Continuity, Completeness, Betweenness and Cone-monotonicity*

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October 28, 2014

Abstract

A non-trivial, transitive and reflexive binary relation on the set of lotteries satisfying independence that also satisfies any two of the following three axioms satisfies the third: completeness, Archimedean and mixture continuity (Dubra (2011)). This paper generalizes Dubra's result in two ways: First, by replacing independence with a weaker betweenness axiom. Second, by replacing independence with a weaker cone-monotonicity axiom. The latter is related to betweenness and, in the case in which outcomes correspond to real numbers, implies monotonicity with respect to first-order stochastic dominance.

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Keywords: Archimedean, mixture continuity, completeness, betweenness, first-order stochastic dominance, decision making under risk

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JEL classification numbers: D01, D81

*We thank Juan Dubra and an anonymous Associate Editor for very useful and insightful comments.

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1 Introduction

Building on a theorem of Schmeidler (1971), Dubra (2011) proved that a nontrivial, transitive and reflexive binary relation on the set of lotteries satisfying independence that also satisfies any two of the following three axioms satisfies the third: completeness, Archimedean and mixture continuity. In this paper we generalize Dubra's result by replacing independence with the weaker betweenness axiom.[1] In addition, we show that if outcomes correspond to real numbers (e.g., monetary prizes) then Dubra's result still holds even if instead of independence we only assume monotonicity with respect to first-order stochastic dominance. In fact, we prove the result replacing independence with a weaker axiom dubbed cone-monotonicity. Cone-monotonicity axiom is weaker than monotonicity with respect to first-order stochastic dominance and independence, and is related to (but not implied by) the betweenness axiom.

In the next section we introduce the analytical framework and the axioms whose interrelation constitute the focal point of this work. The relations among the continuity conditions and completeness with betweenness are analyzed in section 3. In section 4 we introduce the cone-monotonicity axiom and analyze the relations among the continuity conditions and completeness under cone-monotonicity. In section 5 we discuss the relations between cone-monotonicity and betweenness and the relations between cone-monotonicity and monotonicity with respect to first order stochastic dominance. Concluding remarks appear in section 6.

2 The Analytical Framework

Let X be a finite set of k *outcomes*, denote by $\Delta(X)$ the set of all probability measures on X, and by $\text{aff}\Delta(X)$ the affine hull of $\Delta(X)$. For each $p, q \in \Delta(X)$ and $\alpha \in [0,1]$ define $\alpha p + (1-\alpha) q \in \Delta(X)$ by $(\alpha p + (1-\alpha) q)(x) = \alpha p(x) + (1-\alpha) q(x)$, for all $x \in X$.

Let $\succcurlyeq$ be a binary relation on $\Delta(X)$ and denote by $\succ$ and by $\sim$ the asymmetric and symmetric parts of $\succcurlyeq$, respectively. We list below some

[1] Safra (2014) studied the representations of axiomatic theories that include betweenness and depart from the completeness axiom. Karni and Zhou (2014) examined representations of weighted utility theory (which satisfies betweenness) without completeness.

well-known properties that $\succcurlyeq$ might satisfy.

(A.1) **Non-trivial partial order** $\succcurlyeq$ is reflexive and transitive with a non-empty asymmetric part.

The relations among the next three axioms are the focal point of our analysis.

(A.2) **Archimedean** For all $p, q, r \in \Delta(X)$, if $p \succ q$ then there exist $\alpha, \beta \in (0,1)$ such that $\alpha p + (1-\alpha) r \succ q$ and $p \succ \beta q + (1-\beta) r$.

(A.3) **Mixture continuity** For all $p, q, r \in \Delta(X)$ the sets

$$\{\alpha \in [0,1] \mid \alpha p + (1-\alpha) r \succcurlyeq q\} \text{ and } \{\alpha \in [0,1] \mid q \succcurlyeq \alpha p + (1-\alpha) r\}$$

are closed.

(A.4) **Completeness** For all $p, q \in \Delta(X)$, either $p \succcurlyeq q$ or $q \succcurlyeq p$.

