastic dominance $\pi(E)$ is well defined. The question is, does this probability necessarily represent the decision maker's belief of the likelihood the E obtains?

To begin with, recall that both the SEU and the RDU models are special cases of PSC. Moreover, the issue at hand is unrelated to the whether or not the preference relation satisfies the sure thing principle suggesting that the probabilities defined in the PSC models do not necessarily represent the decision maker's beliefs. Perhaps the most compelling argument of why this definition is unsatisfactory is an example given by Robert Aumann in his correspondence with Leonard Savage.[11] In his letter, Aumann describes a man whose life without his beloved wife is "less 'worth living'". The wife falls ill and to survive she must undergo a routine yet dangerous operation. The husband is offered a choice between betting \$100 on his wife's surviving the surgery or betting \$100 on the outcome of a coin flip. Even if the husband believes that his wife has an even chance of surviving the operation, he may still rather bet on her survival. To grasp this note that if the husband bets on the outcome of a coin flip he might win but, in the event that the wife does not survive, the prize is 'somehow worthless' as he will not be able to enjoy it.

Let E be the event "the wife survives", $x = \$100$ and $y = \$0$. Then $x_E y$ is a bet on the wife surviving the operation, and $0.5\delta_x + 0.5\delta_y$ is a bet on the outcome of the coin flip. Even if the husband believes that $\pi(E) = 0.5$, his choice indicates that $x_E y \succ 0.5\delta_x + 0.5\delta_y$. By first-order stochastic dominance, there is $\mu \in (0.5, 1]$ such that $x_E y \sim \mu\delta_x + (1-\mu)\delta_y$. Thus, μ is the subjective probability as defined by the PSC model. Clearly, this does not represent the husband's belief.

More generally, let $E_1, \ldots, E_n$ be a partition of S and $f \in F$. By the representation of $\succcurlyeq$ in PSC theory $V(f) = V\left(\Sigma_{i=1}^n \mu(E_i) f(E_i)\right)$. Let $\pi(E_i) := \mu(E_i)\gamma(E_i)$, where $\gamma : 2^S \to \mathbb{R}_{++}$ and $\Sigma_{i=1}^n \mu(E_i)\gamma(E_i) = 1$. Define $\hat{V}\left(\Sigma_{i=1}^n \pi(E_i) f(E_i), \gamma\right) := V\left(\Sigma_{i=1}^n \mu(E_i) f(E_i)\right)$. Then, $f \longmapsto \hat{V}\left(\Sigma_{i=1}^n \pi(E_i) f(E_i), \gamma\right)$ is another representation of $\succcurlyeq$. The indeterminacy of the subjective probability extends to the probability sophisticated choice model.

2.3.2. Probabilistic sophistication based on event exchangeability

Chew and Sagi (2006) propose an alternative approach to defining subjective probabilities based on event exchangeability. Building on the idea of ethical neutrality, first proposed by Ramsey (1931), they define any two disjoint events E and E' to be *exchangeable* if, for any consequences $x, y \in C$ and $f \in F$, it holds that $x_E y_{E'} f \sim y_E x_{E'} f$, where $x_E y_{E'} f$ denotes the act in F given by $(x_E y_{E'} f)(s) = x$ if $s \in E$, $(x_E y_{E'} f)(s) = y$ if $s \in E'$ and $(x_E y_{E'} f)(s) = f(s)$, otherwise. Presumably, exchangeable events are believed to be equally likely to obtain.

The definition of exchangeability is based on the tacit assumption that the valuations of *all* the consequences are independent of the underlying events. This, in itself, is absurd in many situations as illustrated in the example of Aumann above. Less dramatic, yet equally compelling, is the following example. Let S denote a range of temperatures, say between 0 and 50 degrees Celsius, and suppose that a decision maker, planning a vacation in Bermuda in May, believes that the likelihood of temperature in the range $[5, 10] = E$ is the same as that in the range $[48, 50] = E'$. If the set of consequences includes a bathing suit, x, and a warm coat, y, it is obviously farfetched to suppose that E and E' are exchangeable. Event exchangeability, according to this definition, is not a necessary condition for two events to be conceived as equally likely to obtain.

A different question is whether event exchangeability is a sufficient condition to infer that a decision maker consider exchangeable events equally likely to obtain? To answer this question consider the following simple example. A baseball game between the home team and a visitor team is to take place tomorrow. Let E and E^c denote, respectively, the events "the home team wins" and "the visitor team wins". Suppose that these events are exchangeable. Specifically, let the consequences be monetary payoffs and assume that, for all $x, y \in \mathbb{R}$, $x_E y \sim y_E x$. According to the event exchangeability approach, E and E^c are considered to be equally likely to obtain.

Consider next a fan of the home team who believes that the home team's chance of winning is 1/3. Suppose that the fan is a subjective expected utility maximizer and his utility from winning a bet on the event that the home team wins is twice that of winning the same amount of money when betting that the home team loses. Formally, $u(x; E) = 2u(x, E^c)$, for all $x \in \mathbb{R}$, where u is a von Neumann-Morgenstern utility function. Since E and E^c are exchangeable, we have

$$\frac{1}{3}\left[u(x, E) - u(y, E)\right] + \frac{2}{3}\left[u\left(x, E^c\right) - u\left(y, E^c\right)\right] = 0.$$

Hence,

$$\frac{1}{3}2\left[u\left(x, E^c\right) - u\left(y, E^c\right)\right] = \frac{2}{3}\left[u\left(x, E^c\right) - u\left(y, E^c\right)\right].$$

Even though the events are exchangeable, the conclusion that the decision maker believes that they are equally likely is obviously false. The idea illustrated by this example can be stated more generally. If the implicit valuations of the consequences contingent

[11] The correspondence between Aumann and Savage is reproduced in Drèze (1987) and in the collected writings of Aumann (2000).

on any pair of exchangeable events are rescaling of one another, then the conclusion that the decision maker believes that the events are equally likely to obtain is false.

To conclude, *event exchangeability is neither necessary nor sufficient condition to conclude that a decision maker believes that the exchangeable events are equally likely to obtain.* In other words, even-exchangeability induced subjective probabilities do not necessarily represent decision makers' beliefs.

3. Concluding remarks

In this paper I argue that the subjective probabilities defined in the analytical framework of Savage (1954) depends on an arbitrary choice of a utility function and, consequently, do not necessarily represent the decision maker, or statistician's, belief. The analysis indicates that the problem is the limitation of the analytical framework and not the specific structures of the preference relation.

A more satisfactory approach in the context of the revealed preference methodology require the extension of the analytical framework as in Karni (2011a,b). Alternatively, if verbal expression of preference relations $\succsim$ on the set $\Delta(S \times C)$ of hypothetical state-consequence lotteries is admitted, then the model of Karni and Schmeidler (2016) allows a definition of unique subjective probabilities that represent the decision maker's beliefs. To grasp this claim, consider again the bet $x_E y$ and the state-consequence degenerate lotteries, $\delta_{E,x}$ be defined by $\delta_{E,x}(s,z) = 1$ if $z = x$ and $s \in E$ and $\delta_{E,x}(s,z) = 0$, otherwise. Define $\eta(E)$ by $x_E y \sim \eta(E)\,\delta_{E,x} + (1-\eta(E))\,\delta_{E^c,y}$. Note that this definition is immune to the criticism of Aumann as the payoff $x = \$100$ obtains only in the event E, that the wife survives.

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