Theory and Decision
https://doi.org/10.1007/s11238-024-09995-7

Ambiguity aversion, risk aversion, and the weight of evidence

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Accepted: 28 May 2024

Abstract

Wakker (Theory Decis 29:119–132, 1990) proved that the rank-dependent utility model is a special case of Choquet expected utility model. Invoking this result and applying the certainty equivalent reduction procedure, this paper shows that risk aversion in the rank-dependent utility model implies ambiguity aversion in the corresponding Choquet expected utility model. Consequently, the pattern of choice depicted by Ellsberg's experiments and, more generally, preference for evidence to support their beliefs is an expression of decision makers' risk aversion. In addition, the paper introduces a new procedure of updating capacities by the application of Bayes rule that smooth ambiguity aversion may also be regarded as risk aversion.

Keywords Ambiguity aversion · Risk aversion · Choquet expected utility · Rank-dependent utility · Bayesianism

1 Introduction

Bayesian decision theory, pioneered by Ramsey (1931) and de Finetti (1937), culminated in Savage's (1954) and Anscombe and Aumann's (1963) subjective expected utility (SEU) models. These models presume that decision makers entertain beliefs about the likelihoods of events that are quantifiable by probability measures. Moreover, these beliefs are manifested in, and can be inferred from, decision makers' choice behavior. Using different choice sets, both Savage and Anscombe and Aumann depict structures of preference relations (i.e., patterns of choice behavior)

E. Karni is very grateful to Peter Wakker, Zvi Safra and an anonymous referee for their insightful comments and suggestions.

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Published online: 15 June 2024

that are necessary and sufficient to quantify a decision maker's beliefs by a unique, subjective, probability measure.

Consider a decision maker whose belief that an event, say E, obtains is quantified by a probability $\pi(E)$. Presumably, this belief incorporates the decision maker's information regarding the plausibility of this event. Moreover, it is natural to suppose that the quality of the information and the confidence in the belief it inspires, should affect the decision maker's behavior. Specifically, it is natural to suppose that decision makers prefer taking courses of action about the likely outcomes of which their beliefs are more informed. The subjective expected utility models, however, fail to account for the weight of evidence, or information, that support the decision makers' probabilistic beliefs. In particular, given a posterior probability a decision maker's choices are the same regardless of the evidence supporting that posterior. Put differently, SEU theories accords no weight to the evidence supporting the decision makers' beliefs.

To grasp the issue, consider an urn containing a hundred balls that are known to be either red or black. No other prior information about the color composition of the balls in the urn is available. A ball is about to be drawn at random and a decision maker is contemplating placing a bet on its color. Compare the following two scenarios; in the first scenario, before placing the bet, a sample of balls are drawn from the urn repeatedly, with replacement, and their colors observed. Suppose that after repeated draws, it so happens that the number of red and black balls in the sample is the same. The decision maker concludes that observing either color in the next draw is equally likely. In the second scenario no balls are drawn before the decision maker is required to place the bet. It seems reasonable that, by reason of symmetry (or insufficient reason), the decision maker believes that the events drawing a red ball and drawing a black ball are equally likely. According to subjective expected utility theory, in both scenarios, the decision maker's subjective probabilities of the two events are the same and equal 1/2. Consequently, a subjective expected utility maximizing decision maker displays indifference between betting on red and betting on black, and also between betting on red on a draw from the tested and the untested urn. The greater confidence in his belief about the likely outcome if the ball is drawn from the tested urn is completely disregarded.

This lack of consideration of the weight of evidence supporting the decision making beliefs has been long recognized. In *Treatise on Probability* Keynes discusses a scenario similar to the one described above. In Keynes' words, "…in the first case we know that the urn contains black and white in equal proportions; in the second case the proportion of each color is unknown, and each ball is as likely to be black as white. It is evident that in either case the probability of drawing a white ball is 1/2, but that the weight of the argument in favour of this conclusion is greater in the first case." Keynes (1921 [1973], p. 82).[1]

[1] Zappia (2020) contains a detailed review of this issue including an exchange between Savage and Popper prompted by it. Zappia (2020) mentioned that Popper (1958) "…argued that he found it paradoxical that two apparently similar events should be attributed the same subjective probability even though the evidence supporting judgment in one case was stronger than in the other case."

Ellsberg (1961) argued that, facing choices between the bets described in the example above, decision makers exhibit strict preference for betting on either color of a ball drawn from an urn containing equal number of black and white balls over betting on either color of a ball drawn from an urn in which the color composition of the balls is unknown. These preferences are inconsistent with the existence of additive subjective probabilities à la Savage.

Schmeidler (1986, 1989) proposed a novel decision model, that later became known as Choquet expected utility (CEU), designed to accommodate choice behavior that accounts for the information supporting the decision maker's beliefs.[2] According to the CEU model, decision makers maximize the expectations of a utility function with respect to a non-additive probability measure, or capacity. To capture the preference for being better informed, Schmeidler introduced the notion of ambiguity aversion and characterized it by a convex capacity.[3]

About the same time that Schmeidler developed the CEU model, a class of theories of decision making under risk, dubbed rank-dependent utility (RDU) models were introduced. These include Quiggin's (1982) Anticipated Utility, Yaari's (1987) Dual Theory as well as Wakker (1994). According to these models preference relations on objective risks, or lotteries, are represented by the inner product of a utility function on the set of outcomes and a transformation function on the corresponding probabilities, where the transformation of the objective probabilities depends on the ranks of the outcomes in the set of feasible outcomes.

Wakker (1990) showed that under stochastic dominance the RDU model is a special case of the CEU model. Invoking Wakker's result, I present a hybrid RDU model in which, following Schmeidler (1989), risks are evaluated by their expected utility and uncertainties by the expected utility with respect to a transformation function of the subjective probabilities. I then argue that the representation of risk aversion in that model implies the representation of ambiguity aversion by the corresponding CEU model. I also introduce a new procedure for updating the capacities based on updating the subjective probabilities by Bayes' rule.