3 Continuity, Completeness and Betweenness

The next axiom is a weakening of the independence axiom.[2]

(A.5) **Betweenness** For all $p, q \in \Delta(X)$, $r \in \{p, q\}$ and $\alpha \in (0,1)$,

$$p \succcurlyeq q \iff \alpha p + (1-\alpha) r \succcurlyeq \alpha q + (1-\alpha) r$$

Similar equivalences hold for $\succ$ and $\sim$.

This statement of the betweenness axiom demonstrates to what extent it is a weakening of the independence axiom (as r is not restricted to the set $\{p, q\}$). Moreover, Axiom (A.5) implies the following more common statements of the betweenness property (see Chew 1989 and Dekel 1986): For all $p, q \in \Delta(X)$, $\alpha \in (0,1)$

$$p \succcurlyeq q \implies p \succcurlyeq \alpha p + (1-\alpha) q \succcurlyeq q$$

and

$$p \succ q \implies p \succ \alpha p + (1-\alpha) q \succ q.$$

[2]For clarity, here is the version of the independence we refer to: For all $p, q, r \in \Delta(X)$ and $\alpha \in (0,1)$,

$$p \succcurlyeq q \iff \alpha p + (1-\alpha) r \succcurlyeq \alpha q + (1-\alpha) r.$$

When $\succcurlyeq$ is complete (A.5) is implied by them.

Our first result generalizes the theorem of Dubra (2011) by replacing independence with betweenness. Formally,

Theorem 1 *Suppose that $\succcurlyeq$ is a non-trivial partial order on $\Delta(X)$ satisfying betweenness. Then any two of the three axioms (A.2)-(A.4) imply the third.*

Proof (a) Suppose that $\succcurlyeq$ satisfies Archimedean and completeness. Let $p, q, r \in \Delta(X)$ and consider the set $A = \{\alpha \in [0,1] \mid \alpha p + (1-\alpha) r \succcurlyeq q\}$. Without loss of generality assume that $p \succcurlyeq r$ (by completeness, either $p \succcurlyeq r$ or $r \succcurlyeq p$) and note that, by betweenness, $\alpha p + (1-\alpha) r \succcurlyeq \beta p + (1-\beta) r$ for all $1 \geqslant \alpha \geqslant \beta \geqslant 0$. If $A = \varnothing$ then we are done. Otherwise, define $\alpha^* = \inf\{\alpha \in A\}$. If $\alpha^* \in A$ then, by betweenness, $A = [\alpha^*, 1]$ and hence A is closed. Assume that $\alpha^* \notin A$ and note that, by definition, for every $\varepsilon' > 0$ there exists $\varepsilon \in (0, \varepsilon')$ satisfying $(\alpha^* + \varepsilon) p + (1 - (\alpha^* + \varepsilon)) r \succcurlyeq q$. By betweenness, this implies $[\alpha^* + \varepsilon, 1] \subset A$ and hence $(\alpha^*, 1] \subset A$. Next note that if $\alpha^* p + (1-\alpha^*) r \succcurlyeq q$ does not hold then, by completeness, $q \succ \alpha^* p + (1-\alpha^*) r$ and hence, by Archimedean, there exists $\beta \in (0,1)$ such that $q \succ \beta(\alpha^* p + (1-\alpha^*) r) + (1-\beta) p$. But

$$\begin{aligned}\beta(\alpha^* p + (1-\alpha^*) r) + (1-\beta) p &= (\alpha^* + (1-\beta)(1-\alpha^*)) p \\ &\quad + (1 - (\alpha^* + (1-\beta)(1-\alpha^*))) r\end{aligned}$$

while $\alpha^* + (1-\beta)(1-\alpha^*) \in (\alpha^*, 1]$; a contradiction.

The proof that $\{\alpha \in [0,1] \mid q \succcurlyeq \alpha p + (1-\alpha) r\}$ is closed follows by the same argument.

