

# Objective and Subjective Expected Utility with Incomplete Preferences

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## Abstract

This paper extends the expected utility models of decision making under risk and under uncertainty to include incomplete beliefs and tastes. The main results are two axiomatizations of the multi-prior expected multi-utility representations of preference relation under uncertainty, thereby resolving long standing open questions. The Knightian uncertainty model and expected multi-utility model with complete beliefs are obtained as special cases. In addition, the von Neumann-Morgenstern expected utility model with incomplete preferences is revisited using a “constructive” approach, as opposed to earlier treatments that use convex analysis.

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**Keywords:** Incomplete preferences, Knightian uncertainty, Multi-prior expected multi-utility representations, Incomplete beliefs, Incomplete tastes.

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# 1 Introduction

Facing a choice between alternatives that are not fully understood, or are not readily comparable, decision makers may find themselves unable to express preferences for one alternative over another, or to choose between such alternatives in a coherent manner. This problem was recognized by von Neumann and Morgenstern who stated that “It is conceivable - and may even in a way be more realistic - to allow for cases where the individual is neither able to state which of two alternatives he prefers nor that they are equally desirable.” (von Neumann and Morgenstern [1947] p. 19).<sup>1</sup> Aumann goes further when he says “Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from a normative viewpoint.” (Aumann [1962], p. 446). In the same vein, when discussing the axiomatic structure of what became known as the Choquet expected utility theory, Schmeidler says “Out of the seven axioms listed here the completeness of the preferences seems to me the most restrictive and most imposing assumption of the theory.” (Schmeidler [1989] p.576).<sup>2</sup> A natural way of accommodating such situations while maintaining the other aspects of the theory of rational choice is to relax the assumption that the preference relations are complete.

The objective of studying the representations of incomplete preferences under uncertainty is to identify preference structures on the set of acts that admit multi-prior expected multi-utility representation. Specifically, an act,  $f$ , is preferred over another act,  $g$ , if and only if there is a nonempty set,  $\Phi$ , of pairs  $(\pi, U)$  consisting of a probability measure,  $\pi$ , on the set of states,  $S$ , and affine, real-valued function,  $U$ , on the set,  $\Delta(X)$ , of simple probability

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<sup>1</sup>Later von Neumann and Morgenstern add “We have to concede that one may doubt whether a person can always decide which of two alternatives ... he prefers.” (von Neumann and Morgenstern [1947] p. 28-29). In a letter to H. Wold, dated October 28, 1946, von Neumann discusses the issue of complete preferences. Among others, he says “The general comparability of utilities, i.e., the completeness of their ordering by (one person’s) subjective preferences, is, of course, highly dubious in many important situations.” (Redei (2005)).

<sup>2</sup>Schmeidler goes as far as suggesting that the main contributions of all other axioms is to allow the weakening of the completeness assumption. Yet, he maintains this assumption in his theory.

measures on the set,  $X$ , of outcomes such that

$$\sum_{s \in S} \pi(s) U(f(s)) \geq \sum_{s \in S} \pi(s) U(g(s)), \text{ for all } (\pi, U) \in \Phi. \quad (1)$$

Aumann (1962) was the first to address this issue in the context of expected utility theory under risk (that is, the case in which  $\pi$  in the representation above is a degenerate probability measure that assigns the entire probability mass to one state). Aumann showed that a transitive and reflexive preference relation that satisfies the independence axiom and a weak form of continuity but not the completeness axiom, which "... excludes the possibility that an individual may be willing and able to arrive at a preference decisions only for certain pairs of lotteries, while for others he may be unwilling or unable to arrive at a decision." (p. 446) one lottery is (weakly) preferred over another only if its expected utility is greater for a set of von Neumann-Morgenstern utility functions. If the completeness axioms is added, then set of utility functions reduces to a singleton. More recently, Dubra, Maccheroni and Ok (2004) studied the existence and uniqueness properties of the representations of preference relations over lotteries whose domain is a compact metric space. They show that preference relations whose structures are depicted by the axioms of expected utility theory without the completeness axiom are fully characterized by expected multi-utility representation. Moreover, the utility functions that figure in the representation are unique in the sense that any other expected multi-utility representation of the same preference relation must span a cone whose closure is the same as that of the original representation with possible shifts resulting from adding constant functions.

The issue of incomplete preferences in the context of decision making under uncertainty was first addressed by Bewley (2002), who axiomatized what he refers to as Knightian uncertainty. Bewley's analysis invokes the Anscombe-Aumann (1963) model while departing from the assumption that the preference relation is complete.<sup>3</sup> In Bewley's model, the incompleteness of the preference relations is due, solely, to the incompleteness of beliefs. This incompleteness is represented by a closed convex set of probability measures on the set of states. Accordingly, one act is preferred over another (or the status quo) if its associated subjective expected utility exceeds that of the

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<sup>3</sup>Bewley's original work, under the same title, first appeared as a Cowles Foundation discussion paper no. 807, in (1986).

alternative (or the status quo) according to every probability measures in the set. In terms of the representation (1), Bewley’s work correspond to  $\Phi = \Pi \times \{U\}$ , where  $\Pi$  is a closed convex set of probability measures on the set of states and  $U$  is a von Neumann-Morgenstern utility function.<sup>4</sup>

As in the von Neumann-Morgenstern theory, the incompleteness in the Anscombe-Aumann model may also be due to incomplete tastes. The issue of representation of preference relations that accommodates incompleteness of both beliefs and tastes was studied by Seidenfeld, Schervish, and Kadane (1995), Nau (2006) and, more recently, by Ok, Ortoleva and Riella (2008). Seidenfeld et. al. (1995) axiomatized the case in which the representation entails  $\Phi = \{(\pi, U)\}$ . Ok et. al. (2008) axiomatized a preference structure in which the source of incompleteness is either beliefs or tastes, but not both. Again, in terms of the representation (1), Ok et. al. (2008) axiomatized the cases in which  $\Phi = \Pi \times \{U\}$  or  $\Phi = \{\pi\} \times \mathcal{U}$ .

The axiomatization of incompleteness in both beliefs and tastes (that is, the cases in which the representation (1) involves the sets  $\Phi = \{(\pi, U) \mid U \in \mathcal{U}, \pi \in \Pi^U\}$  or  $\Phi = \Pi \times \mathcal{U}$  remained open questions. In this paper we address these questions and propose axiomatizations of both type of representations. In addition, we introduce new axiomatizations of Knightian uncertainty and of preference relations characterized by complete beliefs and uncertain tastes.<sup>5</sup>

Most of the studies mentioned above use convex analysis as the main analytical tool.<sup>6</sup> In this paper, we revisit the problem using a “constructive” approach, which makes the representation more transparent and easier to understand. To begin with, we obtain a representation of incomplete, von Neumann-Morgenstern, preferences over convex subsets of linear spaces that have a greatest element (that is, an element that is strictly preferred to every other element of the set) and a smallest element (that is, an element that every other element of the set is strictly preferred to it). We note that expected multi-utility representations in the case of risk and additively separable multi-utility representation under uncertainty are obtained

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<sup>4</sup>Seidenfeld et. al.(1995) regard the study of incomplete preferences under uncertainty, and that of multi-prior representations, as motivated, in part, by the interest in robust Bayesian statistics.

<sup>5</sup>Ok et. al. (2008) give an example showing that it is impossible to obtain an identification of incomplete tastes and beliefs in their model. We examine their example in subsection 4.3 below and show why it does not apply to our model.

<sup>6</sup>The exception of Seidenfeld et. al. (1995) who use transfinite induction.

as corollaries of this result.

Our main results consist of necessary and sufficient conditions characterizing the preference structures that admit the aforementioned, multi-prior expected multi-utility representations. The first set of conditions includes the familiar von Neumann-Morgenstern axioms without the completeness axiom. To these we add a dominance axiom, a la Savage's Postulate P7. Specifically, let  $g$  and  $f$  be any two acts and denote by  $f^s$  the constant act whose payoff is  $f(s)$  in every state. Then the axiom requires that if  $g$  is strictly preferred over  $f^s$ , for every  $s$ , then  $g$  be strictly preferred over  $f$ . With this axiom we obtain a representation as in (1) in which the set  $\Phi$  consists of probability-utility pairs,  $(\pi, U)$  such that  $U$  is an element of a convex set,  $\mathcal{U}$ , of affine, real-valued utility functions on the set of simple lotteries,  $\Delta(X)$ , and  $\pi$  is an element of a corresponding convex set,  $\Pi^U$ , of probability measures on  $S$ . The second result involves an axiom belief consistency. To state this axiom we first identify a set,  $\mathcal{M}$ , of all the probability measures on  $S$  that may be involved in the evaluation of acts. Belief consistency requires that if one act, say  $f$ , is strictly preferred over another act,  $g$ , then every constant act obtained by reduction of  $f$  under every compound lottery involving an element of  $\mathcal{M}$  be preferred over the corresponding reduction of  $g$ . The representation in this case is as in (1) above in which the set  $\Phi$  is a product set  $\mathcal{M} \times \mathcal{U}$ , where  $\mathcal{M}$  is a closed and convex set of probability measures on  $S$  and  $\mathcal{U}$  is as above.

Knightian uncertainty and expected multi-utility representation with complete beliefs, are obtained as special cases. The first involves completeness of the conditional (on the states) preference relations, and second a formulation of a new behavioral postulate depicting the completeness of beliefs.<sup>7</sup> By and large, our analysis pertains to decision problems involving choice sets whose elements are either simple lotteries or acts whose consequences are simple lotteries.

The remainder of the paper is organized as follows: In the next section we study the von Neumann-Morgenstern theory without the completeness axiom. Following that, in Section 3 we present our main results. In section 4 we discuss several implications of the main results as well as the significance of the preferential boundedness. of the choice set. The special cases, Knightian uncertainty and its dual, the subjective expected multi-utility model

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<sup>7</sup>Ok et. al. (2008) regard the absence of such formulation as a possible explanation for lack of attention to this case in the literature.

with complete beliefs, are discussed in Section 5. Further discussion and concluding remarks appear in Section 6. The proofs are collected in Section 7.

## 2 The von Neumann-Morgenstern Theory without the Completeness Axiom

### 2.1 The analytical framework and the preference structure

Let  $C$  be a convex subset of a finite dimensional linear space,  $\mathcal{L}$ . A *preference relation* is a binary relation on  $C$  denoted by  $\succ$ . The set  $C$  is said to be  $\succ$ -*bounded* if there exist  $p^M$  and  $p^m$  in  $C$  such that  $p^M \succ p \succ p^m$ , for all  $p \in C - \{p^M, p^m\}$ .

Consider the following axioms depicting the structure of  $\succ$ .

- (A.1) (**Strict partial order**) The preference relation  $\succ$  is transitive and irreflexive.
- (A.2) (**Archimedean**) For all  $p, q, r \in C$ , if  $p \succ q$  and  $q \succ r$  then  $\beta p + (1 - \beta)r \succ q$  and  $q \succ \alpha p + (1 - \alpha)r$  for some  $\alpha, \beta \in (0, 1)$ .
- (A.3) (**Independence**) For all  $p, q, r \in C$  and  $\alpha \in (0, 1]$ ,  $p \succ q$  if and only if  $\alpha p + (1 - \alpha)r \succ \alpha q + (1 - \alpha)r$ .

The difference between the preference structure above and that of expected utility theory is that the induced relation  $\neg(p \succ q)$  is reflexive but not necessarily transitive (hence it is not necessarily a preorder). Moreover, it is not necessarily complete. Thus,  $\neg(p \succ q)$  and  $\neg(q \succ p)$  does not imply that  $p$  and  $q$  are indifferent (i.e., equivalent), rather they may be *noncomparable*. If  $p$  and  $q$  are noncomparable we write  $p \bowtie q$ .

**Definitions 1:** For all  $p, q \in C$ , (a)  $p \succcurlyeq q$  if  $r \succ p$  implies  $r \succ q$ , for all  $r \in C$ , (b)  $p \sim q$  if  $p \succcurlyeq q$  and  $q \succcurlyeq p$ ; and (c)  $p \succeq q$  if  $p \succcurlyeq q$  and  $\neg(p \succ q)$ .

Note that  $\succ$  is not the asymmetric part of  $\succeq$  because of the way  $\succeq$  is defined here.

If  $\succ$  satisfies (A.1)-(A.3) then the derived binary relation  $\succsim$  on  $C$  is a weak order (that is, transitive and reflexive) satisfying the Archimedean and independence axioms that is not necessarily complete.<sup>8</sup> The indifference relation,  $\sim$ , that is, the symmetric part of  $\succsim$ , is an equivalence relation.<sup>9</sup>

Taking the strict preference relation,  $\succ$ , as primitive, it is customary to define the weak preference relations as the negation of  $\succ$ . Formally, given a binary relation  $\succ$  on  $C$ , define a binary relation  $\succsim$  on  $C$  by:  $p \succsim q$  if  $\neg(q \succ p)$ .<sup>10</sup> If the strict preference relation,  $\succ$ , is transitive and irreflexive, then the weak preference relation is complete. Karni (2010a) shows that the weak preference relation, in Definitions 1 agrees with the customary definition if and only if the latter is complete.

The standard practice in decision theory is to take the weak preference relation as primitive and define the strict preference relation as its asymmetric part. Invoking the standard practice, Dubra (2010), showed that if  $C$  is the set of lotteries on a finite set of prizes and the weak preference relation is nontrivial (that is,  $\succ \neq \emptyset$ ) and satisfies (A.3), then any two of the following axioms implied the third, completeness, Archimedean, and mixture continuity.<sup>11</sup> Thus, a nontrivial, partial, preorder satisfying independence must fail to satisfy one of the continuity axioms. Karni (2010a) showed that, if the weak preference relation is as in Definitions 1, then a nontrivial preference relation may satisfy independence, Archimedean, mixture continuity and yet be incomplete. Hence, the approach taken here seems more natural for modeling incomplete preferences as an extension of choice theory with complete preferences.

For every  $p \in C$ , let  $B(p) := \{q \in C \mid q \succ p\}$  and  $W(p) := \{q \in C \mid p \succ q\}$  denote the upper and lower contour sets of  $p$ , respectively. The relation  $\succ$  is convex if the upper contour set is convex. Also, note that  $\succ$ -boundedness of  $C$  implies  $B(p)$  and  $W(p)$  have non empty algebraic interior in the linear space generated by  $C$ .

