Archimedean and Continuity

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Abstract

Let $\succ$ be a complete and transitive binary relation on the set of all lotteries on finite sets of prizes. Then $\succ$ is mixture continuous if and only if it satisfies the Archimedean axiom and a condition called local mixture dominance.

1 Introduction

A preference relation over lotteries (that is, probability distributions over a finite sets of prizes) is representable by an expected utility functional if and only if it is a weak-order satisfying the Archimedean and independence axioms. The independence and Archimedean

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axioms are sufficient but not necessary conditions for mixture continuity of the preference relation. Similarly, betweenness and Archimedean are sufficient but not necessary conditions for mixture continuity.

In this note, I present a weaker condition, dubbed local mixture dominance, that together with the Archimedean axiom constitute necessary and sufficient conditions for mixture continuity of the preference relation and thus of the representation.

2 Mixture Continuity, Local Mixture Dominance and the Archimedean Axiom

Let $\Delta(X)$ be the set of all simple probability measures on an arbitrary set, $X$, of outcomes, and denote by $\succ$ a binary relation on $\Delta(X)$.

**Definition 1:** $\succ$ satisfies mixture continuity if for any $p, q, r \in \Delta(X)$ the sets

$$ S = \{ \alpha \in [0, 1] \mid \alpha p + (1 - \alpha) r \succ q \} \text{ and } T = \{ \alpha \in [0, 1] \mid q \succ \alpha p + (1 - \alpha) r \} $$

are closed in $[0, 1]$.

**Definition 2:** $\succ$ satisfies local mixture dominance at $\alpha$ if for every $p, r \in \Delta(X)$ and every $\gamma \in (0, 1)$ there exists $\beta(\gamma)$ between $\gamma$ and $\alpha$ such that, for all $\gamma'$ between $\alpha$ and $\beta(\gamma)$,

$$ \alpha p + (1 - \alpha) r \succ (\prec) \gamma p + (1 - \gamma) r \Leftrightarrow \gamma' p + (1 - \gamma') r \succ (\prec) \gamma p + (1 - \gamma) r. $$
The binary relation \( \succ \) satisfies *local mixture dominance* if it satisfies local mixture dominance at \( \alpha \) for all \( \alpha \).

**The Archimedean axiom:** For all \( p, q, r \in \Delta(X) \) satisfying \( p \succ q \succ r \) there exist \( \alpha, \beta \in (0,1) \) such that \( \alpha p + (1 - \alpha) r \succ q \succ \beta p + (1 - \beta) r \).

A preference relation is said to satisfy mixture monotonicity if \( 1 \geq \gamma > \gamma' \geq 0 \) implies \( \gamma p + (1 - \gamma) r \succ \gamma' p + (1 - \gamma') r \) for every \( p, r \in \Delta(X) \), with strict preference is \( p \succ r \). Mixture monotonicity is the property implied by independence and betweenness which, together with the Archimedean axiom, imply mixture continuity. It is worth noting that mixture monotonicity is not implied by local mixture dominance.

### 3 The Main Result

Mixture continuity is stronger than the Archimedean axiom. However, a necessary and sufficient condition for a weak order on \( \Delta(X) \) to satisfy mixture continuity is that it satisfies the Archimedean axiom and local mixture dominance.

**Theorem:** Let \( \succ \) be complete and transitive binary relation on \( \Delta(X) \). Then \( \succ \) satisfies local mixture dominance and the Archimedean axiom if and only if it satisfies mixture continuity.
Proof: (Sufficiency) Suppose that $\succ$ is a weak order satisfying local mixture dominance and the Archimedean axiom. For every $p, q, r \in \Delta(X)$ at least one of the sets in Definition 1 is not empty. Without loss of generality, assume that $T$ is not empty. Let $\alpha^*$ be a limit point of $T$ and suppose that $\alpha^* \not\in T$.

The sets

$$M^- = (0, \alpha^*) \cap T \text{ and } M^+ = (\alpha^*, 1) \cap T$$

are bounded and at least one of them is not empty. (If both $M^-$ and $M^+$ were empty then $\alpha^*$ is not a limit point of $T$). By definition, either $\alpha^* = \text{Sup} M^-$ or $\alpha^* = \text{Inf} M^+$, or both.

(If $\alpha^* \neq \text{Sup} M^-$ and $\alpha^* \neq \text{Inf} M^+$, then there is $\alpha' < \alpha^*$ such that $\alpha'$ is an upper bound of $M^-$ and there is $\alpha'' > \alpha^*$ that is a lower bound of $M^+$. Hence $(\alpha', \alpha^*) \cap T = \emptyset = (\alpha^*, \alpha'') \cap T$. Hence $\alpha^*$ is not a limit point of $T$).