(b) Suppose that $\succcurlyeq$ satisfies mixture continuity and completeness. Let $p, q, r \in \Delta(X)$ and suppose that $p \succ q$. Mixture continuity implies that the set $A^q = \{\beta \in [0,1] \mid q \succcurlyeq \beta p + (1-\beta) r\}$ is closed and $p \succ q$ implies that $1 \notin A^q$. Hence, $[0,1] \setminus A^q$ is a non-empty open subset of $[0,1]$. Take $\alpha \in [0,1) \setminus A^q$ and note that, by completeness, $\alpha p + (1-\alpha) r \succ q$.

The proof that there exists $\beta \in (0,1) \setminus A^p$ for which $p \succ \beta q + (1-\beta) r$ is similar.[3]

[3] Note that betweenness was not used in this part. Under completeness, mixture continuity is stronger than Archimedean.

(c) Suppose that $\succsim$ satisfies Archimedean and mixture continuity. We show first that, for all $p \in \Delta(X)$, the sets $\bar{B}(p) = \{q \in \Delta(X) \mid q \succsim p\}$ and $\bar{W}(p) = \{q \in \Delta(X) \mid p \succsim q\}$ are closed and the sets $B(p) = \{q \in \Delta(X) \mid q \succ p\}$ and $W(p) = \{q \in \Delta(X) \mid p \succ q\}$ are open relative to $\Delta(X)$.

Fix $p \in \Delta(X)$. If $\bar{B}(p)$ is a singleton there is nothing to prove. We therefore assume it is not and start by showing that $\bar{B}(p)$ is convex. Take $p^1, p^2 \in \bar{B}(p)$ and let $p^\alpha = \alpha p^2 + (1-\alpha) p^1$ for some $\alpha \in (0,1)$. Note that we cannot assume that either $p^1 \succsim p^2$ or $p^2 \succsim p^1$ hold. If for some $i \in \{1,2\}$ $p^i \sim p$ then, for $j \neq i$ and by transitivity, $p^j \succsim p^i$ and hence, by betweenness, $p^\alpha \succsim p$. Otherwise, assume that $p^i \succ p$, for both $i = 1,2$, and consider the set $A = \{\beta \in [0,1] \mid \beta p^1 + (1-\beta) p^\alpha \succsim p\}$ ($A \neq \varnothing$ by construction). Denote $\beta^* = \inf\{\beta \in A\}$, $p^* = \beta^* p^1 + (1-\beta^*) p^\alpha$ and note that, by mixture continuity, $\beta^* \in A$. If $p^* = p^\alpha$ (that is, if $\beta^* = 0$) then $p^\alpha \succsim p$ and we are done. Assume $p^* \neq p^\alpha$ and note that, by an argument similar to one used in part (a), Archimedean implies that p^* is not strictly preferred to p. Hence $p^* \sim p$. Finally by transitivity, both $p^1 \succsim p^*$ and $p^2 \succsim p^*$ hold and, by betweenness, $p^\gamma \succsim p$ for all $\gamma \in [0,1]$. Hence $p^\alpha \succsim p$ and $\bar{B}(p)$ is convex.

Next choose $q \in \Delta(X)$ in the boundary of $\bar{B}(p)$, let $\{q^n\} \subset \bar{B}(p)$ be a sequence that converges to q, let r belong to the relative interior of $\bar{B}(p)$ and let $N_{\bar{\varepsilon}}(r)$ be an open $\bar{\varepsilon}$-ball around r whose intersection with $\operatorname{aff}\left(\bar{B}(p)\right)$ is a subset of $\bar{B}(p)$. By construction, for a fixed $\varepsilon \in (0, \bar{\varepsilon})$ $r^n = r + \frac{\varepsilon}{\|q^n - q\|}(q - q^n) \in N_{\bar{\varepsilon}}(r) \cap \bar{B}(p)$ and, by the convexity of $\bar{B}(p)$,

$$\bar{q}^n = \frac{\varepsilon}{\varepsilon + \|q^n - q\|} q^n + \frac{\|q^n - q\|}{\varepsilon + \|q^n - q\|} r^n \succsim p$$

Next observe that, as $\bar{q}^n = \frac{\varepsilon}{\varepsilon + \|q^n - q\|} q + \frac{\|q^n - q\|}{\varepsilon + \|q^n - q\|} r$, $\bar{q}^n$ belongs to the line segment connecting r and q and, by construction, the sequence $\bar{q}^n$ converges to q. By mixture continuity, $q \succsim p$ and hence $\bar{B}(p)$ is closed.