**Lemma 1:** Let  $\succ$  be a binary relation on  $C$ . If  $\succ$  satisfies (A.1), (A.2) and (A.3) then it is convex. Moreover, the lower contour set is also convex.

<sup>8</sup>The proof of the claim about the independence is part of the proof of Theorem 1, below. Note that  $\succ$  is not the asymmetric part of  $\succsim$ .

<sup>9</sup>Derived weak orders, close in spirit to Definitions 1, based on a pseudo-transitive weak order appear in Chateauneuf (1987).

<sup>10</sup>See for example Chateauneuf (1987) and Kreps (1988).

<sup>11</sup>A weak preference relation satisfies mixture continuity if, for all  $p, q, r \in \Delta(X)$  the sets  $\{\alpha \in [0, 1] \mid \alpha p + (1 - \alpha)q \succsim r\}$  and  $\{\alpha \in [0, 1] \mid r \succsim \alpha p + (1 - \alpha)q\}$  are closed.

The proof is by two applications of (A.3).<sup>12</sup>

## 2.2 Simple examples

The following examples illustrate some of the ideas expounded on below, and provide simple geometric interpretations. Let  $C = \{p \in [0, 1]^3 \mid \sum_{i=1}^3 p_i = 1\}$  be the two-dimensional simplex in  $\mathbb{R}^4$  and  $\succ$  a preference relation on  $C$ .<sup>13</sup> Suppose that  $C$  is  $\succ$ -bounded and let  $\delta_{x_3}$  and  $\delta_{x_1}$  be the  $\succ$  greatest and smallest elements of  $C$  (see Figure 1 below).

Fix  $p \in C$ , such that  $p \succ \delta_{x_2}$ . The upper and lower contour sets of  $p$ ,  $B(p)$  and  $W(p)$ , are delineated by the preimages of the real-valued, affine, functions on  $C$  defined by  $\bar{U}_1(q) = \inf\{\alpha \in [0, 1] \mid ((1 - \alpha), 0, \alpha) \succ q\}$ ,  $\underline{U}_1(q) = \sup\{\alpha \in [0, 1] \mid q \succ ((1 - \alpha), 0, \alpha)\}$ ,  $\bar{U}_2(q) = \inf\{\alpha \in [0, 1] \mid (0, (1 - \alpha), \alpha) \succ q\}$  and  $\underline{U}_2(q) = \sup\{\alpha \in [0, 1] \mid q \succ (0, (1 - \alpha), \alpha)\}$ . Notice that  $\bar{U}_1(p) \geq \bar{U}_1(q)$  and  $\underline{U}_2(p) \geq \underline{U}_2(q)$ . Consequently,  $q \succ p$  if and only if  $q$  is an interior element of  $B(p)$  (that is, if and only if  $\bar{U}_i(p) > \bar{U}_i(q)$ ,  $i = 1, 2$ ),  $p \succ q$  if and only if  $q \in \bar{B}(p)$ , where  $\bar{B}(p)$  is the closure of  $B(p)$  (that is, if and only if  $\bar{U}_i(p) \geq \bar{U}_i(q)$ ,  $i = 1, 2$ , with equality for some  $i$ ). Hence, the relation  $\succ$  is represented by the doubleton set of functions,  $\mathcal{U} = \{\bar{U}_1, \bar{U}_2\}$ . This is the minimal set of functions that figure in this representation.

For all  $p, q \in C$ ,  $p \bowtie q$  if and only if  $q$  is an element of the intersection of the complements of  $B(p)$  and  $W(p)$ . In this case, it must be that  $U(p) \geq U(q)$  for one  $U$  in  $\mathcal{U}$  and  $U(p) \leq U(q)$  for the other  $U$  in  $\mathcal{U}$ .

<sup>12</sup>Let  $q, r \in B(p)$  and  $\alpha \in [0, 1]$ . To prove the lemma we need to show that  $\alpha q + (1 - \alpha)r \succ p$ . Apply (A.3) twice to obtain,  $\alpha q + (1 - \alpha)r \succ \alpha p + (1 - \alpha)r$  and  $\alpha p + (1 - \alpha)r \succ \alpha p + (1 - \alpha)p$ . The same applies to  $W(p)$ .

<sup>13</sup>For instance, let  $x_i$ ,  $i = 1, 2, 3$  denotes sums of money and suppose that  $x_1 < x_2 < x_3$ . Then  $p = (p_1, p_2, p_3)$  is interpreted to be a lottery that assigns the probability  $p_i$  to winning the prize  $x_i$ .



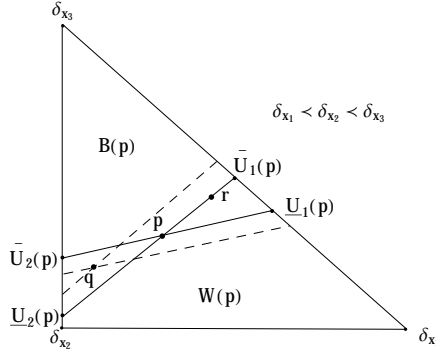


Figure 1 :  $B(p)$  is the "better than" set of  $p$ .  
 $W(p)$  is the "worse than" set of  $p$ . The lotteries  $p$  and  $q$   
are noncomparable and the lotteries  $r$  and  $p$  satisfy  $r \succeq p$ .

Consider next the three-dimensional simplex,  $C$ , in  $\mathbb{R}^4$ . Assume  $C$  is  $\succ$ -bounded and that  $\delta_{x_4} = (0, 0, 0, 1)$  and  $\delta_{x_1} = (1, 0, 0, 0)$  are, respectively, the  $\succ$  greatest and smallest elements of  $C$ . Suppose that the preference relation  $\succ$  is characterized by a full dimensional cone. Let  $\bar{U}$  and  $\underline{U}$  defined as above. In general,  $\bar{U}$  and  $\underline{U}$  are not affine on  $C$ .

For each  $p \in C$ , let  $\mathcal{L}(p)$  be the vector subspace spanned by the vectors  $\delta_{x_4} - p$  and  $p - \delta_{x_1}$ . The restrictions of  $\bar{U}$  and  $\underline{U}$  to the two-dimensional simplex,  $\Delta = C \cap \mathcal{L}(p)$  are affine functions. Hence, the restrictions of the upper and lower contour sets of  $p$  to  $\mathcal{L}(p)$ , that is  $B_p(p) := B(p) \cap \mathcal{L}(p)$  and  $W_p(p) := W(p) \cap \mathcal{L}(p)$  are delineated by the  $\bar{U}$  and  $\underline{U}$  restricted to  $\mathcal{L}(p)$ . These sets, and the corresponding restrictions of  $\bar{U}$  and  $\underline{U}$ , are not necessarily the same for  $\mathcal{L}(p)$  and  $\mathcal{L}(q)$  (see Figure 2). However, for all  $p$  and  $q$  in  $C$ ,  $q \succ p$  if and only if  $q$  in an interior element of  $B(p)$ ;  $q \succcurlyeq p$  if and only if  $q \in \bar{B}(p)$ . These conditions have the same representation as before.

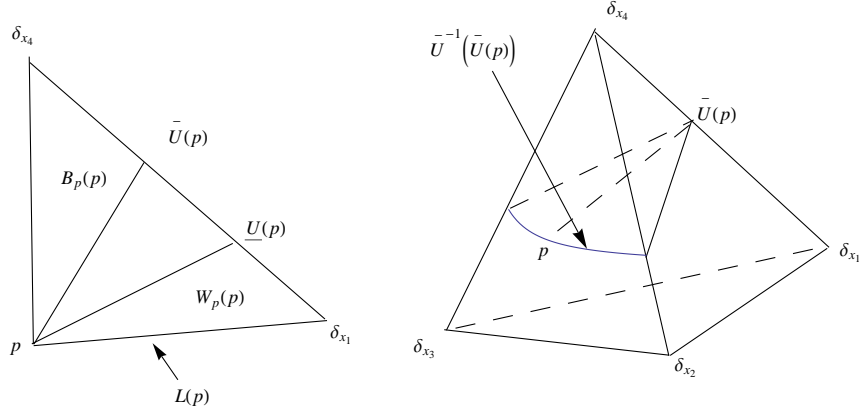


Figure 2

### 2.3 The fundamental representation theorem

We present a general result giving rise to the finite-dimensional expected multi-utility representations under risk, and additively separable multi-utility representation under uncertainty, as immediate implications.

We denote by  $\langle \mathcal{U} \rangle$  the closure of the convex cone generated by all the functions in  $\mathcal{U}$  and all the constant function on  $\mathcal{L}$ .

**Theorem 1** Let  $C$  be a nonempty, convex and full dimensional subset of a finite dimensional linear space,  $\mathcal{L}$ . Let  $\succ$  be a binary relation on  $C$ , then the following conditions are equivalent:

(i)  $C$  is  $\succ$ -bounded and  $\succ$  satisfies (A.1), (A.2) and (A.3)

(ii) There exists nonempty closed convex set of affine functions on  $\mathcal{L}$ ,  $\mathcal{U}$ , such that  $U(p^M) > U(p) > U(p^m)$ , for all  $p \in C - \{p^M, p^m\}$  and for all  $U \in \mathcal{U}$  and  $p, q \in C$ ,

$$q \succcurlyeq p \Leftrightarrow U(q) \geq U(p) \text{ for all } U \in \mathcal{U} \quad (2)$$

and

$$q \succ p \Leftrightarrow U(q) > U(p) \text{ for all } U \in \mathcal{U}. \quad (3)$$

Moreover, if  $\mathcal{V}$  is another set of affine functions that represent  $\succcurlyeq$  and  $\succ$  in the sense of (2) and (3), respectively, then  $\langle \mathcal{V} \rangle = \langle \mathcal{U} \rangle$ .<sup>14</sup>

<sup>14</sup>Shapley and Baucells (1998) has a similar result in their Theorem 1.8.

**Remark 1:** It is shown in the proof and  $q \succeq p$  if and only if  $U(q) \geq U(p)$  for all  $U \in \mathcal{U}$  and  $U(q) = U(p)$  for some  $U \in \mathcal{U}$ .

**Remark 2:** Seidenfeld et. al. (1995) show that a strict partial order, defined by strict first-order stochastic dominance, has expected multi-utility representation, satisfies the independence axiom and violates the Archimedean axiom.<sup>15</sup> To bypass this problem, Seidenfeld et. al. (1995) and subsequent writers invoked alternative continuity axioms that, unlike the Archimedean axiom, require the imposition of a topological structures.<sup>16</sup> We maintained the Archimedean axiom as our continuity postulate at the cost of restricting the upper contour sets associated with the strict preference relation,  $B(p) := \{q \in C \mid q \succ p\}$ , to be algebraically open. (In the example of Seidenfeld et. al. (1995) these sets are closed). Given the trade-off involved, this restriction seems, to us, reasonable. Moreover, restricting the upper contour set in this manner, we follow a long tradition in economic theory.

## 2.4 Expected multi-utility representation for simple probability measures

Let  $X = \{x_1, \dots, x_n\}$  be a finite set of prizes and denote by  $\Delta(X)$  the set of all probability measures on  $X$ . For each  $\ell, \ell' \in \Delta(X)$  and  $\alpha \in [0, 1]$  define  $\alpha\ell + (1 - \alpha)\ell' \in \Delta(X)$  by  $(\alpha\ell + (1 - \alpha)\ell')(x) = \alpha\ell(x) + (1 - \alpha)\ell'(x)$ , for all  $x \in X$ . Then  $\Delta(X)$  is a convex subset of the linear space  $\mathbb{R}^X$ . Let  $\ell^M, \ell^m \in \Delta(X)$  satisfy  $\ell^M \succ \ell \succ \ell^m$ , for all  $\ell \in \Delta(X)$ . Application of Theorem 1 to  $C = \Delta(X)$  yields an expected multi-utility representation.

**Corollary 1 (Expected multi-utility representation)** Let  $\succ$  be a binary relation on  $\Delta(X)$ , then the following conditions are equivalent:

- (i)  $\Delta(X)$  is  $\succ$ -bounded and  $\succ$  satisfies (A.1), (A.2) and (A.3).
- (ii) There exists nonempty, closed and convex set,  $\mathbb{U}$ , of real-valued functions on  $X$  such that

$$\sum_{x \in \text{Supp}(\ell^M)} u(x) \ell^M(x) > \sum_{x \in \text{Supp}(\ell)} u(x) \ell(x) > \sum_{x \in \text{Supp}(\ell^m)} u(x) \ell^m(x),$$

<sup>15</sup>See example 2.1 in their paper.

<sup>16</sup>See Dubra et. al. (2004) and Nau (2006).

for all  $\ell \in \Delta(X) - \{\ell^M, \ell^m\}$ , and  $u \in \mathbb{U}$  and, for all  $p, q \in \Delta(X)$ ,

$$p \succsim q \Leftrightarrow \sum_{x \in \text{Supp}(p)} u(x) p(x) \geq \sum_{x \in \text{Supp}(q)} u(x) q(x), \text{ for all } u \in \mathbb{U}, \quad (4)$$

and

$$p \succ q \Leftrightarrow \sum_{x \in \text{Supp}(p)} u(x) p(x) > \sum_{x \in \text{Supp}(q)} u(x) q(x), \text{ for all } u \in \mathbb{U}. \quad (5)$$

Moreover, if  $\mathbb{V}$  is another set of real-valued, affine, functions on  $\Delta(X)$  that represent  $\succsim$  and  $\succ$  in the sense of (4) and (5), respectively, then  $\langle \mathbb{V} \rangle = \langle \mathbb{U} \rangle$ .

Proof: Let  $C = \Delta(X)$  and  $\mathbb{U} = \{u \in \mathbb{R}^X \mid u \cdot p = U(p), U \in \mathcal{U}\}$ , then the conclusions of the corollary are implied by Theorem 1.