If $\hat{\alpha} \in M^\theta$, $\theta \in \{+, -\}$ and $q \succ \hat{\alpha} p + (1 - \hat{\alpha}) r$ then, by the Archimedean axiom, for some $\beta \in (0, 1)$ and $\gamma = \beta \alpha^* + (1 - \beta) \hat{\alpha} \in M^\theta$, $\gamma p + (1 - \gamma) r \succ q$.

If $\alpha^*$ is the supremum of $M^-$ but not the infimum of $M^+$ then for some $\varepsilon' > 0$, $(\alpha^*, \alpha^* + \varepsilon') \cap T = \emptyset$. By local mixture dominance and transitivity, there exist $\beta (\gamma) \in (\gamma, \alpha^*)$ such that $(\beta (\gamma), \alpha^*) \cap T = \emptyset$. Let $\varepsilon = \min \{\varepsilon', \alpha^* - \beta (\gamma)\}$. Thus $N_\varepsilon (\alpha^*) \cap T = \emptyset$. A contradiction.

If $\alpha^*$ is the infimum of $M^+$ but not the supremum of $M^-$ then a contradiction is produced using similar argument.
If $\alpha^* = \text{Sup} M^- = \text{Inf} M^+$ then, applying local mixture dominance twice, there exist $\varepsilon > 0$ such that $N_\varepsilon (\alpha^*) \cap T = \emptyset$. A contradiction.

If $q \sim \alpha p + (1 - \alpha) r$ for all $\alpha \in T$, then, by transitivity $\alpha^* p + (1 - \alpha^*) r \succ \alpha p + (1 - \alpha) r$ for all $\alpha \in T$. By local mixture dominance, there is a neighborhood $N_\varepsilon (\alpha^*)$ such that $N_\varepsilon (\alpha^*) \cap T = \emptyset$. Hence $\alpha^*$ is not a limit point of $T$. A contradiction.

Thus, if $\alpha^*$ is a limit point of $T$ then $\alpha^* \in T$ and $T$ is closed. By the same argument $S$ is closed.

(Necessity) Suppose that $\succ$ satisfies mixture continuity. That it satisfies the Archimedean axiom is obvious. To show that it satisfies local mixture dominance, fix $\hat{\alpha} \in (0, 1)$ and suppose, by way of negation, that $\succ$ does not satisfy local mixture dominance at $\hat{\alpha}$. Then, for some $p, r \in \Delta (X)$, for some $\bar{\gamma} \in [0, 1]$ there is no $\beta (\bar{\gamma})$ that satisfies the conditions of Definition 2. Let $q (\bar{\gamma}) \in \Delta (X)$ satisfy $q (\bar{\gamma}) \sim \bar{\gamma} p + (1 - \bar{\gamma}) r$. Thus there is no neighborhood, $N_\varepsilon (\hat{\alpha})$ such that $N_\varepsilon (\hat{\alpha}) \subset S_{\bar{\gamma}}$ or $N_\varepsilon (\hat{\alpha}) \subset T_{\bar{\gamma}}$, where $S_{\bar{\gamma}}$ and $T_{\bar{\gamma}}$ are the sets in Definition 1 that correspond to $p, r$ and $q (\bar{\gamma})$. Without loss of generality suppose that $\hat{\alpha} p + (1 - \hat{\alpha}) r \succ q (\bar{\gamma})$ then $\hat{\alpha} \in T^c \subset S$. But $T$ is closed, hence $T^c$ is open. A contradiction. □

4 Concluding Remarks

If local mixture dominance is strengthened by requiring that $\beta (\gamma) = \gamma$ in Definition 2 then it is a sufficient but not necessary condition as is demonstrated by the following counter
Example 1 Let $X = \{0,1\}$ and suppose that $\succ$ is represented by $V(\alpha \delta_0 + (1-\alpha) \delta_1) = \alpha \sin (1/\alpha)$. Then $V$ is continuous in $\alpha$ but does not satisfy the strong version (that is, the version requiring that $\beta(\gamma) = \gamma$) of local mixture dominance at $\alpha = 0$.

It the condition is weakened by not requiring that it holds for all $\gamma$ then the condition is necessary but not sufficient as is made clear by the following example:

Example 2 Let $X = \{0,1\}$ and suppose that $\succ$ is represented by

$$V(\alpha \delta_0 + (1-\alpha) \delta_1) = \begin{cases} 2 & \text{if } \alpha = 0 \\ \frac{3}{2} (1 - \frac{\alpha}{2}) \sin (1/\alpha) & \text{if } \sin x \geq 0 \\ \frac{1}{2} (1 + \alpha) \sin (1/\alpha) & \text{otherwise} \end{cases}$$

Then $V$ is not continuous at $\alpha = 0$ yet $\succ$ satisfies the Archimedean axiom as well as the weak version (that is, it holds for some but not all $\gamma$) of local mixture dominance.

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1 I owe this example to Uzi Segal.
2 I owe this example to Zvi Safra.