A different formalization of ambiguity aversion was proposed by Klibanoff et al. (2005) and Seo (2009). In their models ambiguity aversion is captured by a concave real-valued function on the expected utilities associated with the set of all conceivable priors. I argue that, properly formulated, smooth ambiguity aversion may also be regarded as risk aversion.

The next section sets the stage, briefly describing the rank-dependent utility and Choquet expected utility representations. Section 3 discusses two procedures of reduction of compound lotteries and introduces the lead example. Section 4 includes the statement and proof of the main result. Section 5 applies the main idea

[2] For alternative modelings of Choquet expected utility see Gilboa (1987), Wakker (1989a, b), Nakamura (1990) and Chew and Karni (1994).

[3] Schmeidler (1989) dubbed the preference for betting on events whose probability is supported by more evidence uncertainty aversion. This term was replaced by ambiguity aversion in the nomenclature of decision theory. Schmeidler also discusses several equivalent characterizations of ambiguity aversion. For recent surveys see Gilboa and Marinacci (2016) and Trautmann and van de Kuilen (2015).

to the models based on second-order beliefs. The concluding section discusses the application of the approach to the Ellsberg two-urn experiment reviews the related literature.

2 Setting the stage

2.1 Subjective expected utility and rank-dependent utility

Let S be a finite set of *states* and let X be a nonpoint interval in the real line. Denote by $\mathcal{G}(X)$ the set of cumulative distribution functions (CDF) on X with finite supports, referred to as *lotteries*. The choice set, $A := \{f : S \to \mathcal{G}(X)\}$, of all mappings from S to $\mathcal{G}(X)$, representing alternative courses of action, referred to as *acts*. For all $f, g \in A$ and $\alpha \in [0,1]$, define $(\alpha f + (1-\alpha)g) \in A$ by $(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s)$, for all $s \in S$. Thus, A is a convex set in a linear space.

A *preference relation*, denoted $\succcurlyeq$, is a complete and transitive binary relation on A. The asymmetric and symmetric parts of $\succcurlyeq$ are denoted by $\succ$ and $\sim$, respectively. Anscombe and Aumann (1963) provide necessary and sufficient conditions for the representation of $\succcurlyeq$ by a subjective expected utility (SEU) functional. Formally, for all $f, g \in A$,

$$f \succcurlyeq g \Leftrightarrow \Sigma_{s\in S} U(f(s))\pi(s) \geq \Sigma_{s\in S} U(g(s))\pi(s), \tag{1}$$

where the *utility function* U is affine (i.e., linear in the probabilities) real-valued function on $\mathcal{G}(X)$ and π is an additive probability measure on 2^S. The utility-probability pair (U, π) is *jointly unique* (i.e., given π the function U is unique up to positive affine transformation and given U, π is unique). Define $u : X \to \mathbb{R}$ by $u(x) = U(\delta_x)$, where δ is the Dirac measure.

The RDU models—Quiggin's (1982, 1993) Anticipated Utility,[4] Yaari's (1987) Dual Theory—are theories of decision making under risk (that is, the domain of the preference relation is the set, $\mathcal{G}(X) \subset A$, of constant acts that deliver the same CDF in every state). According to these models a preference relation $\succcurlyeq$ on $\mathcal{G}(X)$ is represented by RDU. Formally, for all $G, F \in \mathcal{G}(X)$,

$$G \succcurlyeq F \Leftrightarrow \int_X u(x)d(\xi \circ G)(x) \geq \int_X u(x)d(\xi \circ F)(x), \tag{2}$$

where the utility, u, is a real-valued function on X, unique up to positive affine transformation, and $\xi : [0,1] \to [0,1]$ is non-decreasing, continuous, and onto function, dubbed *probability transformation function*.

[4] See also Segal (1989, 1993).

2.2 Choquet expected utility and hybrid rank-dependent utility

To analyze the relationships between ambiguity aversion and risk aversion we need to convert the RDU models of decision making under risk to theories of decision making under uncertainty. This conversion was attained by Wakker (1990). Invoking a capacity of the CEU model, Wakker induced a probability transformation function in the RDU model such that when composed with an additive probability measure, it mimics the capacity of the CEU model.

Following Schmeidler (1989), a preference relation, $\hat{\succcurlyeq}$ on A, is said to have a CEU representation if for all $f, g \in A$,

$$f \hat{\succcurlyeq} g \Leftrightarrow \int_S \hat{U}(f(s))d\varphi(s) \geq \int_S \hat{U}(g(s))d\varphi(s), \tag{3}$$

where $\hat{U}$ is an affine, real-valued, function on $\mathcal{G}(X)$, unique up to positive affine transformation, and $\varphi : 2^S \to [0,1]$ is a capacity measure.[5]

It is worth emphasizing that, according to Schmeidler, risks are evaluated by their expected utility using the objective, additive, probability measures (i.e., $\hat{U}(f(s)) = \Sigma_{x\in X}\hat{u}(x)f(s)(x)$) while uncertainties are evaluated by their subjective, non-additive, probability measures. I refer to this distinct treatments of the two sources of variations of outcomes as *Schmeidler's doctrine* and invoke it in the analysis below.