## 2.5 Additively separable multi-utility representation

Consider the Anscombe-Aumann (1963) model. Let  $S$  be a finite set of *states*. Subsets of  $S$  are *events*. Let  $H := \{h : S \rightarrow \Delta(X)\}$  be the set whose elements are *acts*. For all  $h, h' \in H$  and  $\alpha \in [0, 1]$ ,  $\alpha h + (1 - \alpha) h' \in H$  is defined by  $(\alpha h + (1 - \alpha) h')(s) = \alpha h(s) + (1 - \alpha) h'(s)$ , for all  $s \in S$ .<sup>17</sup> Under this definition  $H = \Delta(X)^S$  is a convex subset of the linear space  $\mathbb{R}^{X \times S}$ . Let  $h^M, h^m \in H$  satisfy  $h^M \succ h \succ h^m$ , for all  $h \in H$ .

Applying Theorem 1 to  $H$ , we obtain the following:<sup>18</sup>

**Corollary 2 (Additive multi-utility representation)** Let  $\succ$  be a binary relation on  $H$ , then the following conditions are equivalent:

- (i)  $H$  is  $\succ$ -bounded and  $\succ$  satisfies (A.1), (A.2) and (A.3).
- (ii) There exists nonempty, convex and closed, set  $\mathcal{W}$  of real-valued functions,  $w$ , on  $\Delta(X) \times S$ , affine in its first argument such that

$$\sum_{s \in S} w(h^M(s), s) > \sum_{s \in S} w(h(s), s) > \sum_{s \in S} w(h^m(s), s),$$

<sup>17</sup>For every  $s \in S$ , the convex mixture  $\alpha h(s) + (1 - \alpha) h'(s)$  is defined as in the preceding subsection.

<sup>18</sup>A similar result appears in Nau (2006) for finite  $X$ . Ok, et. al. (2008) show that the same holds when  $X$  is a compact metric space. As mentioned, these authors use a continuity assumption stronger than (A.2).

for all  $h \in H - \{h^M, h^m\}$ , and  $w \in \mathcal{W}$  and, for all  $h, h' \in H$ ,

$$h \succsim h' \Leftrightarrow \sum_{s \in S} w(h(s), s) \geq \sum_{s \in S} w(h'(s), s), \text{ for all } w \in \mathcal{W}, \quad (6)$$

and

$$h \succ h' \Leftrightarrow \sum_{s \in S} w(h(s), s) > \sum_{s \in S} w(h'(s), s), \text{ for all } w \in \mathcal{W}. \quad (7)$$

Moreover, if  $\mathcal{W}'$  is another set of real-valued, affine, functions on  $H$  that represent  $\succsim$  and  $\succ$  in the sense of (6) and (7), respectively, then  $\langle \mathcal{W}'_{\Sigma} \rangle = \langle \mathcal{W}_{\Sigma} \rangle$  where  $\mathcal{W}_{\Sigma} = \{\hat{w} : \Delta(X)^{|S|} \rightarrow \mathcal{R} : \hat{w}(f) = \sum_{s \in S} w(f(s), s), w \in \mathcal{W}\}$ .

The proof is by the application of a standard argument (see Kreps [1988]) to  $U$  in the proof of Theorem 1.

**Remark 3:** Let  $\mathcal{W}_s := \{w(\cdot, s) \mid w \in \mathcal{W}\}$ . By Corollary 1,  $w(h(s), s) = \sum_{x \in \text{Supp}(h(s))} u(x; s) h(x; s)$ , where  $u(\cdot, s)$  is a real-valued function on  $X$ , for all  $s \in S$ .

The representations in Corollary 2 are not the most parsimonious as the set  $\mathcal{W}$  includes functions that are not redundant (that is, their removal does not affect the representation). Henceforth, we can consider a subset of  $\mathcal{W}$  that is sufficient for the representation. We denote the set of these functions by  $\mathcal{W}^o$  and call it the set of essential functions. We also define the sets of essential component functions  $\mathcal{W}_s^o := \{w(\cdot, s) \mid w \in \mathcal{W}^o\}$ ,  $s \in S$ . As part of the proof of Theorem 2 below, we show that, under additional assumptions to be specified, the component functions corresponding to the essential functions in  $\mathcal{W}^o$ , are positive linear transformations of one another (under suitably chosen  $\mathcal{W}^o$ ).

### 3 The Main Results

Our main results are extensions of the Anscombe-Aumann (1963) model to include incomplete preferences. As mentioned earlier, the incompleteness in this model may stem from two distinct sources, namely, beliefs and tastes. We present below two models in which these sources of incompleteness are represented by sets of priors and utilities, respectively. In the first, more

general model, the beliefs and tastes are not entirely separated, and the representation involves sets of priors that are “utility dependent.” The second model entails total separation of beliefs and tastes and the representation involves sets of priors and utility functions that are independent.

### 3.1 Multi-prior expected multi-utility representation I - The general case

#### 3.1.1 Dominance

For each  $f \in H$  and every  $s \in S$ , let  $f^s$  denote the constant act whose payoff is  $f(s)$  in every state. Formally,  $f^s(s') = f(s)$  for all  $s' \in S$ . The next axiom, which is a special case of Savage’s (1954) postulate P7, requires that if an act,  $g$ , is strictly preferred over every constant act,  $f^s$ , obtained from the act  $f$ , then  $g$  be strictly preferred over  $f$ . Formally,

**(A.4) (Dominance)** For all  $f, g \in H$ , if  $g \succ f^s$  for every  $s \in S$ , then  $g \succ f$ .

Axiom (A.4) appears and discussed in Fishburn (1970).<sup>19</sup> Later we shall observe that, in conjunction with the other axioms, axiom (A.4) implies that if a decision maker prefers one act over another under all conceivable beliefs about the likelihoods of the states, then he prefers the former act over the latter.

#### 3.1.2 Multi-prior expected multi-utility representation

Theorem 2 below shows that a preference relation satisfies the axioms (A.1)-(A.4) if and only if there is a non-empty convex set of affine utility functions on  $\Delta(X)$  and, corresponding to each utility function, a convex set of probability measures on  $S$  such that when presented with a choice between two acts the decision maker prefers the acts that yield higher expected utility according to every utility function and every probability measure in the corresponding set. Let the set of probability-utility pairs that figure in the representation be  $\Phi := \{(\pi, U) \mid U \in \mathcal{U}, \pi \in \Pi^U\}$ . Each  $(\pi, U) \in \Phi$  define a hyperplane  $w := \pi \cdot U$ . Denote by  $\mathcal{W}$  the set of all these hyperplanes, and define  $\langle \Phi \rangle = \langle \mathcal{W} \rangle$ .

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<sup>19</sup>See Fishburn (1970), p. 179.

**Theorem 2** Let  $\succ$  be a binary relation on  $H$ , then the following conditions are equivalent:

(i)  $H$  is  $\succ$ -bounded and  $\succ$  is nonempty satisfying (A.1) - (A.4).

(ii) There exists a nonempty, closed and convex, set,  $\mathcal{U} \in \mathcal{B}$ , of real-valued, affine, functions on  $\Delta(X)$ , and closed and convex sets  $\Pi^U$ ,  $U \in \mathcal{U}$ , of probability measures on  $S$  such that,

$$\sum_{s \in S} U(h^M(s)) > \sum_{s \in S} U(h(s)) \pi(s) > \sum_{s \in S} U(h^m(s)),$$

for all  $h \in H$ ,  $(\pi, U) \in \Phi$ , and, for all  $h, h' \in H$ ,

$$h \succcurlyeq h' \Leftrightarrow \sum_{s \in S} U(h(s)) \pi(s) \geq \sum_{s \in S} U(h'(s)) \pi(s), \text{ for all } (\pi, U) \in \Phi, \quad (8)$$

and

$$h \succ h' \Leftrightarrow \sum_{s \in S} U(h(s)) \pi(s) > \sum_{s \in S} U(h'(s)) \pi(s), \text{ for all } (\pi, U) \in \Phi, \quad (9)$$

where  $\Phi = \{(\pi, U) \mid U \in \mathcal{U}, \pi \in \Pi^U\}$ .

Moreover, if  $\Phi' = \{(\pi', V) \mid V \in \mathcal{V}, \pi' \in \Pi^V\}$  is an another set of real-valued, affine, functions on  $\Delta(X)$  and sets of probability measures on  $S$  that represent  $\succcurlyeq$  and  $\succ$  in the sense of (8) and (9), respectively, then,  $\langle \Phi' \rangle = \langle \Phi \rangle$  and  $\pi(s) > 0$  for all  $s$ .

Note that, since the set of priors representing the beliefs may depend on the utility function that capture the tastes, the representation in Theorem 2 does not completely separate beliefs from tastes .

## 3.2 Multi-prior expected multi-utility representation II - The product case

### 3.2.1 Beliefs consistency

To grasp the meaning of the next axiom it is helpful to consider the state-space representation of preferences in subjective expected utility theory, and the maxmin expected utility model of Gilboa and Schmeidler (1989). In these models, the relevant sets of subjective probabilities (a singleton, in the first instance) are defined by the hyperplanes supporting the upper counter

sets at certainty.<sup>20</sup> We employ the same reasoning below, to identify all the probability measures on  $S$  that may be involved in the evaluation of the acts. Specifically, these measures are the normalized normals of the supporting hyperplanes of the upper (and lower) contour sets of acts representing certain outcomes.<sup>21</sup> Let  $\mathcal{M}$  denote the set of all these measures.

Consider an act-probability pair  $(f, \alpha) \in H \times \Delta(S)$ . For every  $f \in H$  and  $\alpha \in \Delta(S)$ , let  $f^\alpha$  be the constant act defined by  $f^\alpha(s) = \sum_{s' \in S} \alpha_{s'} f(s')$  for all  $s \in S$ . Then,  $f^\alpha$  is the constant act obtained by reduction of the compound lottery  $(f, \alpha)$ . The axiom asserts that  $g \succ f$  is sufficient for the reduction of  $(g, \alpha)$  to be preferred over the reduction of  $(f, \alpha)$  for all  $\alpha \in \mathcal{M}$ .

To formalize this discussion, fix  $x \in X$  and denote by  $h^x$  the constant act defined by  $h^x(s) = \delta_x$ , for all  $s \in S$ . (That is,  $h^x \in \mathbb{R}^{|S| \times |X|}$  and may be written as  $h^x = (\delta_x, \dots, \delta_x)$ , where  $\delta_x \in \mathbb{R}^X$  is the degenerate lottery that puts weight 1 on some  $x \in X$ ). Then  $B(h^x)$  is a convex set in the linear space  $\mathbb{R}^{|S| \times |X|}$ .

Let  $\mathcal{T}$  be the set of the supporting hyperplanes of  $B(h^x)$  at  $h^x$ . That is,  $\mathcal{T}$  is the set of hyperplanes  $T(\xi_T, h^x) = \{y \in \mathbb{R}^{|S| \times |X|} \mid \xi_T \cdot (y - h^x) = 0\}$  such that  $B(h^x) \in \cap_{T \in \mathcal{T}} T^+$ , where  $T^+ = \{y \in \mathbb{R}^{|S| \times |X|} \mid \xi_T \cdot (y - h^x) \geq 0\}$ , is the positive half-space define by  $T$ , and  $\xi_T$  denotes the normal of  $T$ .<sup>22</sup> For each  $T \in \mathcal{T}$ , let  $\beta_T \in \mathbb{R}^{|S|}$  be defined by  $\beta_T(s) = \sum_{x \in X} \xi_T(x, s)$ , and let  $\alpha_T \in \Delta(S)$  be defined by  $\alpha_T(s) = \beta_T(s) / \sum_{s' \in S} \beta_T(s')$ . Define  $\mathcal{M} = \{\alpha_T \in \Delta(S) \mid T \in \mathcal{T}\}$ .

The next axiom requires that if an act  $g$  is strictly preferred over another act  $f$  then the constant act  $g^\alpha$  is preferred over the constant act  $f^\alpha$ , for all  $\alpha \in \mathcal{M}$ . Formally,

**(A.5) (Beliefs consistency)** For all  $f, g \in H$ ,  $g \succ f$  implies that  $g^\alpha \succ f^\alpha$ , for all  $\alpha \in \mathcal{M}$ .

Notice that the necessity of this condition is implied by Theorem 2. Hence, taken together, axioms (A.1) - (A.5) amount to the condition that each of these measures combines with each of the utility functions in the process of assessing the merits of the alternative acts.

<sup>20</sup>To be exact, the probabilities are the normalized normals of these hyperplanes.

<sup>21</sup>Note that, some of these measures, may be non-essential, which means that they do not restrict the preference relation.

<sup>22</sup>Since  $B(h^x)$  is a convex set in the linear space  $\mathbb{R}^{|S| \times |X|}$ , the set  $\mathcal{T}$  is well-defined and convex.



### 3.2.2 Multi-prior expected multi-utility product representation

Our second main result is a representation theorem that totally separates beliefs from tastes. Specifically, it shows that a preference relation satisfies (A.1)-(A.5) if and only if there is a nonempty, convex set,  $\mathcal{U}$ , of utility functions on  $\Delta(X)$  and a nonempty convex, set,  $\mathcal{M}$ , of probability measures on  $S$  such that when presented with a choice between two acts the decision maker prefers one act over another if and only if the former act yields higher expected utility according to every combination of a utility function and a probability measure in these sets.

**Theorem 3** Let  $\succ$  be a binary relation on  $H$ , then the following conditions are equivalent:

- (i)  $H$  is  $\succ$ -bounded and  $\succ$  is nonempty satisfying (A.1) - (A.5).
- (ii) There exists a nonempty, closed and convex set  $\mathcal{U} \in \mathcal{B}$ , of real-valued functions on  $\Delta(X)$ , and a closed convex  $\mathcal{M}$ , of probability measures on  $S$  such that

$$\sum_{s \in S} U(h^M(s)) \pi(s) > \sum_{s \in S} U(h(s)) \pi(s) > \sum_{s \in S} U(h^m(s)) \pi(s),$$

for all  $(\pi, U) \in \mathcal{M} \times \mathcal{U}$  and  $h \in H$ , and, for all  $h, h' \in H$ ,

$$h \succcurlyeq h' \Leftrightarrow \sum_{s \in S} U(h(s)) \pi(s) \geq \sum_{s \in S} U(h'(s)) \pi(s), \text{ for all } (\pi, U) \in \mathcal{M} \times \mathcal{U}, \quad (10)$$

and

$$h \succ h' \Leftrightarrow \sum_{s \in S} U(h(s)) \pi(s) > \sum_{s \in S} U(h'(s)) \pi(s), \text{ for all } (\pi, U) \in \mathcal{M} \times \mathcal{U}, \quad (11)$$

Moreover, if  $\mathcal{V}$  and  $\mathcal{M}'$ , is another pair of sets of real-valued, affine, functions on  $\Delta(X)$  and probability measures on  $S$  that represent  $\succcurlyeq$  and  $\succ$  in the sense of (10) and (11), respectively, then  $V(U) = b_U U + a_U$ ,  $b_U > 0$  for all  $U \in \mathcal{U}$ ,  $\mathcal{M} = \mathcal{M}'$ , and  $\pi(s) > 0$  for all  $s$ .