The proof that $\bar{W}(p)$ is closed follows by a similar argument.

Consider next the set $B(p)$.[4] If $B(p) = \varnothing$ then there is nothing to prove. Suppose that it is not and observe that, by arguments similar to the above, $B(p)$ is convex. Take $q \in B(p)$ and $r^1, \ldots r^{k-1} \in \Delta(X)$ such that $\{r^i - q\}_{i=1}^{k-1}$ span $\operatorname{aff}(\Delta(X))$. By Archimedean, for each r^i there exists $\alpha^i \in (0,1)$ satisfying $q + \alpha^i (r^i - q) \succ p$. Next, if there exists $\gamma^i > 0$ such

[4] In this part of the proof we follow in the footsteps of Dubra (2011).

that $q-\gamma^i(r^i-q)\in\Delta(X)$ then, by Archimedean, there exists $\beta^i\in(0,\gamma^i)$ satisfying $q-\beta^i(r^i-q)\succ p$. If such γ^i does not exist (this happens if q belongs to the boundary of $\Delta(X)$) then for this i we define $\beta^i=1$. Let $\alpha=\min\{\alpha^i,\beta^i\}_{i=1}^{k-1}$ and note that, by convexity, the convex hull of $\{q+\alpha(r^i-q)\}_{i=1}^{k-1}$ contains an open ball $N_\varepsilon(q)$ around q. Hence, q is an interior point of $B(p)$ and $B(p)$ is open.

By similar argument, $W(p)$ is open.

Since $\Delta(X)$ is a connected topological space these observations, in conjunction with the theorem of Schmeidler (1971), imply that $\succcurlyeq$ is complete.

□

4 Continuity, Completeness and Cone-monotonicity

For our second result, we replace the betweenness axiom with an axiom dubbed cone-monotonicity. Although this axiom seems weaker than betweenness (and indeed is satisfied by some betweenness relations), this implication does not always hold (see the discussion below). As we explain later, all relations that are monotone with respect to the partial relation of first-order stochastic dominance satisfy cone-monotonicity.[5]

(A.5) **Cone-monotonicity** Every $p\in\text{int}\Delta(X)$ has a non-empty cone $C^p\subset\text{aff}\Delta(X)-\{p\}$, open relative to $\text{aff}\Delta(X)-\{p\}$, such that for all $r\in\Delta(X)$,

$$\begin{array}{rrcl} & r-p\in C^p & \Longrightarrow & r\succ p, \\ & p-r\in C^p & \Longrightarrow & p\succ r \\ \text{and} & r-p\in C^p & \Longleftrightarrow & r-p\in C^r \end{array}$$

Let $r>_{C^p}p$ denote $r-p\in C^p$.

Theorem 2 *Suppose that $\succcurlyeq$ is a non-trivial partial order on $\Delta(X)$ satisfying cone-monotonicity. Then, on $\text{int}\Delta(X)$, any two of the three axioms (A.2)-(A.4) imply the third. If, in addition, mixture continuity holds on $\Delta(X)$ then completeness holds on $\Delta(X)$.*

[5]All our cones are assumed to be 'nonnegative' (i.e., closed under nonnegative scalar multiplications).

Note that non-triviality is implied by cone-monotonicity. It is left in the statement of the theorem for ease of exposition.