Given a preference relation $\hat{\succcurlyeq}$ on A that has a CEU representation, define a binary relation $\succcurlyeq^*$on 2^S as follows: $C \succcurlyeq^* B$ if and only if $\varphi(C) \geq \varphi(B)$. Consider a decision maker whose preference relation $\hat{\succcurlyeq}$ on A incorporates beliefs that are represented by an additive subjective probability measure, π on 2^S, such that $C \succcurlyeq^* B$ if and only if $\pi(C) \geq \pi(B)$. Thus, $\varphi(C) \geq \varphi(B)$ if and only if $\pi(C) \geq \pi(B)$. Therefore, there exists a strictly increasing and onto probability transformation function $\zeta : [0,1] \to [0,1]$ defined by $\varphi(E) = (\zeta \circ \pi)(E)$, for all $E \in 2^S$.[6] Then

$$\int_S \hat{U}(f(s))d\varphi(s) = \int_S \hat{U}(f(s))d(\zeta \circ \pi)(s). \tag{4}$$

Adding to Schmeidler's CEU model the assumption that the capacity is a transform of subjective probabilities, including that such subjective probabilities exist, is nontrivial restrictive assumption. It allows me to present the main idea in the most transparent manner. Exploration of the implications of weakening this assumption is beyond the scope of this paper.

For each $f \in A$ let $c_f := \left(c_{f(s)}\right)_{s\in S}$, where $c_{f(s)} \in X$ is the certainty equivalent of $f(s)$ (i.e., $\hat{U}(f(s)) = \hat{U}\left(\delta_{c_{f(s)}}\right)$).[7] Then, by (3), $f \hat{\sim} c_f$.

[5] A capacity measure is a set function φ on a measurable space (S, Σ) such that $\varphi(\emptyset) = 0$, $\varphi(S) = 1$ and $\varphi(A) \leq \varphi(B)$ for all $A, B \in \Sigma$ such that $A \subseteq B$.

[6] Wakker (1990, Lemma 6) gives necessary and sufficient condition for the existence of such probability transformation function.

[7] If $\hat{U}$ is on X is continuous and increasing then the certainty equivalent is unique.

For each $f \in A$, define $E_f(x) = \{s \in S \mid c_{f(s)} \leq x)\}$, $x \in X$, and let $G_f(x) := \pi\left(E_f(x)\right)$, for all $x \in X$. Let H_f denote the decumulative distribution function induced by f (i.e., $H_f(x) := 1 - G_f(x)$, for all $x \in X$). Then, $H_f(x) = \pi\left(S \backslash E_f(x)\right)$ and, for each $f \in A$,[8]

$$\begin{aligned}\int_S u\left(c_{f(s)}\right) d(\zeta \circ \pi)(s) &= \int_X u(x)(\zeta \circ \pi)\left(S \backslash E_f(x)\right) dx \\ &= \int_X u(x) d\left(-\zeta \circ H_f\right)(x).\end{aligned} \tag{5}$$

Let $\xi(\tau) = 1 - \zeta(1 - \tau)$, then

$$-\int_X u(x) d\left(\zeta \circ H_f\right)(x) = \int_X u(x) d\left(\left(\xi \circ G_f\right)(x) - 1\right). \tag{6}$$

Hence,

$$\begin{aligned}\int_S U(f(s)) d(\zeta \circ \pi)(s) &= \int_S u\left(c_{f(s)}\right) d(\zeta \circ \pi)(s) \\ &= \int_X u(x)(\zeta \circ \pi)\left(E_f(x)\right) dx \\ &= \int_X u(x) d\left(\xi \circ G_f\right)(x).\end{aligned} \tag{7}$$

Therefore, we have what I refer to as *hybrid RDU* model (henceforth HRDU) according to which, invoking Schmeidler's doctrine, risks (i.e., constant acts) are evaluated by their expected utility and non-constant acts are evaluated by their expected utility with respect to a transformed additive subjective probability measure that mimics the capacity of the corresponding CEU model.

3 The weight of evidence and risk aversion

3.1 Two reduction procedures

Resolutions of uncertainties may occur in stages as information may be obtained sequentially. If acting upon the receipt of early information is not feasible, the situation is represented by compound lotteries that may involve subjective and objective uncertainties. A prime example of such compound lotteries is the Anscombe and Aumann (1963) model in which the decision maker's beliefs about the likely outcomes of a horse race are represented by subjective probabilities and the payoffs, contingent on the outcomes of the horse race, are 'roulette lotteries' that assign objective probabilities to a set of prizes.

[8] See Wakker (1990) for a detailed exposition.

The literature dealing with decision making under uncertainty includes alternative procedures of reducing compound lotteries to equivalent (i.e., indifferent) one stage lotteries. To discuss the reduction procedures formally, let $\mathcal{G}(X)$ denote the set of one stage lotteries. A two-stage compound lottery is a probability distribution on $\mathcal{G}(X)$. Formally, $L_2 = \left[\pi(1), G_1; \pi(2), G_2; \ \ldots ; \pi(n), G_n\right]$, where $G_s \in \mathcal{G}(X)$ and $\pi(\cdot)$ is a subjective probability measure on S. Denote by $\mathcal{L}_2$ the set of two-stage compound lotteries.

The reduction of compound lotteries axiom requires that the probabilities of the ultimate outcomes, or prizes, be the product of the probabilities of reaching the nodes along the path leading to the outcome. For our purpose we only need to consider two-stage compound lotteries.

Reduction of compound lotteries axiom (RCLA): For all $L_2 \in \mathcal{L}_2$, $L_2 \sim \left[\Sigma_{s\in S}\pi(s)g_s\left(x_1\right), x_1; \ \ldots ; \Sigma_{s\in S}\pi(s)g_s\left(x_m\right), x_m\right]$, where $g_s\left(x_i\right) := G_s\left(x_i\right) - G_s\left(x_{i-1}\right)$, for all $s \in S$ and $i = 1, \ldots, m$.

An alternative reduction procedure calls for the replacement of the second-stage lottery by its certainty equivalent to obtain one stage lottery. Formally, denote by G_s the constant act that assigns the lottery $G_s \in \mathcal{G}(X)$ to every $s \in S$. The certainty equivalent of G_s is $c\left(G_s\right) \in \mathbb{R}$ defined by $\delta_{c(G_s)} \sim G_s$.