**Remark 4:** The set  $\mathcal{M}$  is the convex hull of union over  $\mathcal{U}$  of the sets of probability measures,  $\Pi^U$ , that figure in Theorem 2.

## 4 Discussion

### 4.1 State-independent preferences

Consider the following additional notations and definitions. For each  $h \in H$  and  $s \in S$  let  $h_{-s}p$  the act that is obtained by replacing the  $s$ -th coordinate of  $h$ ,  $h(s)$ , with  $p$ . Define the conditional preference relation,  $\succ_s$  on  $\Delta(X)$ , by  $p \succ_s q$  if there exists  $h_{-s}$  such that  $h_{-s}p \succ h_{-s}q$ , for all  $p, q \in \Delta(X)$ . A state  $s$  is said to be *nonnull* if  $p \succ_s q$ , for some  $p, q \in \Delta(X)$ , and it is *null* otherwise.

**Definition 3:** A preference relation displays state-independence if  $\succ_s = \succ_{s'}$ , for all nonnull  $s, s' \in S$ .

**Lemma 2:** Let  $\succ$  be a nonempty binary relation on  $H$ , and suppose that  $H$  is  $\succ$ -bounded. If  $\succ$  satisfies (A.1) - (A.4), then it displays state-independent preferences. Moreover, all states are non-null and  $h^M = (\delta_{x_1}, \dots, \delta_{x_1})$  and  $h^m = (\delta_{x_2}, \dots, \delta_{x_2})$  for some  $x_1, x_2 \in X$ .

We denote  $p^M = \delta_{x_1}$  and  $p^m = \delta_{x_2}$ . The proof is an immediate implication of Theorem 2, and is omitted. Note that if the preference relation satisfies (A.1) - (A.3) and is complete, then state-independence is equivalence to monotonicity (that is, if  $h(s) \succcurlyeq h'(s)$  (as constant acts) for every  $s \in S$  then  $h \succcurlyeq h'$ ). This equivalence does not hold when the preference relation is incomplete (see Ok et. al. (2008)).

### 4.2 Coherent beliefs

It is noteworthy that the axiomatic structure of the preference relation depicted by (A.1) - (A.4) implies that the decision maker's beliefs are coherent. To define the notion of coherent beliefs, let  $h^p$  denote the constant act whose payoff is  $h^p(s) = p$  for every  $s \in S$ . For each event  $E$ ,  $pEq \in H$  is the act whose payoff is  $p$  for all  $s \in E$  and  $q$  for all  $s \in S - E$ . Denote  $paq$  the constant act whose payoff, in every state, is  $\alpha p + (1 - \alpha)q$ . A *bet* on an event  $E$  is the act  $pEq$ , whose payoffs satisfy  $h^p \succ h^q$ .

Suppose that the decision maker considers the constant act  $paq$  preferable to the bet  $pEq$ . This is interpreted to imply that he believes  $\alpha$  exceeds the likelihood of  $E$ . This belief is coherent if the same holds for any other bet

on  $E$  and the corresponding constant acts (that is, if  $h^{p_1} \succ h^{q_1}$ , then the constant acts  $p_1 \alpha q_1$  is preferable to the bet  $p_1 E q_1$ ). The same logic applies when the bet  $p E q$  is preferable to the constant act  $p \alpha q$ . Formally,

**Definition 4:** A preference relation  $\succ$  on  $H$  exhibits coherent beliefs if, for all events  $E$  and  $p, q, p', q' \in \Delta(X)$  such that  $h^p \succ h^q$  and  $h^{p'} \succ h^{q'}$ ,  
 $p \alpha q \succ p E q$  if and only if  $p' \alpha q' \succ p' E q'$  and  $p E q \succ p \alpha q$  if and only if  $p' E q' \succ p' \alpha q'$ .

**Lemma 3:** Let  $\succ$  be a nonempty binary relation on  $H$  satisfying (A.1) - (A.4). Suppose that  $H$  is  $\succ$ -bounded, then  $\succ$  exhibits coherent beliefs.

The proof is an immediate implication of Theorem 2, and is omitted.

### 4.3 On the significance of $\succ$ -boundedness

Ok et. al. (2008), give an example showing the existence of a preference relation that satisfies independence, monotonicity, continuity and state-independence and yet, has no multi-prior expected multi-utility, representation. Let  $X = \{x, y\}$ ,  $S = \{s_1, s_2\}$  and  $\mathcal{U} = \{U, V\}$ , where  $U, V$  are defined by tables below:

$U$	$s_1$	$s_2$	$V$	$s_1$	$s_2$
$x$	3	0	$x$	0	1
$y$	0	1	$y$	2	0

Define  $p \succeq q$  if and only if  $U(p) \geq U(q)$  and  $V(p) \geq V(q)$ , for all  $p, q \in \Delta(X)^S$ . This preference relation has no multi-prior expected multi-utility representation.

This example does not apply to our setting. Specifically, in this preference relation there is no greatest element. Assume the existence of the greatest element,  $p^M$ , and extend the tables as follows:

$U$	$s_1$	$s_2$	$V$	$s_1$	$s_2$
$p^M$	4	4	$p^M$	4	4
$x$	3	0	$x$	0	1
$y$	0	1	$y$	2	0

We show next that preference relation defined by these tables does not satisfy monotonicity. Suppose  $p = (\frac{29}{35}, \frac{1}{35}, \frac{5}{35})$ ,  $q = (\frac{27}{35}, \frac{7}{35}, \frac{1}{35})$ ,  $r = (\frac{31}{35}, \frac{1}{35}, \frac{3}{35})$  and  $t = (\frac{30}{35}, \frac{4}{35}, \frac{1}{35})$  where the first number is the weigh on  $p^M$  and the second number is the weigh on  $x$ . Then constant act  $p$  is preferred over  $q$  since  $U(p) = \frac{29}{35} \times 4 + \frac{1}{35} \times 3 + \frac{29}{35} \times 4 + \frac{5}{35} \times 1 = \frac{240}{35} > \frac{226}{35} = U(q)$  and  $V(p) = \frac{243}{35} > \frac{225}{35} = V(q)$ . Also, constant act  $r$  is preferred to constant act  $t$  since  $U(r) = \frac{254}{35} > \frac{253}{35} = U(t)$  and  $V(r) = \frac{255}{35} > \frac{246}{35} = V(t)$ . But  $U(p, r) - U(q, t) = -\frac{4}{35} < 0$  implying that  $(p, q) \succeq (q, t)$  does not hold. Hence,  $\succeq$  does not satisfy monotonicity and, therefore, our model is not inconsistent with this example.

## 5 Special Cases

Consider next two special cases. The first, known as Knightian uncertainty, involves incomplete beliefs and complete tastes. The second involves incomplete tastes and complete beliefs.

### 5.1 Knightian uncertainty

Consider the extension of the Anscombe-Aumann (1963) model to include incomplete preferences, and suppose that the incompleteness is due, entirely, to incomplete beliefs. In other words, the decision maker knows his risk attitudes and is capable of comparing and expressing his preferences between any two lotteries in every state. This case was dealt with by Bewley (2002) and is referred to as Knightian uncertainty.<sup>23</sup>

The model of Knightian uncertainty requires a formal definition of complete tastes. The next axiom formalizes the completeness of tastes:<sup>24</sup>

**(A.6) (Complete tastes)** For every  $s \in S$  and  $p, q \in \Delta(X)$ , either  $p \succ_s q$  or  $q \succ_s p$ .

Note that Lemma 2 and (A.6) imply that the preference relation  $\succ$  is complete on the set of constant acts. This is the assumption of Bewley (2002). The next theorem is our version of Knightian uncertainty.

<sup>23</sup>See also Ok et. al. (2008).

<sup>24</sup>In the presence of (A.4), this axiom is equivalent to the strict preference relation  $\succ$  being complete on the subset of constant acts. This is sometimes called the partial completeness axiom (see Ok [2007]).

**Theorem 4** Let  $\succ$  be a binary relation on  $H$ , then the following conditions are equivalent:

- (i)  $H$  is  $\succ$ -bounded,  $\succ$  is nonempty and satisfies (A.1) - (A.4) and (A.6).
- (ii) There exists a nonempty, closed, convex set,  $\mathcal{M}$ , of probability measures on  $S$  and a real-valued, affine, function  $U$  on  $\Delta(X)$  such that,

$$\sum_{s \in S} U(h^M(s)) \pi(s) > \sum_{s \in S} U(h(s)) \pi(s) > \sum_{s \in S} U(h^m(s)) \pi(s),$$

for all  $h \in H, \pi \in \mathcal{M}$  and, for all  $h, h' \in H$ ,

$$h \succcurlyeq h' \Leftrightarrow \sum_{s \in S} U(h(s)) \pi(s) \geq \sum_{s \in S} U(h'(s)) \pi(s), \text{ for all } \pi \in \mathcal{M}, \quad (12)$$

and

$$h > h' \Leftrightarrow \sum_{s \in S} U(h(s)) \pi(s) \succ \sum_{s \in S} U(h'(s)) \pi(s), \text{ for all } \pi \in \mathcal{M}. \quad (13)$$

Moreover,  $U$  is unique up to positive linear transformation,  $\mathcal{M}$  is unique, and for all  $\pi \in \mathcal{M}$ ,  $\pi(s) > 0$  for any  $s$ .

## 5.2 Subjective expected multi-utility representation with complete beliefs

Consider next the extension of the Anscombe-Aumann (1963) model to the dual case of Knightian uncertainty in which incompleteness of the decision-maker's preferences is due solely to the incompleteness of his tastes. This situation was modeled in Ok et. al. (2008) using an axiom they call reduction.<sup>25</sup> We propose here an alternative formulation based on the idea of completeness of beliefs. This idea is captured by the following axiom:<sup>26</sup>

- (A.7) (Complete beliefs)** For all events  $E$  and  $\alpha \in [0, 1]$ , either  $p^M \alpha p^m \succ p^M E p^m$  or  $p^M E p^m \succ p^M \alpha' p^m$  for all  $\alpha > \alpha'$ .

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<sup>25</sup>The reduction axiom of Ok et. al. (2008) requires that, for every  $h \in H$ , there exists a probability measure,  $\mu$ , on  $S$  such that  $h^\mu \sim h$ .

<sup>26</sup>Unlike weak reduction of Ok et. al. (2008), neither complete beliefs nor complete tastes involve an existential clause.

A preference relation  $\succ$  displays complete beliefs if it satisfies (A.7). If the beliefs are complete then the incompleteness of the preference relation on  $H$  is due entirely to the incompleteness of tastes. The next theorem is the subjective expected multi-utility version of the Anscombe-Aumann (1963) model corresponding to the situation in which the decision maker's beliefs are complete.<sup>27</sup>

**Theorem 5** Let  $\succ$  be a binary relation on  $H$  and  $\succcurlyeq$  the induced binary relation given in Definition 1, then the following conditions are equivalent:

- (i)  $H$  is  $\succ$ -bounded,  $\succ$  is nonempty and satisfies (A.1) - (A.4) and (A.7).
- (ii) There exists a nonempty convex and closed set,  $\mathcal{U} \in \mathcal{B}$ , of real-valued, affine, functions on  $\Delta(X)$  and a probability measure  $\pi$  on  $S$  such that,

$$\sum_{s \in S} U(h^M(s)) \pi(s) > \sum_{s \in S} U(h(s)) \pi(s) > \sum_{s \in S} U(h^m(s)) \pi(s),$$

for all  $U \in \mathcal{U}$  and  $h \in H$ , and, for all  $h, h' \in H$ ,

$$h \succcurlyeq h' \Leftrightarrow \sum_{s \in S} U(h(s)) \pi(s) \geq \sum_{s \in S} U(h'(s)) \pi(s) \text{ for all } U \in \mathcal{U}. \quad (14)$$

and

$$h \succ h' \Leftrightarrow \sum_{s \in S} U(h(s)) \pi(s) > \sum_{s \in S} U(h'(s)) \pi(s) \text{ for all } U \in \mathcal{U}. \quad (15)$$

Moreover, if  $\mathcal{V}$  is another set of real-valued, affine, functions on  $\Delta(X)$  that represent  $\succcurlyeq$  in the sense of (14) and (15), respectively, then  $\langle \mathcal{V} \rangle = \langle \mathcal{U} \rangle$ . The probability measure,  $\pi$ , is unique and  $\pi(s) > 0$  if and only if  $s$  is nonnull.

**Remark 5:** For every event  $E$ , define the *upper probability* of  $E$  is  $\pi^u(E) = \inf\{\alpha \in [0, 1] \mid p^M \alpha p^m \succ p^M E p^m\}$  and the *lower probability* of  $E$  is  $\pi^l(E) = \sup\{\alpha \in [0, 1] \mid p^M E p^m \succ p^M \alpha p^m\}$ . Lemma 3 asserts that the upper and lower probabilities are well-defined.<sup>28</sup> Theorem 5 implies that a preference relation  $\succ$  satisfying (A.1) - (A.4) displays complete beliefs if and only if  $\pi^u(E) = \pi^l(E)$ , for every  $E$ .

<sup>27</sup>See Ok et. al. (2008) theorem 4, for their version of this result.

<sup>28</sup>Notice that  $\succ$  displays complete beliefs if and only if there exists a probability measure  $\mu$  on  $S$  such that  $p^M \mu(E) p^m \succcurlyeq p^M E p^m \succcurlyeq p^M \mu(E) p^m$  for every event  $E$ .