Proof (a) Suppose that $\succcurlyeq$ satisfies Archimedean and completeness on $\mathrm{int}\Delta(X)$. Let $p, q, r \in \mathrm{int}\Delta(X)$ and consider the set $A = \{\alpha \in [0,1] \mid \alpha p + (1-\alpha) r \succcurlyeq q\}$. If A is either empty or finite then we are done. Otherwise let α^* be an accumulation point of A (that is, there exists a sequence $\{\alpha^n\} \subset A \setminus \{\alpha^*\}$ that converges to α^*) and denote $p^n = \alpha^n p + (1-\alpha^n) r$ and $p^* = \alpha^* p + (1-\alpha^*) r$. Assume, by way of negation, that $p^* \succcurlyeq q$ does not hold. By completeness, $q \succ p^*$. As $p^* \in \mathrm{int}\Delta(X)$, cone-monotonicity implies the existence of $\bar{p} \in \mathrm{int}\Delta(X)$ satisfying $\bar{p} >_{C^p} p^*$. By Archimedean, there exists $\beta \in (0,1)$ such that $q \succ \beta\bar{p} + (1-\beta) p^*$ and, since C^p is a cone, $\beta\bar{p} + (1-\beta) p^* >_{C^p} p^*$. By cone-monotonicity, $\beta\bar{p} + (1-\beta) p^* >_{C^{\beta\bar{p}+(1-\beta)p^*}} p^*$ and hence, by the openness of $C^{\beta\bar{p}+(1-\beta)p^*}$, there exists an open ε-ball $N_\varepsilon(p^*)$ satisfying $N_\varepsilon(p^*) \cap \mathrm{aff}\Delta(X) \subset \Delta(X)$ such that $\beta\bar{p} + (1-\beta) p^* >_{C^{\beta\bar{p}+(1-\beta)p^*}} p'$ for all $p' \in N_\varepsilon(p^*) \cap \mathrm{aff}\Delta(X)$. By cone-monotonicity $\beta\bar{p} + (1-\beta) p^* \succ p'$ and, by transitivity, $q \succ p'$, for all $p' \in N_\varepsilon(p^*) \cap \mathrm{aff}\Delta(X)$. But, for sufficiently large n, $p^n \in N_\varepsilon(p^*) \cap \mathrm{aff}\Delta(X)$ and hence $q \succ p^n$; a contradiction.

The proof that $\{\alpha \in [0,1] \mid q \succcurlyeq \alpha p + (1-\alpha) r\}$ is closed follows by the same argument.

(b) The proof that Archimedean is implied by mixture continuity and completeness is identical to that of Theorem 1 part (b) (note that the betweenness property is not needed there and that the proof holds for the entire $\Delta(X)$).

(c) Suppose that $\succcurlyeq$ satisfies Archimedean and mixture continuity on $\mathrm{int}\Delta(X)$. We begin by showing that, for all $p \in \mathrm{int}\Delta(X)$, the sets $\bar{B}^{int}(p) = \{q \in \mathrm{int}\Delta(X) \mid q \succcurlyeq p\}$ and $\bar{W}^{int}(p) = \{q \in \mathrm{int}\Delta(X) \mid p \succcurlyeq q\}$ are closed relative to $\mathrm{int}\Delta(X)$, and the sets $B^{int}(p) = \{q \in \mathrm{int}\Delta(X) \mid q \succ p\}$ and $W^{int}(p) = \{q \in \mathrm{int}\Delta(X) \mid p \succ q\}$ are open relative to $\mathrm{int}\Delta(X)$. Then we use Schmeidler's theorem to derive completeness of $\succcurlyeq$ on $\mathrm{int}\Delta(X)$. Finally, we show that, if mixture continuity holds on $\Delta(X)$, $\succcurlyeq$ on $\Delta(X)$ is also complete.