Certainty equivalence reduction axiom (CERA): For all $L_2 \in \mathcal{L}_2$, $L_2 \sim \left[\pi(1), \delta_{c(G_1)}; \pi(2), \delta_{c(G_2)}; \ \ldots ; \pi(n), \delta_{c(G_n)}\right]$.

Preference relations that admit expected utility representations are consistent with both reduction procedures. By contrast, preference relations that admit rank-dependent utility representations with nonlinear probability transformation functions are consistent with CERA but not with RCLA (see Segal, 1990).

3.2 An example

Consider an urn drawn at random from a population of urns each containing 100 balls that are either red or black and no further information is available. There are 101 feasible color compositions defining the states. Imagine the possibly of asking decision makers to choose among acts whose payoffs are contingent on the color composition of the balls in the urns before selecting an urn and verifying its content (i.e., color composition). To formalize this decision problem, let $S = \{0, 1, \ldots, 100\}$, where $s \in S$ is the number of red balls in the urn. Consider a bet that pays off \$100 if a ball drawn at random is red and \$0 otherwise. Then, from the decision maker's viewpoint, the bet is an act, $f_R := \left(f_R(s)\right)_{s\in S}$ whose payoffs are state-contingent lotteries $f_R(s) = \left(\$100, s/100; \$0, (1-s)/100\right)$. Let $c\left(f_R(s)\right)$ denote the certainty equivalent of $f_R(s)$ (i.e., the decision maker is indifferent between the constant act that pays $c\left(f_R(s)\right) \in \mathbb{R}$ in every state and the constant act that pays off \$100 with probability $s/100$ and \$0 otherwise). Define the induced act $c_R = \left(c\left(f_R(0)\right), c\left(f_R(1)\right), \ldots, c\left(f_R(100)\right)\right)$.

Let π denote the decision maker's subjective prior belief on S, and denote by $\pi(\cdot \mid y^n)$, his posterior belief on S, given sample $y^n = \left(y_1, \ldots, y_n\right)$, $n \geq 1$, drawn at random from the urn with replacement. Let y^0 indicates that no ball is drawn. Then $\pi(\cdot \mid y^0) = \pi(\cdot)$.

Consider the decision maker's choice between the acts, $f_R \mid_{y^n}$ and $f_R \mid_{y^m}$, (i.e., a choice between betting on red conditional on observing the samples y^n and y^m, of sizes n and m, respectively.[9] Clearly, $\pi(\cdot \mid y^n)$ is a mean-preserving squeeze of $\pi(\cdot \mid y^m)$ on $\left(c\left(f_R(0)\right), c\left(f_R(1)\right), \dots, c\left(f_R(100)\right)\right)$.[10] By definition every risk-averse decision maker prefers $f_R \mid_{y^n}$ over $f_R \mid_{y^m}$.

Let there be two identical urns, say A and B, containing 100 balls each that are either red or black. Suppose that random samples of sizes $2m$ and $2n$, $m < n$, are to be drawn from A and B, respectively, with replacement. Assume that the prior $\pi(\cdot \mid y^0)$ is symmetric and consider the event that both samples happen to contain equal number of red and black balls.

Consider a decision maker whose preference relation has a HRDU representation as in (4) and (7) facing a choice between two bets; $f_R \mid_A$, that pays off \$100 if the next ball drawn at random from urn A is red, and $f_R \mid_B$, that pays off \$100 if the next ball drawn at random from urn B is red. By Bayes rule, given a state s and a sample of size $r = 2k$, by the binomial distribution, the probability of k red balls is

$$\begin{aligned}\Pr(k \mid s) &= \frac{r!}{k!(r-k)!}\pi\left(s \mid y^0\right)^k\left(1-\pi\left(s \mid y^0\right)\right)^{r-k} \\ &= \frac{r!}{k!(r-k)!}\left[\pi\left(s \mid y^0\right)\left(1-\pi\left(s \mid y^0\right)\right)\right]^k.\end{aligned}$$

By Bayes rule, the posterior probabilities satisfy, for all $r = 2k, k = 1, 2, \dots,$

$$\pi(s \mid k) = \frac{\left[\pi\left(s \mid y^0\right)\left(1-\pi\left(s \mid y^0\right)\right)\right]^k \pi\left(s \mid y^0\right)}{\Sigma_{s' \in S}\left[\pi\left(s' \mid y^0\right)\left(1-\pi\left(s' \mid y^0\right)\right)\right]^k \pi\left(s' \mid y^0\right)}.$$

An expected utility-maximizing decision maker would be indifferent between the two bets. To grasp this suffices it to note that an expected utility maximizer abides by the reduction of compound lotteries axiom. Letting $u(100) = 1$ and $u(0) = 0$, an expected utility maximizing decision maker evaluates the bets as follows:

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9 Ellsberg's two-urn experiment is a version of this example with $m = 0$ corresponding to the ambiguous urn and $n \to \infty$ corresponding to the unambiguous urn.

10 Formally, given $(x(1), \dots, x(\mid S \mid)) \in X^n$ define $\Pi\left(x \mid y^k\right) = \sum_{s \in \{s' \in S \mid x(s') \leq x\}} \pi\left(s \mid y^k\right)$, $\forall x \in X$. Then, $\sum_{s=0}^{t}\left[\Pi\left(c\left(f_R(s)\right) \mid y^n\right) - \Pi\left(c\left(f_R(s)\right) \mid y^m\right)\right] \leq 0$ for all $t \in \{0, 1, \dots, 100\}$ and $\sum_{s=0}^{100}\left[\Pi\left(c\left(f_R(s)\right) \mid y^n\right)\right.$ $\left.-\Pi\left(c\left(f_R(s)\right) \mid y^m\right)\right] = 0$, where we invoke the fact that $c\left(f_R(i)\right) < c\left(f_R(i+1)\right), i = 0, \dots, 99$, for all monotonic increasing utility functions.