## 6 Concluding Remarks

Choice theoretic models that depart from the completeness axiom are potentially useful in a variety of applications. Seidenfeld et. al. (1995) emphasized their potential role in establishing a choice-based foundations of Bayesian robustness analysis in statistics. Possible economic applications include medical decision making, where the states involved are the decision maker's health status. Considering a state of health that he have not experienced, a decision maker may find himself unable to compare risky medical treatments. Another example is the need to choose among alternative pension plans. In general, such plans are complicated and difficult to compare. It may well be the case that a decision maker find the plans incomparable. Behaviorally this may result in inertia, that is, if assigned randomly to a particular plan, a decision maker may be reluctant to switch to another plan.

In this paper we studied alternative axiomatizations of strict partial orders, depicting choice behavior under risk and under uncertainty, and their corresponding representations. Taking a “constructive” approach, we revisited some known results, offering variations on the axiomatic structure. In addition, we extended the analysis to obtain some new results, the most important of which are two axiomatizations of the multi-prior expected multi-utility representation.

Following conventional usage, we interpreted Knightian uncertainty, or the multi-prior expected utility as expression of incomplete *beliefs*. We note, however, that, as is the case in the Anscombe and Aumann (1963) and other models based on the analytical framework of Savage (1954), the probabilities in this model are based on the convention that constant acts are constant utility acts.<sup>29</sup> Because this convention is not implied by the axiomatic structure, it may well be that what is referred to as incomplete beliefs is in fact an expression of incomplete tastes of a particular kind, namely, variations in the utility functions across states that does not involve variations in the risk attitudes. By contrast, what is usually referred to as incomplete tastes pertains to situations in which the decision maker is not sure about his risk attitudes within states.

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<sup>29</sup>See discussion in Seidenfeldt et. al. (1995) and Karni (2009).

## 7 Proofs

### 7.1 Proof of theorem 1

(i)  $\Rightarrow$  (ii). If  $\succ$  is empty then, by definition,  $p \succcurlyeq q$  and  $q \succcurlyeq p$ , for all  $p, q \in C$ . Let  $\mathcal{U}$  be the set of all constant real-valued functions on  $C$ .

Henceforth, assume that  $\succ$  is not empty.

*Claim 1.*  $\succcurlyeq$  satisfies independence (that is, for all  $p, q, r \in C$  and  $\alpha \in (0, 1]$ ,  $p \succcurlyeq q$  implies  $\alpha p + (1 - \alpha)r \succcurlyeq \alpha q + (1 - \alpha)r$ ).

*Proof of claim 1:* Suppose that  $p \succcurlyeq q$ . Let  $s \succ \alpha p + (1 - \alpha)r$ , we need to show that  $s \succ \alpha q + (1 - \alpha)r$ . Fix  $t \succ p$  such that  $s \succ \alpha t + (1 - \alpha)r$ . To establish that such  $t$  exists, let  $B(p) = \{\ell \in C \mid \ell \succ p\}$ , for all  $p \in C$ , and define

$$T(\alpha p + (1 - \alpha)r) = \{\alpha \ell + (1 - \alpha)r \in C \mid \ell \in B(p)\}.$$

Then  $B(\alpha p + (1 - \alpha)r) \supset T(\alpha p + (1 - \alpha)r)$ .

Let  $C^0$  denote the algebraic interior of  $C$ . Consider first the case in which  $p \in C^0$ . Given  $s \in B(\alpha p + (1 - \alpha)r)$ , let  $\ell(\gamma) = \gamma s + (1 - \gamma)(\alpha p + (1 - \alpha)r)$ . Apply (A.3) twice to obtain  $s \succ \ell(\gamma) \succ \alpha p + (1 - \alpha)r$  for all  $\gamma \in (0, 1)$ .

Let  $\varepsilon \in (0, 1)$  and define  $N_\varepsilon(\alpha p + (1 - \alpha)r) = \{\ell(\gamma) \in C \mid \gamma \in (0, \varepsilon)\}$  then, for sufficiently small  $\varepsilon$ ,  $N_\varepsilon(\alpha p + (1 - \alpha)r) \subset T(\alpha p + (1 - \alpha)r)$ . Thus  $\ell(\gamma) \in B(p)$ . Let  $t = \ell(\gamma^0)$  for some  $\gamma^0 \in (0, \varepsilon)$ .

Suppose next that  $p \in C - C^0$ . Let  $p' \in C^0$  such that  $p' \succeq p$  and  $s \succ \alpha p' + (1 - \alpha)r$ . (To show that such  $p'$  exists, note that, by (A.3) and definition 1,  $\alpha p'' + (1 - \alpha)r \succeq \alpha p + (1 - \alpha)r$  for all  $p''$  satisfying  $p'' \succeq p$ . In particular,  $\alpha p'' + (1 - \alpha)r \succeq p$ . If  $s \succ \alpha p'' + (1 - \alpha)r$  let  $p' = p''$ . If  $\alpha p'' + (1 - \alpha)r \succ s \succ \alpha p + (1 - \alpha)r$  then, by (A.2), there exist  $\beta \in (0, 1)$  such that

$$s \succ \alpha(\beta p'' + (1 - \beta)p) + (1 - \alpha)r.$$

Let  $p' = \beta p'' + (1 - \beta)p$ . Then  $p'$  satisfies the requirements mentioned above.) Repeat the above argument replacing  $p$  with  $p'$ . Then  $t \succ p'$ . If  $p' \succeq p$  then, in particular,  $p' \succcurlyeq p$ . Hence, by definition 1,  $t \succ p$ .

By (A.3)  $\alpha t + (1 - \alpha)r \succ \alpha p + (1 - \alpha)r$ . But  $s \succ \alpha t + (1 - \alpha)r$ , hence, apply (A.3) twice to obtain,  $s \succ \beta(\alpha t + (1 - \alpha)r) + (1 - \beta)(\alpha p + (1 - \alpha)r)$ . Thus  $s \succ \alpha(\beta t + (1 - \beta)p) + (1 - \alpha)r$ . By (A.3),  $\beta t + (1 - \beta)p \succ p$  implying that  $\beta t + (1 - \beta)p \succ q$ . Thus  $s \succ \alpha(\beta t + (1 - \beta)p) + (1 - \alpha)r \succ \alpha q + (1 - \alpha)r$ , where the last preference follows from (A.3). Hence,  $\alpha p + (1 - \alpha)r \succcurlyeq \alpha q + (1 - \alpha)r$ . ♣



A preference relation  $\succsim$  is said to satisfy *mixture monotonicity* if  $p \succ q$  and  $0 \leq \alpha < \beta \leq 1$  imply that  $\beta p + (1 - \beta) q \succ \alpha p + (1 - \alpha) q$ .

*Claim 2:*  $\succsim$  satisfies mixture monotonicity.

The proof, by standard argument, is an implication of (A.3).<sup>30</sup>

*Claim 3.* Let  $p^M$  and  $p^m$  be the greatest and smallest elements of  $C$ , respectively. Then, for each  $q \in C$ , there exist  $\underline{\alpha}(q), \bar{\alpha}(q) \in (0, 1)$  such that  $\alpha p^M + (1 - \alpha) p^m \succ q$  for all  $\alpha > \bar{\alpha}(q)$  and  $q \succ \alpha p^M + (1 - \alpha) p^m$  for all  $\alpha < \underline{\alpha}(q)$ .

*Proof of claim 3:* Let  $S_q^+ = \{\alpha \in [0, 1] \mid \alpha p^M + (1 - \alpha) p^m \succ q\}$ . Since  $S_q^+$  is not empty (e.g.,  $1 \in S_q^+$ ) and bounded, the infimum of  $S_q^+$  exists. Let  $\bar{\alpha}(q) = \inf S_q^+$ . By mixture monotonicity,  $\alpha > \bar{\alpha}(q)$  implies  $\alpha \in S_q^+$ .

Next we show that  $\bar{\alpha}(q) \notin S_q^+$ . Suppose, by way of negation, that  $\bar{\alpha}(q) \in S_q^+$  then, by (A.2), there is  $\beta \in (0, 1)$  such that  $\beta (\bar{\alpha}(q) p^M + (1 - \bar{\alpha}(q)) p^m) + (1 - \beta) p^m \succ q$ . Hence  $\beta \bar{\alpha}(q) p^M + (1 - \beta \bar{\alpha}(q)) p^m \succ q$ . By mixture monotonicity,  $\alpha p^M + (1 - \alpha) p^m \succ q$  for all  $\alpha > \beta \bar{\alpha}(q)$ . But  $\bar{\alpha}(q) > \beta \bar{\alpha}(q)$ , thus  $\bar{\alpha}(q)$  is not a lower bound of  $S_q^+$ . A contradiction.

Let  $\underline{\alpha}(q)$  be the supremum of  $S_q^- := \{\alpha \in [0, 1] \mid q \succ \alpha p^M + (1 - \alpha) p^m\}$ . By similar argument,  $\alpha \in S_q$  for all  $\underline{\alpha}(q) > \alpha$ , and  $\underline{\alpha}(q) \notin S_q^-$ . ♣

*Claim 4.* For all  $q \in C$ ,  $\bar{\alpha}(q) p^M + (1 - \bar{\alpha}(q)) p^m \succeq q$  and  $q \succeq \underline{\alpha}(q) p^M + (1 - \underline{\alpha}(q)) p^m$ .

*Proof of claim 4:* Let  $r \succ \bar{\alpha}(q) p^M + (1 - \bar{\alpha}(q)) p^m$  then, by (A.2), there is  $\beta \in (0, 1)$  such that  $r \succ [\beta (1 - \bar{\alpha}(q)) + \bar{\alpha}(q)] p^M + (1 - \beta (1 - \bar{\alpha}(q)) + \bar{\alpha}(q)) p^m$ . But  $\bar{\alpha}(q) < \bar{\alpha}(q) + \beta (1 - \bar{\alpha}(q))$ , hence

$$[\beta (1 - \bar{\alpha}(q)) + \bar{\alpha}(q)] p^M + (1 - \beta (1 - \bar{\alpha}(q)) + \bar{\alpha}(q)) p^m \in S_q^+.$$

Thus, by transitivity,  $r \succ q$ . Hence, by Definition 1,  $\bar{\alpha}(q) p^M + (1 - \bar{\alpha}(q)) p^m \succsim q$ .

Moreover,  $\bar{\alpha}(q) \notin S_q^+$  implies that  $\neg(q \succ \bar{\alpha}(q) p^M + (1 - \bar{\alpha}(q)) p^m)$ . Hence, by Definition 1,  $\bar{\alpha}(q) p^M + (1 - \bar{\alpha}(q)) p^m \succeq q$ .

The proof that  $q \succeq \underline{\alpha}(q) p^M + (1 - \underline{\alpha}(q)) p^m$  is by the same argument. ♣

For every  $p \in C$ , let  $\mathcal{L}(p)$  be the linear subspace spanned by the vectors  $(\bar{\alpha}(p) - p)$  and  $(p^M - p^m)$ .<sup>31</sup>

<sup>30</sup>See Kreps (1988).

<sup>31</sup>That is,  $\mathcal{L}(p) = \{q \in C \mid q = \xi (\bar{\alpha}(p) - p) + \lambda (p^M - p^m), \xi, \lambda \in \mathbb{R}\}$ .

*Claim 5.* The functions  $\underline{\alpha}(\cdot)$  and  $\bar{\alpha}(\cdot)$  are affine on  $\mathcal{L}(p) \cap C$ .

*Proof of claim 5:* Let  $p, q \in \mathcal{L}(p)$  and suppose that  $p \succneq q$ . Define  $\varphi(p) = \bar{\alpha}(p)p^M + (1 - \bar{\alpha}(p))p^m$ . Then  $\varphi(p) - p$  is parallel to  $\varphi(q) - q$ . To see this, suppose, by way of negation that  $\varphi(p) - p$  is not parallel to  $\varphi(q) - q$ , then

$$\{q + \lambda(\varphi(p) - p) \mid \lambda \geq 0\} \cap \langle p^m, p^M \rangle = \varphi(q) + \mu(p^M - p^m), \quad (16)$$

where  $\langle p^m, p^M \rangle := \{\lambda p^M + (1 - \lambda)p^m \mid \lambda \in \mathbb{R}\}$  is the line that goes through  $p^M$  and  $p^m$ .

Two vectors being nonparallel implies that  $\mu \neq 0$ . Without loss of generality, assume that  $\mu > 0$ . By claim 1 and (A.3),  $\beta p + (1 - \beta)q \succneq q$ . For sufficiently small  $\beta$ ,  $\varphi(q) + \mu(p^M - p^m) \succ \varphi(\beta p + (1 - \beta)q) \succneq \varphi(q)$ . Then there exists  $r \in (\varphi(\beta p + (1 - \beta)q), \varphi(q))$ . This means  $r \succ \beta p + (1 - \beta)q$  and  $\neg(r \succ q)$ . This contradicts  $\beta p + (1 - \beta)q \succneq q$ .

The affinity of  $\underline{\alpha}(\cdot)$  is proved similarly. ♣

*Claim 6.* The function  $\bar{\alpha}(\cdot)$  is convex on  $C$ .

*Proof of claim 6:* Let  $p$  and  $q$  be such that  $\mathcal{L}(p) \neq \mathcal{L}(q)$  and  $\bar{\alpha}(p) = \bar{\alpha}(q) = \hat{\alpha}$ . Thus,  $\hat{\alpha}p^M + (1 - \hat{\alpha})p^m \succneq p, q$ . By Claim 1,  $\beta p + (1 - \beta)(\hat{\alpha}p^M + (1 - \hat{\alpha})p^m) \succneq \beta p + (1 - \beta)q$ , for all  $\beta \in [0, 1]$ . Moreover, since  $\hat{\alpha}p^M + (1 - \hat{\alpha})p^m \in \mathcal{L}(p)$ , by Claim 5

$$\bar{\alpha}(\beta p + (1 - \beta)(\hat{\alpha}p^M + (1 - \hat{\alpha})p^m)) = \beta \bar{\alpha}(p) + (1 - \beta) \bar{\alpha}(\hat{\alpha}p^M + (1 - \hat{\alpha})p^m) = \hat{\alpha}.$$

Hence, by Definition 1,  $\hat{\alpha} \geq \bar{\alpha}(\beta p + (1 - \beta)q)$ . Thus,  $\bar{\alpha}(\beta p + (1 - \beta)q) \leq \hat{\alpha} = \beta \bar{\alpha}(p) + (1 - \beta) \bar{\alpha}(q)$ . ♣

Let  $\{e^i \in \mathcal{L} \mid i = 1, \dots, n\}$  be the canonical basis of  $\mathcal{L}$ . By mixture monotonicity,  $p^M = e^i$  and  $p^m = e^j$ , for some  $i, j$ . Without loss of generality, let  $p^M = e^n$  and  $p^m = e^1$ . For every  $i \in \{1, \dots, n - 2\}$  and  $\lambda \in [0, 1]$ , define  $q(i, \lambda) = \lambda e^i + (1 - \lambda)e^{i+1}$ , and  $q(n - 1, \lambda) = \lambda e^{n-1} + (1 - \lambda)e^1$ . For every  $p \in C$ , let  $\mathcal{L}(p, q(i, \lambda))$  be the linear subspace spanned by  $p^M, p$  and  $q(i, \lambda)$ .