Fix $p \in \mathrm{int}\Delta(X)$ and note that, by cone-monotonicity, the intersection of C^p and $\mathrm{int}\Delta(X)$ is open and non-empty and hence $\bar{B}^{int}(p)$ is not a singleton. If q is an interior boundary point of $\bar{B}^{int}(p)$, that is if $q \in \partial\bar{B}^{int}(p) \cap \mathrm{int}\Delta(X)$ (∂ denotes the boundary), then let $\{q^n\} \subset \bar{B}^{int}(p)$ be a sequence that converges to q. By cone-monotonicity the intersection of

C^q and $\text{int}\Delta(X)$ is non-empty and there exists r satisfying $r >_{C^q} q$. Using cone-monotonicity again, $r >_{C^r} q$ and, by the openness of C^r, there exists an open ε-ball $N_\varepsilon(q) \subset \Delta(X)$ such that $r >_{C^r} q'$, for all $q' \in N_\varepsilon(q)$. This implies that for n sufficiently large, $r >_{C^r} q^n$. Hence, by cone-monotonicity $r \succ q^n$ and, by transitivity, $r \succcurlyeq p$. Since C^q is a cone, for all $\alpha \in (0,1]$, $r^\alpha = \alpha r + (1-\alpha) q >_{C^q} q$ and hence, by similar arguments, $r^\alpha \succcurlyeq p$. Therefore, the set $\{\alpha \in [0,1] \mid \alpha r + (1-\alpha) q \succcurlyeq p\}$ contains the interval $(0,1]$ and hence, by mixture continuity, includes 0. Therefore $q \in \bar{B}^{int}(p)$ and $\bar{B}^{int}(p)$ is closed in $\text{int}\Delta(X)$.

The proof that $\bar{W}^{int}(p)$ is closed in $\text{int}\Delta(X)$ follows by similar arguments.

Next consider the set $B^{int}(p)$ and note, again, that $p \in \text{int}\Delta(X)$ implies $B^{int}(p) \neq \varnothing$. Choose $q \in B^{int}(p)$ and r satisfying $q >_{C^q} r$. By Archimedean there exists $\alpha \in (0,1)$ such that $q^\alpha = \alpha q + (1-\alpha) r \succ p$. Since C^q is a cone, $q >_{C^q} q^\alpha$ which implies $q >_{C^{q^\alpha}} q^\alpha$. Hence there exists an open ε-ball $N_\varepsilon(q) \subset \Delta(X)$ such that $q' >_{C^{q^\alpha}} q^\alpha$ for all $q' \in N_\varepsilon(q)$. By cone-monotonicity and transitivity $q' \succ p$. Hence $N_\varepsilon(q) \subset B^{int}(p)$ and $B^{int}(p)$ is open in $\text{int}\Delta(X)$.

By similar arguments, $W^{int}(p)$ is open in $\text{int}\Delta(X)$.

Since $\text{int}\Delta(X)$ is a connected topological space, by Schmeidler (1971), these observations imply that $\succcurlyeq$ is complete on $\text{int}\Delta(X)$.

Finally, we show that if mixture continuity holds on $\Delta(X)$ then completeness extends to the entire set $\Delta(X)$.

Let $q \in \partial\Delta(X)$, $p, r \in \text{int}\Delta(X)$ and consider the set $A = \{\alpha \in [0,1] \mid \alpha q + (1-\alpha) r \succcurlyeq p\}$. If 1 is an accumulation point of A then, by mixture continuity, $q \succcurlyeq p$. Otherwise, by completeness on $\text{int}\Delta(X)$, there exists $\bar{\alpha} < 1$ such that $p \succcurlyeq \alpha q + (1-\alpha) r$, for all $\alpha \in (\bar{\alpha}, 1)$. By mixture continuity, $p \succcurlyeq q$. Hence either $q \succcurlyeq p$ or $p \succcurlyeq q$.

Next let $q, p \in \partial\Delta(X)$, $r \in \text{int}\Delta(X)$ and consider the set $A = \{\alpha \in [0,1] \mid \alpha q + (1-\alpha) r \succcurlyeq p\}$. If 1 is an accumulation point of A then, by mixture continuity, $q \succcurlyeq p$. Otherwise, since all points $\alpha q + (1-\alpha) r$ are interior points when $\alpha \in [0,1)$ then, by the preceding argument, there exists $\bar{\alpha} < 1$ such that $p \succcurlyeq \alpha q + (1-\alpha) r$ for all $\alpha \in (\bar{\alpha}, 1)$ and again, by mixture continuity, $p \succcurlyeq q$. Hence $\succcurlyeq$ is complete on $\Delta(X)$. $\square$