$$U\left(f_R \mid_A\right) = \Sigma_{s=0}^{100} \frac{s}{100} \pi(s \mid y^m) = \frac{1}{2} = \Sigma_{s=0}^{100} \frac{s}{100} \pi(s \mid y^n) = U\left(f_R \mid_B\right).$$ [11]

However, a risk-averse HRDU maximizing decision maker would strictly prefer to bet on red from urn B. To grasp this, recall that a HRDU maximizing decision maker who is not an expected utility maximizer does not abide by the RCLA, displaying CERA instead. Moreover, risk-aversion in the HRDU model requires that the probability transformation function, ξ, that figures in the representation (7) be concave (see Chew et al., 1987). Since $\pi(\cdot \mid y^m)$ is a mean-preserving spread of $\pi(\cdot \mid y^n)$, the HRDU valuations of the bets are:

$$V\left(f_R \mid_A\right) = \sum_{s \in S} u\left(c\left(f_R(s)\right)\right)\left[\xi\left(\Sigma_{t=0}^{s} \pi(t \mid m/2)\right) - \xi\left(\Sigma_{t=0}^{s-1} \pi(t \mid m/2)\right)\right]$$

and

$$V\left(f_R \mid_B\right) = \sum_{s \in S} u\left(c\left(f_R(s)\right)\right)\left[\xi\left(\Sigma_{t=0}^{s} \pi(t \mid n/2)\right) - \xi\left(\Sigma_{t=0}^{s-1} \pi(t \mid n/2)\right)\right].$$

Hence, by risk-aversion, $V\left(f_R \mid_A\right) < V\left(f_R \mid_B\right)$. The weight of evidence tilts the bet in favor of the option in which the belief is based on more informative experiment. This observation underscores the point on which I elaborate in the analysis below, that it is risk-aversion in the context of the HRDU model that captures the effect of the weight of evidence on the decision maker's preferences.

[11] To grasp this claim observe that, by symmetry, $\pi\left(s \mid y^0\right) = \pi\left(100 - s \mid y^0\right)$, for all $s \in S$. Thus,

$$\pi\left(s \mid y^0\right)\left(1 - \pi\left(s \mid y^0\right)\right) = \pi\left(100 - s \mid y^0\right)\left(1 - \pi\left(100 - s \mid y^0\right)\right),$$

and

$$\begin{aligned} U\left(f_R \mid_A\right) &= \frac{1}{100} \sum_{s=0}^{100} \frac{\left[\pi\left(s \mid y^0\right)\left(1 - \pi\left(s \mid y^0\right)\right)\right]^{r/2} \pi\left(s \mid y^0\right) s}{\Sigma_{s' \in S}\left[\pi\left(s' \mid y^0\right)\left(1 - \pi\left(s' \mid y^0\right)\right)\right]^{r/2} \pi\left(s' \mid y^0\right)} \\ &= \frac{1}{100} \sum_{s=0}^{50} \frac{\left[2\pi\left(s \mid y^0\right)\left(1 - \pi\left(s \mid y^0\right)\right)\right]^{r/2} \pi\left(s \mid y^0\right)\left(\frac{\pi\left(s \mid y^0\right)\left(1 - \pi\left(s \mid y^0\right)\right)s + \pi\left(100 - s \mid y^0\right)\left(1 - \pi\left(100 - s \mid y^0\right)\right)(100 - s)}{2\pi\left(s \mid y^0\right)\left(1 - \pi\left(s \mid y^0\right)\right)}\right)}{\Sigma_{s' \in S}\left[\pi\left(s' \mid y^0\right)\left(1 - \pi\left(s' \mid y^0\right)\right)\right]^{r/2} \pi\left(s' \mid y^0\right)} \\ &= \frac{50}{100} \sum_{s=0}^{50} \frac{2\left[\pi\left(s \mid y^0\right)\left(1 - \pi\left(s \mid y^0\right)\right)\right]^{r/2} \pi\left(s \mid y^0\right)}{\Sigma_{s' \in S}\left[\pi\left(s' \mid y^0\right)\left(1 - \pi\left(s' \mid y^0\right)\right)\right]^{r/2} \pi\left(s' \mid y^0\right)} = \frac{1}{2}. \end{aligned}$$

This holds for all $r = 2k, k = 0, 1 \ldots$

4 Risk aversion and ambiguity aversion

4.1 Preliminaries

A preference relation $\succcurlyeq$ on $\mathcal{G}(X)$ is said to display risk aversion if, for all $F, G \in \mathcal{G}(X)$, $F \succcurlyeq G$ whenever G differs from F by a mean-preserving spread. It is said to display strict risk aversion if $F \succ G$.

The theory of risk aversion in the general RDU model was developed in Chew et al. (1987). In particular, Chew et al. showed that a preference relation $\succcurlyeq$ on $\mathcal{G}(X)$ that admits a RDU presentation $G \to \int_X u(x)d(\xi \circ G)(x)$ displays risk aversion if and only if both the utility function, u, and the probability transformation function, ξ, are concave. It displays strict risk aversion if and only if it displays risk aversion and either u or ξ or both are strictly concave.

The notion of ambiguity aversion was first formulated and characterized by Schmeidler (1986) and was further developed in Schmeidler (1989). Formally, a preference relation $\hat{\succcurlyeq}$ on $A$is said to display ambiguity aversion if, for all $f, g \in A$ and $\alpha \in (0,1]$, $f \hat{\succcurlyeq} g$ implies that $\alpha f + (1-\alpha)g \hat{\succcurlyeq} g$. It is said to display strict ambiguity aversion if $\alpha f + (1-\alpha)g \hat{\succ} g$. If $\hat{\succcurlyeq}$ admits CEU representation then ambiguity aversion is characterized by a convex capacity (i.e., a capacity satisfying $\varphi(B \cup C) + \varphi(B \cap C) \geq \varphi(B) + \varphi(C)$, for all $B, C \in 2^S$). Equivalently, it is characterized by concavity of the CEU functional $I(f) = \int_S U(f(s))d\varphi(s)$.[12]

To set the stage for the main result we need the following.