Let  $J = \{p \in C \mid e^M \succ p \succ e^i, i = 1, \dots, n - 1\}$ . For every  $\lambda \in [0, 1]$ , define

$$\bar{\alpha}^{i\lambda}(p) = \inf\{\alpha \in [0, 1] \mid \alpha e^M + (1 - \alpha)q(i, \lambda) \succ p\}.$$

By the same argument as above,  $\bar{\alpha}^{i\lambda}(\cdot)$  is a convex function on  $J$  whose restriction to  $\mathcal{L}(p, q(i, \lambda))$  is affine.

Fix  $\bar{p} \in J$  and let  $Q = \{q \in J \mid \bar{\alpha}^{i\lambda}(q) = \bar{\alpha}^{i\lambda}(\bar{p})\}$ . The every  $p \in C$  may be expressed as  $p = \varsigma p^M + (1 - \varsigma)q$  for some  $q \in Q$  and  $1 \geq \varsigma$ . Extend  $\bar{\alpha}^{i\lambda}(\cdot)$  to  $C$  by defining  $\bar{\alpha}^{i\lambda}(p) = 1 - (1 - \varsigma) \bar{\alpha}^{i\lambda}(\bar{p})$ .

By convexity,  $\bar{\alpha}^{i\lambda}(\cdot)$  is differentiable everywhere, except possibly at a countable number of points. For every  $p \in C$  at which  $\bar{\alpha}^{i\lambda}(\cdot)$  is differentiable, denote by  $\nabla \bar{\alpha}^{i\lambda}(p)$  the gradient vector of  $\bar{\alpha}^{i\lambda}(\cdot)$  at  $p$ .<sup>32</sup> The affinity of  $\bar{\alpha}^{i\lambda}(\cdot)$  on  $\mathcal{L}(p, q(i, \lambda))$  implies that  $\nabla \bar{\alpha}^{i\lambda}(p) = \nabla \bar{\alpha}^{i\lambda}(r)$  for all  $r \in \mathcal{L}(p, q(i, \lambda))$ .

Define  $G = \{\nabla \bar{\alpha}_p^{i\lambda} \in \mathbb{R}^n \mid i = 1, \dots, n-1, \lambda \in [0, 1], p \in Q\}$ . For each  $u \in \mathbb{R}^n$  define a function  $U : C \rightarrow \mathbb{R}$  by  $U(q) = u \cdot q$ . Let  $\mathcal{U} := \{U \mid u \in G\}$ , then, by definition,  $U \in \mathcal{U}$  implies that  $U$  is affine.

By definition of  $\bar{\alpha}^{i\lambda}(\cdot)$ ,  $q \succ p$  implies  $\bar{\alpha}^{i\lambda}(q) > \bar{\alpha}^{i\lambda}(p)$ , for all  $\bar{\alpha}^{i\lambda}(\cdot)$ , and  $q \succcurlyeq p$  implies  $\bar{\alpha}^{i\lambda}(q) \geq \bar{\alpha}^{i\lambda}(p)$ , for all  $\bar{\alpha}^{i\lambda}(\cdot)$ . But  $\bar{\alpha}^{i\lambda}(q) > \bar{\alpha}^{i\lambda}(p)$  if and only if  $\bar{\alpha}_r^{i\lambda}(q) > \bar{\alpha}_r^{i\lambda}(p)$  for all  $i = 1, \dots, n-1, \lambda \in [0, 1]$  and  $r \in Q$ . Thus, by definition  $q \succ p$  implies  $U(p) > U(q)$  for all  $U \in \mathcal{U}$  and  $q \succcurlyeq p$  implies  $U(p) \geq U(q)$ .

To show the converse, let  $U(q) \geq U(p)$  for all  $U \in \mathcal{U}$ , and suppose, by way of negation, that not  $q \succcurlyeq p$ . If  $p \succ q$  then, by necessity,  $U(p) > U(q)$ , for all  $U \in \mathcal{U}$ , which is a contradiction. Suppose that  $q \bowtie p$  and let  $\partial B(r)$  denote the boundary of the upper contour set,  $B(r)$ , of  $r$ .

*Claim 7:* For all  $q, p \in C$ ,  $q \bowtie p$  implies that  $\partial B(q) \cap \partial B(p) \neq \emptyset$ .

*Proof of claim 7:* Since  $B(p)$  and  $B(q)$  are full dimensional cones,  $q \bowtie p$  implies that there exist  $d \in \partial B(q)$  such that the ray  $\langle d, q \rangle := \{\xi(d - q) \mid \xi > 0\}$ , intersects  $B(p)$ . Let  $r = \langle d, q \rangle \cap B(p)$ . Thus,  $r \in \partial B(q) \cap \partial B(p)$ . ♣

Choose  $r \in \partial B(q) \cap \partial B(p)$ , then  $r \succcurlyeq p$  and  $r \succcurlyeq q$ . Let  $t = \langle p, r \rangle \cap \partial C$  then, by definition,  $U(t) = U(p) = U(r)$  for some  $U \in \mathcal{U}$ . Moreover,  $U(t) > U(q)$  implying that  $U(r) > U(q)$ . Hence,  $U(p) > U(q)$ . A contradiction.

Hence,  $q \succcurlyeq p$  if and only if  $U(q) \geq U(p)$ , for all  $U \in \mathcal{U}$ . By the same argument,  $q \succ p$  if and only if  $U(q) > U(p)$  for all  $U \in \mathcal{U}$ . This complete the proof that (i) implies (ii).

The proof the (ii) implies (i) is straightforward.

To prove the uniqueness, let  $\mathcal{V} = \{\sum_{j=1}^r \lambda_j U_j + d \mid \lambda_j > 0, \sum_{j=1}^r \lambda_j = 1, d \in \mathbb{R}, U_j \in \mathcal{U} \text{ and } r \in N\}$ , then, obviously,  $V(p) > V(q)$  for all  $V \in \mathcal{V}$  if and only

<sup>32</sup>Denote by  $G_{\bar{\alpha}^{i\lambda}}$  the epigraph of  $\bar{\alpha}^{i\lambda}(\cdot)$ , then  $G_{\bar{\alpha}^{i\lambda}}$  and  $C$  are convex sets, hence, so is  $G_{\bar{\alpha}} \cap C$ . For every  $p \in C$  such that  $e^n \succ p \succ e^i$ , for all  $i = 1, \dots, n-1$ , and let  $H(u_{\bar{\alpha}^{i\lambda}, p}, \bar{\alpha}^{i\lambda}(p))$  be a supporting hyperplane of  $G_{\bar{\alpha}^{i\lambda}}$  at  $p$ . That such a hyperplane exists follows from the fact that the algebraic interior,  $G_{\bar{\alpha}^{i\lambda}}^o$ , of  $G_{\bar{\alpha}^{i\lambda}}$  is nonempty (e.g.,  $(\frac{1}{2}\bar{\alpha}(p) + \frac{1}{2}1, p) \in G_{\bar{\alpha}}^o$ ).

if  $U(p) > U(q)$  for all  $U \in \mathcal{U}$ . The affinity of  $U \in \mathcal{U}$  implies the affinity of  $V$ .

Let  $\mathcal{V}$  be a set of real-valued, affine, functions on  $C$  and suppose that  $\mathcal{V}$  represents  $\succsim$  in the sense of (2). Suppose that there is  $v^0 \in \mathcal{V}$  and  $v^0$  is not in the cone spanned by  $\mathcal{U}$ . Because the cone spanned by  $\mathcal{U}$  is closed and has a non-empty interior, by the separating hyperplane theorem, there is  $t \in \mathbb{R}^n$  such that  $u \cdot \frac{t}{\|t\|} < 0 < v^0 \cdot \frac{t}{\|t\|}$  for all  $u \in \mathcal{U}$ . Take  $p, q \in C$  such that  $\frac{t}{\|t\|} = p - q$ . Then  $u \cdot q > u \cdot p$  and  $v^0 \cdot p > v^0 \cdot q$ . A contradiction.  $\blacksquare$

## 7.2 Proof of theorem 2

(i)  $\Rightarrow$  (ii). Axioms (A.1), (A.2), (A.3) and Corollary 2 imply that every  $w \in \mathcal{W}$  may be expressed as  $|S|$ -tuple  $(w(\cdot, 1), \dots, w(\cdot, |S|)) \in \mathcal{W}_1 \times \dots \times \mathcal{W}_{|S|}$  and, for all  $h, f \in H$ ,  $h \succsim f$  if and only if  $\sum_{s \in S} w(h(s), s) \geq \sum_{s \in S} w(f(s), s)$  for all  $w \in \mathcal{W}$ .

Let  $B := \{\lambda(h' - h) \mid h' \succsim h, h', h \in H, \lambda \geq 0\}$ . Then  $B$  is a closed convex cone with non-empty interior in finite dimensional linear space. By theorem V.9.8 in Dunford and Schwartz (1957), there is a dense set,  $D$ , in its boundary such that each point of  $D$  has a unique tangent. Let  $\mathcal{W}^o$  be the collection of all the supporting hyperplanes corresponding to this dense set. Without loss of generality we assume that each function in  $\mathcal{W}^o$  has unit normal vector. It is easy to see that  $\mathcal{W}^o$  represents  $\succsim$ .

For every  $f \in H$  let  $H^c(f)$  be the convex-hull of  $\{f^s \mid s \in S\}$ . Then,  $f^\alpha \in H^c(f)$  is the constant act defined by  $f^\alpha = \sum_{s \in S} \alpha_s f^s$  for all  $\alpha \in \Delta(S)$ . Now, (A.3) implies that  $g \succ f^s$  for every  $s \in S$  if and only if  $g \succ f^\alpha$  for every  $\alpha \in \Delta(S)$ . Sufficiency is immediate since, for all  $s \in S$ ,  $\delta_s \in \Delta(S)$ . To prove necessity, suppose that  $g \succ f^s$  for all  $s \in S$ . Apply (A.3) twice to obtain

$$g = \alpha g + (1 - \alpha) g \succ \alpha g + (1 - \alpha) f^s \succ \alpha f^{s'} + (1 - \alpha) f^s,$$

for all  $\alpha \in [0, 1]$  and  $s, s' \in S$ . Let  $f^{s'} \alpha f^s := \alpha f^{s'} + (1 - \alpha) f^s$ , then, by repeated application of (A.3), we have  $g \succ \alpha' (f^{s'} \alpha f^s) + (1 - \alpha') f^{s''}$ , for all  $\alpha \in [0, 1]$  and  $s, s', s'' \in S$ . By the same argument,  $g \succ f^\alpha$  for all  $f^\alpha \in H^c(f)$ . Hence, an equivalent statement of (A.4) is,

(A.4') For all  $f, g \in H$ ,  $g \succ f^\alpha$  for every  $\alpha \in \Delta(S)$  implies  $g \succ f$ .

Before presenting the main argument of the proof we provide some useful facts.

*Claim 1:* For all  $f, g \in H$ , if  $g \succsim f^\alpha$  for all  $\alpha \in \Delta(S)$  then  $g \succsim f$ .

The proof is immediate application of (A.4), the preceding argument, and Definitions 1. Henceforth, when we invoke axiom (A.4) we will use it in either the, equivalent, strict preference form (A.4') or the weak preference form given in Claim 1, as the need may be.

To state the next result we invoke the following notations. For each  $h \in H$  and  $s \in S$ , let  $h_{-s}p$  the act that is obtained by replacing the  $s$ -th coordinate of  $h$ ,  $h(s)$ , with  $p$ . Let  $h^p$  denote the constant act whose payoff is  $h^p(s) = p$ , for every  $s \in S$

*Claim 2:* If  $h^p \succcurlyeq h^q$  then  $h^p \succcurlyeq h_{-s}^p q$ , for all  $s \in S$ .

*Proof of claim 2:* For any  $\alpha \in \Delta(S)$ ,  $(h_{-s}^p q)^\alpha$  is a convex combination of  $h^p$  and  $h^q$ . To be exact,  $(h_{-s}^p q)^\alpha = (1 - \alpha_s) h^p + \alpha_s h^q$ . By (A.3), applied to  $\succcurlyeq$ , we have  $h^p \succcurlyeq \alpha_s h^p + (1 - \alpha_s) h^q \succcurlyeq h^q$ , (that is,  $h^p \succcurlyeq (h_{-s}^p q)^\alpha$  for all  $\alpha \in \Delta(S)$ ). Hence, by (A.4) and Claim 1,  $h^p \succcurlyeq h_{-s}^p q$ . ♣

We now turn to the main argument. In particular, we show that the component functions,  $\{w_s\}_{s \in S}$ , of each essential function,  $w \in \mathcal{W}^o$ , that figures in the representation are positive linear transformations of one another.

*Lemma:* If  $\hat{w} \in \mathcal{W}^o$  then, for all non-null  $s, t \in S$ ,  $\hat{w}(\cdot, s)$  and  $\hat{w}(\cdot, t)$  are positive linear transformations of one another.