5 Cone-Monotonicity, Betweenness and Stochastic Dominance

To analyze the relationships between cone-monotonicity and betweenness we make the following definition. A binary relation $\succcurlyeq$ on $\Delta(X)$ is *non-trivial at* $p \in \Delta(X)$ if there exists $q \in \Delta(X)$ such that either $p \succ q$ or $q \succ p$ hold. The following proposition serves to clarify the relation between cone-monotonicity and betweenness.

Proposition 1 *Suppose that $\succcurlyeq$ is a non-trivial partial order on $\Delta(X)$ satisfying betweenness, mixture continuity and Archimedean. Let $p \in \text{int}\Delta(X)$ and denote*

$$C(p) = \{\lambda(q - p) \mid \lambda > 0, q \in \Delta(X) \text{ and } q \succ p\}$$

If $\succcurlyeq$ is *non-trivial at p then the cone $C(p)$ is non-empty, open relative to* $\text{aff}\Delta(X) - \{p\}$, *and satisfies*

$$\begin{aligned} B(p) &= (p + C(p)) \cap \Delta(X) \\ W(p) &= (p - C(p)) \cap \Delta(X) \end{aligned}$$

where $B(p) = \{q \in \Delta(X) \mid q \succ p\}$ and $W(p) = \{q \in \Delta(X) \mid p \succ q\}$.

Proof Fix $p \in \text{int}\Delta(X)$ at which $\succcurlyeq$ is non-trivial. First note that, by betweenness, if p belongs to the open line segment (q', q'') then $q' \succ p \Leftrightarrow p \succ q''$. Hence $B(p)$ is non-empty.

It can be shown (see Safra 2014 for details) that, by betweenness, $r \in B(p)$ if and only if $p + \lambda(r - p) \in B(p)$ for all $\lambda > 0$ and $p + \lambda(r - p) \in \Delta(X)$. Then, from the proof of Theorem 1 part (c) it follows that $B(p)$ is convex and open. Hence, the equivalences

$$B(p) = (p + C(p)) \cap \Delta(X), \quad W(p) = (p - C(p)) \cap \Delta(X)$$

are satisfied. □

Proposition 1 implies that a partial order that satisfies betweenness, Archimedean and is non-trivial at every $p \in \text{int}\Delta(X)$ also satisfies cone-monotonicity (just define $C^p = C(p)$). It is immediate to verify that, for such relations (on $\text{int}\Delta(X)$), Theorem 1 is implied by Theorem 2.[6] In addition, it is easy to verify that, for non-trivial partial orders, independence implies cone-monotonicity (independence implies that non-triviality at a given point extends to the entire $\Delta(X)$ and that all cones $C(p)$ are identical).

The next example demonstrates that non-triviality at every $p \in \text{int}\Delta(X)$ does not follow from the betweenness property.

Example Let $X = \{0, 1, 2\}$ and consider the incomplete partial order satisfying betweenness defined by

$$p \succcurlyeq q \quad \Longleftrightarrow \quad V_j(p) \geqslant V_j(q) \quad \text{for} \quad j = 1, 2$$

where

$$V_1(p) = \frac{\sum_i p(i)\, w(i)\, i}{\sum_i p(i)\, w(i)}, \quad w(1) = 1, \; w(0) = w(2) = 0.5$$

is a weighted utility function that ranks 2 at the top and 0 at the bottom and

$$V_2(p) = \sum_i p(i)(2 - i)$$

in an expected utility function that ranks 0 at the top and 2 at the bottom. As can be seen in Figure 1, $\succcurlyeq$ is not non-trivial at p for all $p \in \{\alpha\delta_1 + (1-\alpha)(0.5\delta_2 + 0.5\delta_0) \mid \alpha \in [0, 1]\}$.