Lemma 1 *Let π be a subjective probability measure on S that figure in the SEU representation* (1), *then, for all $f, g \in A, \alpha \in [0,1]$ and $x \in X$,*

$$\pi\left(E_{\alpha f+(1-\alpha)g}(x)\right) = \alpha\pi\left(E_f(x)\right) + (1-\alpha)\pi\left(E_g(x)\right).$$

Proof By the theorem of Anscombe and Aumann (1963), the affinity of the utility function implies that, for all $f, g \in A$ and $\alpha \in [0,1]$,

$$\Sigma_{s\in S}U(\alpha f(s) + (1-\alpha)g(s))\pi(s) = \alpha\Sigma_{s\in S}U(f(s))\pi(s) + (1-\alpha)\Sigma_{s\in S}U(g(s))\pi(s). \tag{8}$$

□

Equivalently, by the certainty equivalence reduction axiom,

$$\Sigma_{s\in S}u\left(c_{\alpha f+(1-\alpha)g}(s)\right)\pi(s) = \alpha\Sigma_{s\in S}u\left(c_f(s)\right)\pi(s) + (1-\alpha)\Sigma_{s\in S}u\left(c_g(s)\right)\pi(s). \tag{9}$$

But, for all $f \in A$,

$$\Sigma_{s\in S}u\left(c_f(s)\right)\pi(s) = \int_X u(x)d\pi\left(E_f(x)\right). \tag{10}$$

[12] See the Proposition in Schmeidler (1989).

Hence, by (9) and (10), for all $f, g \in A$ and $\alpha \in [0, 1]$,

$$\begin{aligned}\int_X u(x)d\pi\left(E_{\alpha f+(1-\alpha)g}(x)\right) &= \alpha \int_X u(x)d\pi\left(E_f(x)\right) + (1-\alpha)\int_X u(x)d\pi\left(E_g(x)\right) \\ &= \int_X u(x)d\left(\alpha\pi\left(E_f(x)\right) + (1-\alpha)\pi\left(E_g(x)\right)\right).\end{aligned} \tag{11}$$

By the uniqueness of π, we get that

$$\pi\left(E_{\alpha f+(1-\alpha)g}(x)\right) = \alpha\pi\left(E_f(x)\right) + (1-\alpha)\pi\left(E_g(x)\right), \tag{12}$$

for all $f, g \in A$, $\alpha \in [0, 1]$ and $x \in X$. ∎

4.2 The main result

The main result of this paper is the demonstration that risk aversion in the HRDU model implies ambiguity aversion in the corresponding CEU model.

Theorem 1 *Let* $\hat{\succcurlyeq}$ *and* $\succcurlyeq$ *on* A *have CEU and HRDU representations* $f \to \int_S U(f(s))d\varphi(s)$ *and* $f \to \int_S U(f(s))d(\zeta\circ\pi)(s)$, *respectively, and suppose that* $\int_S U(f(s))d\varphi(s) = \int_S U(f(s))d(\xi\circ\pi)(s)$. *If* $\succcurlyeq$ *displays (strict) risk aversion then* $\hat{\succcurlyeq}$ *displays (strict) ambiguity aversion.*

Proof By the hypothesis, (4), (5), (6) and (7),

$$\int_S U(f(s))d\varphi(s) = \int_X U(f(s))d(\zeta\circ\pi)\left(E_f(x)\right) = \int_X u(x)d\left(\xi\circ G_f\right)(x). \tag{13}$$

Corollary 2 of Chew et al. (1987) asserts that a necessary condition for $\succcurlyeq$ to display (strict) risk aversion is that ξ is (strictly) concave. Thus, if $\succcurlyeq$ displays (strict) risk aversion then

$$\begin{aligned}\int_X u(x)d\left(\xi\circ\left(\alpha G_f + (1-\alpha)G_g\right)\right)(x) \geq (>)\alpha \int_X u(x)d\left(\xi\circ G_f\right)(x) \\ + (1-\alpha)\int_X u(x)d\left(\xi\circ G_g\right)(x).\end{aligned} \tag{14}$$

The convexity of A and the affinity of U imply that

$$U(\alpha f(s) + (1-\alpha)g(s)) = \alpha U(f(s)) + (1-\alpha)U(g(s)), \quad \forall s \in S. \tag{15}$$

By the lemma and the definition of $G_f(x)$,

$$\int_S (\alpha U(f(s)) + (1-\alpha)U(g(s)))d\varphi(s) = \int_X u(x)d(\zeta \circ \pi)\left(E_{\alpha f+(1-\alpha)g}(x)\right)$$
$$= \int_X u(x)d\zeta\left(\alpha\pi\left(E_f(x)\right) + (1-\alpha)\pi\left(E_g(x)\right)\right) \tag{16}$$
$$= \int_X u(x)d\left(\xi\circ\left(\alpha G_f + (1-\alpha)G_g\right)\right)(x),$$

and

$$\alpha \int_S U(f(s))d\varphi(s) + (1-\alpha)\int_S U(f(s))d\varphi(s) = \alpha \int_X u(x)d\left(\xi\circ G_f\right)(x) + (1-\alpha)\int_X u(x)d\left(\xi\circ G_g\right)(x). \tag{17}$$

Thus, (13) and (14) imply that

$$\int_S (\alpha U(f(s)) + (1-\alpha)U(g(s)))d\varphi(s) \geq (>)\alpha \int_S U(f(s))d\varphi(s) + (1-\alpha)\int_S U(g(s))d\varphi(s). \tag{18}$$

By the Proposition of Schmeidler (1989), $\hat{\succcurlyeq}$ displays (strict) ambiguity aversion if and only if $I(f) := \int_S u(f(s))d\varphi(s)$ is concave. Hence, if $\succcurlyeq$ displays risk aversion then $\hat{\succcurlyeq}$ displays ambiguity aversion. ■

Ambiguity aversion in the CEU model constrains only the capacity. Thus, it is consistent with convex utility function. In the HRDU model convex utility function is inconsistent with risk aversion. Hence, it is not the case that ambiguity aversion under CEU implies risk aversion under RDU.