*Proof of Lemma:* By way of negation, suppose that there exist  $s, t$  such that  $\hat{w}(\cdot, s)$  and  $\hat{w}(\cdot, t)$  are not positive linear transformations of one another. Then there are  $p, q \in \Delta(X)$  such that  $\hat{w}(p, s) > \hat{w}(q, s)$  and  $\hat{w}(q, t) > \hat{w}(p, t)$ . Without loss of generality, let  $p$  in the interior of  $\Delta(X)$  be such that  $\hat{w}(h^p) > \hat{w}(h^q)$ . Define  $q(\lambda) = \lambda p + (1 - \lambda)q$  for  $\lambda \in (0, 1)$ , then  $\hat{w}(p, s) > \hat{w}(q(\lambda), s)$  and  $\hat{w}(q(\lambda), t) > \hat{w}(p, t)$ . Following Ok et. al. (2008), we use the following construction. Let  $f_\lambda \in H$  be defined as follows:  $f_\lambda(s') = p$  if  $s' = s$ ,  $f_\lambda(s') = q(\lambda)$  if  $s' = t$ , and, for  $s' \neq s, t$ ,  $f_\lambda(s') = p$  if  $w(p, s') \geq w(q(\lambda), s')$ , and  $f_\lambda(s') = q(\lambda)$  otherwise.

Clearly,  $\sum_{s \in S} \hat{w}(f_\lambda(s), s) > \sum_{s \in S} \hat{w}(f_\lambda^\alpha(s), s)$ , for all  $\alpha \in \Delta(S)$ . Since  $f_\lambda$  involves only  $p$  and  $q(\lambda)$ ,  $\{f_\lambda^\alpha \mid \alpha \in \Delta(S)\} = \{\alpha h^p + (1 - \alpha) h^{q(\lambda)} \mid \alpha \in [0, 1]\}$ .

Since  $\hat{w} \in \mathcal{W}^o$ , there exists  $g \in H$  such that  $g \succcurlyeq h^p$ ,  $\hat{w}(g) = \hat{w}(h^p)$  and  $\hat{w}$  is the unique supporting hyperplane at  $g$ .

*Claim 3:* There exist  $\beta^* > 0$  such that  $h^p + \beta^*(g - h^p) \succcurlyeq h^{q(\lambda)}$

*Proof of claim 3:* Suppose not. Then, for any  $n \in \{1, 2, \dots\}$ , there exists  $w_n \in \mathcal{W}^o$  such that  $w_n(h^p + n(g - h^p)) < w_n(h^{q(\lambda)})$ . Since  $w_n$  is linear, we can regard  $w_n$  as a vector and  $w_n(f)$  as the inner product  $w_n \cdot f$ . Hence, we have

$$n w_n \cdot (g - h^p) < w_n \cdot (h^{q(\lambda)} - h^p), \text{ for all } n. \quad (17)$$

Since  $\|w_n\| = 1$ , we can find convergent subsequence  $\{w_{n_k}\}$ . Without loss of generality we assume that  $\{w_n\}$  itself is convergent and  $w_n \rightarrow w^* \in cl(\mathcal{W}^o)$ . The right-hand side of inequality (17) converges to  $w^* \cdot (h^{q(\lambda)} - h^p)$ . If  $w^* \cdot (g - h^p) > 0$  then the left-hand side of inequality (17) will explode to  $+\infty$  as  $n \rightarrow \infty$ . A contradiction. Hence,  $w^*(g) = w^*(h^p)$ . Also,  $w_n(h^p) \leq w_n(h^p + n(g - h^p)) < w_n(h^{q(\lambda)})$  implies  $w^*(h^p) \leq w^*(h^{q(\lambda)})$ . Since  $\hat{w}(h^p) > \hat{w}(h^{q(\lambda)})$ ,  $\hat{w} \neq w^*$ . This contradicts the uniqueness of the supporting hyperplane at  $g \in H$ . This completes the proof of claim 3.

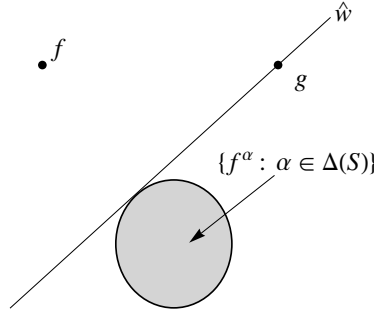


Figure 3

Let  $g_\lambda = h^p + \beta(g - h^p)$ . Then  $g_\lambda \succcurlyeq h^p$  and  $g_\lambda \succcurlyeq h^{q(\lambda)}$ . By choosing  $\lambda$  close to 1, we can find  $g_\lambda$  that is feasible (i.e.,  $g_\lambda(s) \in \Delta(X)$  for all  $s \in S$ ). By virtue of being on the hyperplane defined by  $w$ ,  $\sum_{s \in S} w(g_\lambda(s), s) = w(h^p)$ . Since  $g_\lambda \succcurlyeq h^p, h^{q(\lambda)}$ , we have  $g_\lambda \succcurlyeq (f_\lambda)^\alpha$ , for all  $\alpha \in \Delta(S)$ . Hence, by (A.4) and Claim 1,  $g_\lambda \succcurlyeq f_\lambda$ . But  $\sum_{s \in S} \hat{w}(f_\lambda(s), s) > \hat{w}(h^p) = \sum_{s \in S} \hat{w}(g_\lambda(s), s)$ , which is a contradiction. (See Figure 3). Hence, if  $\hat{w}(\cdot, s)$  and  $\hat{w}(\cdot, t)$  are not positive linear transformation of one another then  $\hat{w} \notin \mathcal{W}^o$ . This completes the proof of the Lemma. ♣

The representation follows by standard argument. For each  $w \in \mathcal{W}^o$ , define  $U^w(\cdot) = w(\cdot, 1)$  and, for all  $s \in S$ , let  $w(\cdot, s) = b_s^w U^w(\cdot) + a_s^w$ ,  $b_s^w > 0$ . Define  $\pi^w(s) = b_s^w / \sum_{s' \in S} b_{s'}^w$ , for all  $s \in S$ . Let  $\mathcal{U}$  be the collection of distinct  $U^w$  and for each  $U \in \mathcal{U}$ , let  $\Pi^U = \{\pi^w \mid \forall w \text{ such that } U^w = U\}$ .

To see we can also represent strict relation  $\succ$ , we need to consider the following. Any supporting hyperplane  $w$  of  $B$  can be expressed as a limit point of sequence  $\{w_n\}$  from  $\mathcal{W}^o$ . Since any  $w_n$  has the property that each of its component is a positive linear transformation of one another,  $w$  has the same property. If we add all those  $w$ 's into  $\mathcal{W}^o$ , then the new set of functions will represent both  $\succcurlyeq$  and  $\succ$ .

(ii)  $\Rightarrow$  (i). Axioms (A.1) - (A.3) are implied by Corollary 2. Axioms (A.4) is an immediate implication of the representation.  $\blacksquare$

### 7.3 Proof of theorem 3

(i)  $\Rightarrow$  (ii). Suppose that  $\succ$  on  $H$  satisfies (A.1) - (A.5). We show first that  $\mathcal{M}$  is a convex set. Let  $\xi_T$  and  $\xi_{T'}$  be the normals corresponding to  $T, T' \in \mathcal{T}$ . Let

$$\hat{T} := \{z \in \mathbb{R}^{|S| \times |X|} \mid (\lambda \xi_T + (1 - \lambda) \xi_{T'}) \cdot (z - h^x) = 0\}. \quad (18)$$

Then  $\hat{T}$  is an hyperplane in  $\mathbb{R}^{|S| \times |X|}$  that passes through  $h^x$ . We need to show that  $B(h^x) \subset \hat{T}^+$ . Take any  $f \in B(h^x)$ , since  $B(h^x) \subset T^+ \cap T'^+$ ,  $\xi_T \cdot (f - h^x) \geq 0$  and  $\xi_{T'} \cdot (f - h^x) \geq 0$ . Thus,  $(\lambda \xi_T + (1 - \lambda) \xi_{T'}) \cdot (f - h^x) \geq 0$ . Hence,  $\hat{T}$  is a supporting hyperplane of  $B(h^x)$  at  $h^x$ .

By definition,  $\beta_{\hat{T}}(s) = \sum_{x \in X} (\lambda \xi_T + (1 - \lambda) \xi_{T'})(x, s) = \lambda \beta_T(s) + (1 - \lambda) \beta_{T'}(s)$ , for all  $s \in S$ . Define  $\alpha_{\hat{T}} \in \Delta(S)$  by  $\alpha_{\hat{T}}(s) = \beta_{\hat{T}}(s) / \sum_{s' \in S} \beta_{\hat{T}}(s')$ , for all  $s \in S$ . Then,  $\alpha_{\hat{T}} = \lambda \alpha_T + (1 - \lambda) \alpha_{T'}$ ,  $\lambda \in (0, 1)$  and, by definition,  $\alpha_{\hat{T}} \in \mathcal{M}$ . Hence  $\mathcal{M}$  is convex.

By (A.5),  $g \succ f$  implies that  $g^\alpha \succ f^\alpha$ , for all  $\alpha \in \mathcal{M}$ . By Theorem 2,  $g^\alpha \succ f^\alpha$  for all  $\alpha \in \mathcal{M}$  if and only if  $U(g^\alpha) > U(f^\alpha)$  for all  $U \in \mathcal{U}$  and  $\alpha \in \mathcal{M}$ . By the affinity of  $U \in \mathcal{U}$ ,  $U(g^\alpha) > U(f^\alpha)$  for all  $U \in \mathcal{U}$  and  $\alpha \in \mathcal{M}$  if and only if  $\sum_{s \in S} U(g(s)) \alpha(s) > \sum_{s \in S} U(f(s)) \alpha(s)$ , for all  $(\alpha, U) \in \mathcal{M} \times \mathcal{U}$ . Hence,  $g \succ f$  implies  $\sum_{s \in S} U(g(s)) \alpha(s) > \sum_{s \in S} U(f(s)) \alpha(s)$ , for all  $(\alpha, U) \in \mathcal{M} \times \mathcal{U}$ .

To prove the inverse implication, note that, by Theorem 2,  $g^\alpha \succ f^\alpha$  if and only if  $U(g^\alpha) > U(f^\alpha)$  for all  $U \in \mathcal{U}$ . Thus, by the affinity of  $U \in \mathcal{U}$ ,  $g^\alpha \succ f^\alpha$  for all  $\alpha \in \mathcal{M}$  if and only if  $\sum_{s \in S} U(g(s)) \alpha(s) > \sum_{s \in S} U(f(s)) \alpha(s)$ , for all  $(\alpha, U) \in \mathcal{M} \times \mathcal{U}$ . But, by Theorem 2,  $\sum_{s \in S} U(g(s)) \alpha(s) > \sum_{s \in S} U(f(s)) \alpha(s)$ , for all  $(\alpha, U) \in \{(\alpha, U) \mid U \in \mathcal{U}, \alpha \in \Pi^U\}$  if and only if  $g \succ f$ . Since  $\mathcal{M} \supseteq \cup_{U \in \mathcal{U}} \Pi^U$ , this implies the representation (9) in Theorem 3. The representation (8) follows by similar argument.

(ii)  $\Rightarrow$  (i). The necessity of the  $\succ$ -boundedness of  $H$  and (A.1)-(A.4) follows from Theorem 2. To prove the necessity of (A.5), we need to show that  $\mathcal{M} = \mathcal{M}'$ , where  $\mathcal{M}'$  is defined as in (A.5).  $\mathcal{M} \subseteq \mathcal{M}'$  is easy to show. To show that  $\mathcal{M}' \subseteq \mathcal{M}$ , suppose there exist  $\pi \in \mathcal{M}' - \mathcal{M}$ . For any  $U \in \mathcal{U}$ , the hyperplane corresponding  $(\pi, U)$  is a supporting hyperplane of “better than” sets in  $H$ . This hyperplane can be expressed as a convex combination of essential hyperplanes supporting the “better than” sets. By

Theorem 2, those essential hyperplanes correspond to probability-utility pairs  $(\pi_\lambda, U_\lambda)_{\lambda \in \Lambda} \subset \mathcal{M} \times \mathcal{U}$ , where  $\Lambda$  is the index set of the essential functions that figure in Theorem 2. Thus,  $\pi$  is a convex combination of  $(\pi_\lambda)_{\lambda \in \Lambda}$ . Since  $\mathcal{M}$  is convex, we have  $\pi \in \mathcal{M}$  which is a contradiction.

The uniqueness is implied by the uniqueness of the representations in Theorem 2. ■

## 7.4 Proof of theorem 4

(i)  $\Rightarrow$  (ii). By Corollary 2,  $p \succcurlyeq_s q$  if and only if  $\sum_{x \in X} w(x, s) p(x) \geq \sum_{x \in X} w(x, s) q(x)$ , for all  $w \in \mathcal{W}$ . But the preference relation  $\succcurlyeq_s$  on  $\Delta(X)$  is a complete, (A.6), weak order satisfying the Archimedean, (A.2), and independence axioms (A.3). Thus, by the von Neumann Morgenstern expected utility theorem it has an expected utility representation,  $p \mapsto \sum_{x \in X} u_s(x) p(x)$ , where  $u_s(\cdot)$  is unique up to positive linear transformation.

Let  $t \in S$  be nonnull (that such a state exists is implied by the non-emptiness of  $\succ$ ) and define  $u(\cdot) = u_t(\cdot)$ . By Lemma 2, the functions  $\{u_s(\cdot)\}_{s \in S}$  are positive linear transformations of one another. Moreover, by the uniqueness of the von Neumann-Morgenstern expected utility theorem, for each  $w \in \mathcal{W}$ ,  $w(\cdot, s) = b_{ws} u(\cdot) + a_{ws}$ , where  $b_{ws} > 0$  if  $s$  is nonnull and  $b_{ws} = 0$  otherwise. Define  $\pi_w(s) = b_{ws}/b_w$ , where  $b_w = \sum_{s \in S} b_{ws}$ . Then conclusion follows from the representation (6), where  $\Pi = \{\pi_w(s) \mid w \in \mathcal{W}\}$ .

The proof that (ii)  $\Rightarrow$  (i) is straightforward.

The uniqueness follows from the uniqueness in Theorem 1. ■

## 7.5 Proof of theorem 5

First we show that (A.7) assures a unique probability over  $S$ .

*Claim 1:* Under (A.7),  $\pi^u(E) = \pi^l(E)$ .