Place Figure 1 here

To analyze the relationships between cone-monotonicity and monotonicity with respect to first-order stochastic dominance, we assume that $X \subset \mathbb{R}$ and, without loss of generality, let $x_1 < x_2 < \cdots < x_k$. For each $p \in \Delta(X)$, we denote by p_i the probability the lottery p assigns to x_i, $i = 1..., k$. Then $p \in \Delta(X)$ is said to *strongly dominate* $q \in \Delta(X)$ *with respect to first-order stochastic dominance* if, for all j=1, ..., $k-1$

$$\sum_{i=1}^{j} p_i < \sum_{i=1}^{j} q_i$$

[6] Since (given completeness) mixture continuity is stronger than Archimedean, part (b) in the proof of Theorem 1, the only one that does not assume Archimedean, requires neither betweenness nor cone-monotonicity.

We denote this relation by $p >_1 q$ and say that a preference relation $\succcurlyeq$ on $\Delta(X)$ satisfies *monotonicity* if, for all $p, q \in \Delta(X)$,

$$p >_1 q \Longrightarrow p \succ q$$

To see that monotonicity implies cone-monotonicity, let

$$C = \left\{ h \in \mathbb{R}^k \mid \sum_{i=1}^{k} h_i = 0,\ \sum_{i=1}^{j} h_i < 0,\ j = 1, ..., k-1 \right\}$$

and note that cone-monotonicity is satisfied for $C^p \equiv C$, for all $p \in \Delta(X)$. Hence, when $X \subset \mathbb{R}$, Theorem 2 applies to partial orders that satisfy monotonicity. Note that such orders are quite common, as monotonicity is usually assumed in most economic applications. It should also be mentioned that the partial relation of first-order stochastic dominance satisfies independence (indeed, this was utilized in the proof of Theorem 2, parts (a) and (c)). More on this can be found in Dubra and Ok (2002).

6 Concluding Remarks

The implication of continuity for completeness, as appeared in the theorem of Schmeidler (1971) is inherited by the theorem of Dubra (2011) and the two theorems in this paper. Karni (2011) showed that, starting with a strict preference relation, these results depend crucially on the definition of the weak preference relation. In particular, if the weak preference relation is defined as in Galaabaatar and Karni (2013) (that is, for all $q, p \in \Delta(x)$, q is weakly preferred over p if every $r \in \Delta(X)$ that is strictly preferred over q is strictly preferred over p), then Archimedean and mixture monotonicity no longer imply completeness, regardless of whether the preference relation satisfies independence. If we start with a weak preference relation, as we do in this note, a different definition of the strict preference relation is required to overcome the difficulty posed by incompleteness. Consider, for instance, a reflexive and transitive preference relation $\succcurlyeq$ with a non-empty asymmetric part that satisfies the independence axiom. Such preference relation has a representation as follows:[7] For all $p, q \in \Delta(X)$, $p \succcurlyeq q$ if and only if $\sum_{x\in X} u(x)\,p(x) \geq \sum_{x\in X} u(x)\,q(x)$, for all $u \in \mathcal{U}$, where $\mathcal{U}$ is a set of real-valued functions on X. For each $u \in \mathcal{U}$, define an induced preference relation on $\Delta(X)$ as follows: $p \succcurlyeq_u q$ if and only if $\sum_{x\in X} u(x)\,p(x) \geq \sum_{x\in X} u(x)\,q(x)$ and let $\succ_u$ be the asymmetric part of $\succcurlyeq_u$. Then $\succcurlyeq = \cap_{u\in\mathcal{U}} \succcurlyeq_u$ and $\hat{\succ} = \cap_{u\in\mathcal{U}} \succ_u$. For all $p \in \mathrm{int}\Delta(X)$, the set $\{q \in \Delta(X) \mid q \hat{\succ} p\}$ is equal to the relative interior of $\{q \in \Delta(X) \mid q \succcurlyeq p\}$ and, as a result, Archimedean and mixture monotonicity no longer imply completeness. Similar argument applies if the independence axiom is replaced by betweenness or cone monotonicity.

[7] See, for example, Shapley and Baucells (2008).

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