It is worth underscoring, that the like Schmeidler's CEU model, the hybrid RDU evaluates objective risks by their expected utility with respect to the objective probabilities and subjective risks by their expected utility with respect to the transformed subjective probabilities.

Schmeidler's rank-dependent CEU, uses top-down integration, weighing "good news event" of receiving anything better than an outcome. Quiggin's rank-dependent RDU, uses bottom-up integration, weighing "bad news events" of receiving an outcome or anything worse. Thus, convexity of the capacity corresponds with concavity of probability transformation. Using bottom-up integration in the RDU model, risk-aversion a la Chew et al. (1987) would require that the probability transformation function be convex.

4.3 Bayesian updating

The issue of updating ambiguous beliefs has been thoroughly studied, and various axiomatic-based updating rules have been proposed for Choquet expected utility (e.g., Eichenberger et al., 2007; Gilboa & Schmeidler, 1993).

The approach taken in this paper suggests a new and natural procedure for updating the capacities by the application of Bayes rule in situations in which the Choquet expected utility has an equivalent HRDU representation. The idea is as follows: Update the additive probability measure using Bayes rule and update the corresponding capacity by setting the capacity of each event equal to the transformation of the updated additive probability of the same event. Formally, let y^n denote a sample of observations. For each event E, set $\varphi(E \mid y^n) = \zeta(\pi(E \mid y^n))$, where

$$\pi(E \mid y^n) = \frac{\pi(y^n \mid E)\pi\left(E \mid y^0\right)}{\pi(y^n \mid E)\pi(E) + \pi(y^n \mid S\backslash E)(1 - \pi\left(E \mid y^0\right)}.$$

In particular, $\varphi\left(E_f(x) \mid y^n\right) = \zeta(\pi\left(E_f(x) \mid y^n\right) = \xi \circ G_f(x \mid y^n), x \in X.$

5 Ambiguity and risk aversion with second-order beliefs

Klibanoff et al. (2005) and Seo (2009) propose alternative axiomatizations of a model of decision making under uncertainty in which ambiguity is expressed by the set of conceivable priors and ambiguity attitudes are captured by a real-valued function on the reals representing the expected utilities of acts under these priors. A decision maker's beliefs about the likelihoods of the priors is represented by a probability measure referred to as *second-order belief.* Formally, a preference relation exhibiting smooth ambiguity aversion has the representation

$$f \mapsto \int_{\Pi} v\left(\int_S [\Sigma_{x \in X} u(x) f(s)(x)] d\pi(s)\right) d\Phi(\pi), \tag{19}$$

where $\Pi := \{\pi \in [0,1]^{|S|} \mid \Sigma_{s \in S}\pi(s) = 1\}$, is the set of all probability distributions on the set of states, v is a real-valued function on $\mathbb{R}$, u is a real-valued function on X, and Φ a probability measure on Π, representing the decision maker's second order beliefs. In the usual interpretation, risk aversion is characterized by the concavity of u and ambiguity aversion by the concavity of v.

Adopting the approach of the preceding analysis I show below that smooth ambiguity aversion may be interpreted as (extra) layer of risk aversion. To begin with, I use the CERA to translate the ambiguity to risk. With this objective in mind, consider the risk represented by a choice of an act, f, under a prior $\pi \in \Pi$, and define the certainty equivalent of this risk, denoted $c_f(\pi)$, as the solution of the equation

$$u\left(c_f(\pi)\right) = \int_S [\Sigma_{x \in X} u(x) f(s)(x)] d\pi(s). \tag{20}$$

Then the representation (19) may be written as

$$f \mapsto \int_{\Pi} (v \circ u)\left(c_f(\pi)\right) d\Phi(\pi). \tag{21}$$

For every $x \in \mathbb{R}$, define $E_f(x) = \{\pi \in \Pi \mid c_f(\pi) \leq x\}$ and let $G_f(x) := \Phi(E_f(x))$, $x \in \mathbb{R}$. Then

$$\int_{\Pi} v\left(\int_S [\Sigma_{x\in X} u(x) f(s)(x)] d\pi(s)\right) d\Phi(\pi) = \int_X (v \circ u)(x) dG_f(x). \tag{22}$$

Given an act f let $\hat{G}_f$ be a mean-preserving spread of G_f (i.e., G_f is the CDF conditioned on a larger sample) then the decision maker displays risk aversion if $G_f \succcurlyeq \hat{G}_f$ if and only if the composition $v \circ u$ is a concave function. In particular, the concavity of v, which presumably captures the decision maker's ambiguity aversion may also be an expression of a second layer of risk aversion. Indeed, since v is a concave transformation of u, by a theorem of Pratt (1964), $v \circ u$ exhibits greater risk aversion than u.

Consider next the effect of the weight of evidence. Returning to the lead example, the utility of betting on red from urn A is

$$V(f_R \mid_A) = \Sigma_{i=1}^{100} (v \circ u)(c_f(\pi_i)) \left[G_f(c_f(\pi_i) \mid m/2) - G_f(c_f(\pi_{i-1}) \mid m/2)\right],$$

and that of betting on red from urn B is

$$V(f_R \mid_B) = \Sigma_{i=1}^{100} (v \circ u)(c_f(\pi_i)) \left[G_f(c_f(\pi_i) \mid n/2) - G_f(c_f(\pi_{i-1}) \mid n/2)\right].$$

But $n > m$ implies that $G_f(\cdot \mid n/2)$ is a mean-preserving squeeze of $G_f(\cdot \mid m/2)$, and (strict) risk aversion implies that $V(f_R \mid_B) > V(f_R \mid_A)$. The weight of evidence makes the betting on red from the less ambiguous urn preferable.