*Proof of claim 1:* Mixture monotonicity implies that  $\pi^u(E) \geq \pi^l(E)$ . Suppose  $\pi^u(E) > \pi^l(E)$ . Then there exists  $\alpha_1, \alpha_2$  such that  $\pi^u(E) > \alpha_1 > \alpha_2 > \pi^l(E)$ . Since  $\pi^u(E) > \alpha_1$  implies  $p^M \alpha_1 p^m \succ p^M E p^m$  does not hold, (A.7) implies  $p^M E p^m \succ p^M \alpha_2 p^m$  which is a contradiction to  $\alpha_2 > \pi^l(E)$ . Therefore,  $\pi^u(E) = \pi^l(E)$ . ♣



Define  $\pi(E) := \pi^u(E) = \pi^l(E)$ . Next, we show that  $\pi$  is a probability measure.

*Claim 2:* Under (A.7),  $\pi : 2^S \rightarrow [0, 1]$  is a probability measure.

*Proof of claim 2:* By definition  $\pi(S) = 1$ . Since  $S$  is a finite set, it is enough to show that  $\pi(E \cup \{s\}) = \pi(E) + \pi(s)$  for all  $E \subseteq S$  and for all  $s \notin E$ .

First, we show  $\pi(E \cup \{s\}) \leq \pi(E) + \pi(s)$ . Without loss of generality, assume that  $\pi(E) + \pi(s) < 1$ . Pick any  $\varepsilon > 0$  such that  $\pi(E) + \pi(s) + 2\varepsilon < 1$ . Then there exist  $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1]$  such that  $\pi(E) < \beta_1 < \alpha_1 < \pi(E) + \varepsilon$  and  $\pi(s) < \beta_2 < \alpha_2 < \pi(s) + \varepsilon$ .

If we can show that  $p^M(\alpha_1 + \alpha_2)p^m \succ p^M(E \cup \{s\})p^m$ ,<sup>33</sup> then we have  $\pi(E \cup \{s\}) < \alpha_1 + \alpha_2 < \pi(E) + \pi(s) + 2\varepsilon$  which implies  $\pi(E \cup \{s\}) \leq \pi(E) + \pi(s)$ . Suppose that  $p^M(\alpha_1 + \alpha_2)p^m \succ p^M(E \cup \{s\})p^m$  does not hold. Then, by (A.7),  $p^M(\beta_1 + \beta_2)p^m \prec p^M(E \cup \{s\})p^m$ .

We know that  $p^M\beta_1p^m \succ p^MEp^m$  and  $p^M\beta_2p^m \succ p^M\{s\}p^m$  imply that, for all  $w \in \mathcal{W}$ ,

$$\beta_1 \sum_{s \in S} w(p^M, s) + (1 - \beta_1) \sum_{s \in S} w(p^m, s) > \sum_{t \in E} w(p^M, t) + \sum_{t \notin E} w(p^m, t) \quad (19)$$

and

$$\beta_2 \sum_{s \in S} w(p^M, s) + (1 - \beta_2) \sum_{s \in S} w(p^m, s) > w(p^M, s) + \sum_{t \neq s} w(p^m, t). \quad (20)$$

Adding above two inequalities we obtain that, for all  $w \in \mathcal{W}$ ,

$$(\beta_1 + \beta_2) \sum_{s \in S} w(p^M, s) + (1 - \beta_1 - \beta_2) \sum_{s \in S} w(p^m, s) + \sum_{s \in S} w(p^m, s) > w(p^M(E \cup \{s\})p^m) + \sum_{s \in S} w(p^m, s) \quad (21)$$

hence, for all  $w \in \mathcal{W}$ ,

$$(\beta_1 + \beta_2) \sum_{s \in S} w(p^M, s) + (1 - \beta_1 - \beta_2) \sum_{s \in S} w(p^m, s) > \sum_{s \in S} w(p^M(E \cup \{s\})p^m, s). \quad (22)$$

But this is obviously a contradiction to  $p^M(\beta_1 + \beta_2)p^m \prec p^M(E \cup \{s\})p^m$ . Thus,  $\pi(E \cup \{s\}) \leq \pi(E) + \pi(s)$ .

Suppose  $\pi(E \cup \{s\}) < \pi(E) + \pi(s)$ . Then there exist  $\alpha$  such that  $\pi(E \cup \{s\}) < \alpha < \pi(E) + \pi(s)$ . Since  $0 \leq \alpha - \pi(E) < \pi(s)$ , we can find  $\alpha_1 < \alpha$

<sup>33</sup>Recall Lemma 2. We have  $h^M = (p^M, \dots, p^M)$  and  $h^m = (p^m, \dots, p^m)$ .

such that  $\alpha - \pi(E) < \alpha_1 < \pi(s)$ . Thus, we have  $\alpha - \alpha_1 \in (0, \pi(E))$  and  $\alpha_1 < \pi(s)$ . Therefore, by using the same argument above, we can have,

$$p^M \{s\} p^m \succ p^M \alpha_1 p^m \text{ and } p^M E p^m \succ p^M (\alpha - \alpha_1) p^m \Rightarrow p^M (E \cup \{s\}) p^m \succ p^M \alpha p^m \quad (23)$$

This is a contradiction to  $\pi(E \cup \{s\}) < \alpha$ . ♣

Now we enter the proof of Theorem 5. Suppose  $\alpha > \pi(E)$ . Then, by Corollary 2,

$$p^M \alpha p^m \succ p^M E p^m \Leftrightarrow \sum_{s \in S} w(p^M \alpha p^m, s) > \sum_{s \in S} w(p^M E p^m, s), \quad \forall w \in \mathcal{W}. \quad (24)$$

But (24) implies that, for all  $w \in \mathcal{W}$ ,

$$\alpha \sum_{s \in S} w(p^M, s) + (1 - \alpha) \sum_{s \in S} w(p^m, s) > \sum_{s \in E} w(p^M, s) + \sum_{s \notin E} w(p^m, s), \quad (25)$$

which, in turn, implies that, for all  $w \in \mathcal{W}$ ,

$$\alpha \sum_{s \notin E} w(p^M, s) + (1 - \alpha) \sum_{s \in E} w(p^m, s) > (1 - \alpha) \sum_{s \in E} w(p^M, s) + \alpha \sum_{s \notin E} w(p^m, s). \quad (26)$$

But (26) implies that, for all  $w \in \mathcal{W}$ ,

$$\frac{\alpha}{1 - \alpha} > \frac{\sum_{s \in E} w(p^M, s) - \sum_{s \in E} w(p^m, s)}{\sum_{s \notin E} w(p^M, s) - \sum_{s \notin E} w(p^m, s)}, \quad \forall \alpha > \pi(E). \quad (27)$$

Hence,

$$\frac{\pi(E)}{1 - \pi(E)} \geq \frac{\sum_{s \in E} w(p^M, s) - \sum_{s \in E} w(p^m, s)}{\sum_{s \notin E} w(p^M, s) - \sum_{s \notin E} w(p^m, s)}, \quad \forall w \in \mathcal{W}. \quad (28)$$

For all  $\alpha < \pi(E)$ , we can repeat the same argument. Therefore, we get, for all  $w \in \mathcal{W}$ ,

$$\frac{\alpha}{1 - \alpha} < \frac{\sum_{s \in E} w(p^M, s) - \sum_{s \in E} w(p^m, s)}{\sum_{s \notin E} w(p^M, s) - \sum_{s \notin E} w(p^m, s)}, \quad \forall \alpha < \pi(E). \quad (29)$$

Hence,

$$\frac{\pi(E)}{1 - \pi(E)} \leq \frac{\sum_{s \in E} w(p^M, s) - \sum_{s \in E} w(p^m, s)}{\sum_{s \notin E} w(p^M, s) - \sum_{s \notin E} w(p^m, s)}, \quad \forall w \in \mathcal{W}. \quad (30)$$

Thus, we conclude that

$$\frac{\pi(E)}{1 - \pi(E)} = \frac{\sum_{s \in E} w(p^M, s) - \sum_{s \in E} w(p^m, s)}{\sum_{s \notin E} w(p^M, s) - \sum_{s \notin E} w(p^m, s)}, \quad \forall w \in \mathcal{W}. \quad (31)$$

But Lemma 3 implies that whenever  $h^x \succ h^m$ ,  $p^M \alpha p^m \succ p^M E p^m$  if and only if  $\delta_x \alpha p^m \succ \delta_x E p^m$ . Thus, for all  $w \in \mathcal{W}$ ,

$$\frac{\pi(E)}{1 - \pi(E)} = \frac{\sum_{s \in E} w(p^M, s) - \sum_{s \in E} w(p^m, s)}{\sum_{s \notin E} w(p^M, s) - \sum_{s \notin E} w(p^m, s)} = \frac{\sum_{s \in E} w(\delta_x, s) - \sum_{s \in E} w(p^m, s)}{\sum_{s \notin E} w(\delta_x, s) - \sum_{s \notin E} w(p^m, s)}, \quad (32)$$

Let  $S = \{s_1, s_2, \dots, s_n\}$  and  $E = \{s_i\}$ . By equation (32), we have, for all  $w \in \mathcal{W}$ ,

$$\begin{aligned} \frac{1 - \pi(s_i)}{\pi(s_i)} &= \frac{\sum_{s \neq s_i} w(\delta_x, s) - \sum_{s \neq s_i} w(p^m, s)}{w(\delta_x, s_i) - w(p^m, s_i)} \Rightarrow \\ \frac{1}{\pi(s_i)} &= \frac{\sum_s w(\delta_x, s) - \sum_s w(p^m, s)}{w(\delta_x, s_i) - w(p^m, s_i)} \Rightarrow \\ \frac{\pi(s_i)}{\pi(s_j)} &= \frac{w(\delta_x, s_i) - w(p^m, s_i)}{w(\delta_x, s_j) - w(p^m, s_j)}, \quad \forall i, j, \Rightarrow \text{By taking } j = 1, \\ w(\delta_x, s_i) &= \frac{\pi(s_i)}{\pi(s_1)} (w(\delta_x, s_1) - w(p^m, s_1)) + w(p^m, s_i) \Rightarrow \\ w(p, s_i) &= \frac{\pi(s_i)}{\pi(s_1)} w(p, s_1) - \frac{\pi(s_i)}{\pi(s_1)} w(p^m, s_1) + w(p^m, s_i). \end{aligned} \quad (33)$$

Suppose  $h, g \in \Delta(X)^S$ . Then, for all  $w \in \mathcal{W}$ ,

$$h \succcurlyeq g \Leftrightarrow \sum_s w(h(s), s) \geq \sum_s w(g(s), s), \quad (34)$$

By using equation (33), we can easily show.

$$\sum_s w(h(s), s) \geq \sum_s w(g(s), s) \Leftrightarrow \sum_i \pi(s_i) w(h(s_i), s_1) \geq \sum_i \pi(s_i) w(g(s_i), s_1), \quad \forall w \in \mathcal{W}. \quad (35)$$

Define  $\mathcal{U} = \{w(\cdot, s_1) | w \in \mathcal{W}\}$ . Then, (34) and (35) implies

$$h \succeq g \Leftrightarrow \sum_{s \in S} \pi(s)U(h(s)) \geq \sum_{s \in S} \pi(s)U(g(s)), \quad \forall U \in \mathcal{U}. \quad (36)$$

By repeating exactly the same argument, we can show

$$h \succ g \Leftrightarrow \sum_{s \in S} \pi(s)U(h(s)) > \sum_{s \in S} \pi(s)U(g(s)), \quad \forall U \in \mathcal{U}. \quad (37)$$

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## References

- [1] Aumann, Robert J. (1962) “Utility Theory without the Completeness Axiom,” *Econometrica* 30, 445-462.
- [2] Anscombe, Francis J. and Aumann, Robert J. (1963) “A Definition of Subjective Probability,” *Annals of Mathematical Statistics* 43, 199–205.
- [3] Bewley, Truman F. (2002) “Knightian Decision Theory: Part I,” *Decision in Economics and Finance*, 25, 79-110.
- [4] Chateauneuf, Alain (1987) “Continuous Representation of a Preference Relation on a Connected Topological Space,” *Journal of Mathematical Economics*, 16, 139-146.
- [5] Dubra, Juan (2010) “Continuity and Completeness under Risk,” unpublished manuscript
- [6] Dubra, Juan, Maccheroni, Fabio and Ok, Efe A. (2004) “Expected Utility Theory without the Completeness Axiom,” *Journal of Economic Theory*, 115, 118-133.
- [7] Dunford, Nelson and Jacob T. Schwartz (1957) *Linear Operators*, Part I. Interscience Publishers, NY.
- [8] Fishburn, Peter C. (1970) *Utility Theory for Decision Making*. John Wiley and Sons, Inc. New York.

- [9] Gilboa, Itzhak, and David Schmeidler (1989) “Maxmin Expected Utility with Non-Unique Prior,” *Journal of Mathematical Economics*, 18, 141-153.
- [10] Karni, Edi (2010) “A Theory of Bayesian Decision Making with Action-Dependent Subjective Probabilities,” *Economic Theory*, forthcoming.
- [11] Karni, Edi (2010a) “Continuity, Completeness and the Definition of Weak Preferences,” unpublished manuscript.
- [12] Kreps, David M. (1988) *Notes on the Theory of Choice* Westview Press, Boulder.
- [13] Nau, Robert (2006) “The Shape of Incomplete Preferences,” *Annals of Statistics*, 34, 2430-2448.
- [14] Ok, Efe, A. (2007) *Real Analysis with Economic Applications*. Princeton University Press, Princeton, N.J.
- [15] Ok, Efe A. Ortoleva, Pietro and Riella, Gil (2008) “Incomplete Preferences under Uncertainty: Indecisiveness in Beliefs vs. Tastes,” unpublished manuscript
- [16] Redei, Miklos (ed.) (2005) *John von Neumann: Selected Letters*. History of Mathematics Vol. 27. American Mathematical Society.
- [17] Savage, Leonard J. (1954) *The Foundations of Statistics*. John Wiley and Sons, New York.
- [18] Schmeidler, David (1989) “Subjective Probability and Expected Utility without Additivity,” *Econometrica*, 57, 571-588.
- [19] Shapley, Lloyd S. and Baucells, Manel J. (1998) “Multiperson Utility,” working paper
- [20] Seidenfeld, Teddy, Schervish, Mark, J. and Kadane, Joseph, B. (1995) “A Representation of Partially Ordered Preferences,” *Annals of Statistics*, 23, 2168-2217.
- [21] von Neumann, John and Morgenstern, Oscar (1947) *Theory of games and Economic Behavior* Princeton University Press, Princeton.