6 Discussion

Decision makers' preference to base their beliefs about the likely realization of the events that underlie the risks they are facing on more information about probabilities, dubbed ambiguity aversion, is captured by a convex capacity in the CEU model. By contrast, risk aversion depicts the decision maker's preference to avoid larger spreads of payoffs of the risky prospects, is captured by the concavity of the utility and the CDF transformation functions in the RDU models. Generally speaking, ambiguity aversion and risk aversion are distinct concepts. In this paper I propose a hybrid RDU model that mimics the CEU model and show that, under CERA, risk aversion displayed by the HRDU implies ambiguity aversion displayed by the corresponding CEU model, both describing attitudes towards mean-preserving spreads of the ultimate payoffs. The spread may be accounted for by objective, irreducible, risks or by uncertainty, reducible through the acquisition of information. The hybrid RDU and CEU models treat objective and subjective risks as distinct and evaluate them differently.

6.1 Ellsberg revisited

Consider the Ellsberg (1961) two urn experiment mentioned in the introduction. To simplify the exposition, consider a decision maker whose utility function is linear and normalized so that $u(\$0) = 0$ and $u(\$100) = 1$.[13] Suppose that the decision maker faces a choice between betting on the event, R, (i.e., being paid \$100 if the ball drawn at random from the ambiguous urn is red and \$0 otherwise) and betting on the event B, (i.e., being paid \$100 if the ball drawn at random from the ambiguous urn is black and \$0 otherwise). According to the CEU model symmetry and ambiguity aversion imply that $\varphi(R) = \varphi(B) < 1/2$. By the additivity of the prior probability measure $\pi(\cdot \mid y^0)$, we have $\pi(R \mid y^0) = \pi(B \mid y^0) = 1/2$, and by (4), $\varphi(E) = (\zeta \circ \pi)(E), E \in \{A, B\}$.

According to the analysis in this paper, a bet on any color from the ambiguous urn is modeled as two-stage compound lottery in which the beliefs about the likely realization of the states (i.e., color-composition of the selected urn) in the first stage are represented by a convex capacity measure. Risk aversion in the corresponding HRDU model is captured by a concave transformation function of additive probability measures that ranks the probability of event in the same way that the convex capacity does. The *state-dependent second stage risks* represented by the bets, either or red or black, are replaced by their *state-dependent certainty equivalents under expected utility*. For instance, because no ambiguity is associated with the unambiguous urn, the capacity assigned to the event red ball drawn at random from that urn is $\varphi(R) = 1/2$. Consequently, $\pi(R \mid y^0) = \varphi(R)$. Thus, the second-stage risk of betting on red from the unambiguous urn is reduced (by CERA) to a certainty equivalent outcome, $c(R)$, equal to $u^{-1}(1/2)$ (i.e., $0\pi(B \mid y^0) + 1\pi(R \mid y^0) = 1/2 = u(c(R))$). If betting on the color of a ball from the ambiguous urn displays ambiguity averse behavior according to the CEU model then, as is shown by the theorem, the representation of corresponding HRDU model displays concave CDF transformation function. Let $0\pi(B \mid y^0) + 1\pi(R \mid y^0) = s/100$, be the expected value of betting on red if the state (i.e., the number of red balls in the ambiguous urn) is $s = 0, 1, \ldots, 100$, then the certainty equivalent a bet on red, $\hat{c}(R)$, is given by the equation

$$u(\hat{c}(R)) = \frac{1}{100} \sum_{s \in 1}^{100} s\left[\xi\left(\Sigma_{t=0}^{s} \pi(t \mid y^0)\right) - \xi\left(\Sigma_{t=0}^{s-1} \pi(t \mid y^0)\right)\right] < 1/2,$$

where the inequality is an implication of the concavity of ξ. Thus, given our way of modeling ambiguity the CEU model and the corresponding HRDU model agree on the ranking of bets on the same color from the ambiguous and unambiguous urns.

[13] In the case of RDU theory this corresponds to Yaari's (1987) dual theory. The CEU representation is: $f \mapsto \int_S \left[\Sigma_{x \in X} x f(s)(x)\right] d\varphi(s)$.

6.2 Related literature

Segal (1987) studied the necessary conditions that the probability transformation function in the RDU model must satisfy if a decision maker is to prefer betting on a color from the unambiguous urn over betting on the same color from the ambiguous urn in Ellsberg's (1961) thought experiment. These conditions are more restrictive than the concavity of the probability transformation function. The explanation of the discrepancy between Segal's conclusions and those of this paper is to be found in the definitions of the certainty equivalents of unambiguous lotteries. In this paper, I adopted the Schmeidler doctrine, defining the certainty equivalents of the second-stage lotteries according to the expected utility model (i.e., using additive probabilities). By contrast, Segal invokes the RDU model to represent both the subjective and objective uncertainty. Specifically, using our notations, Segal's (1987) certainty equivalent of a lottery, ℓ, that pays off \$$x$ with probability p and \$0with probability $1-p$ is $c(\ell)=u^{-1}(u(0)\xi(p)+u(x)(1-\xi(p)))$. Consequently, according to Segal, the certainty-equivalent reduction of the two-stage compound lottery that consists of subjective uncertainty in the first stage and objective uncertainty in the second that corresponds to the ambiguous urn in Ellsberg's thought experiment, involves the probability transformation function twice, which is the source of the additional conditions in Segal's results.

For long I have been puzzled by the fact that institutions designed to better allocate risk-bearing (such as insurance and financial markets) are prevalent whereas no institutions appear to have been designed to improve the allocation of ambiguity. Perhaps the answer is that ambiguity aversion is an aspect of risk aversion and therefore the allocation of ambiguity bearing does not require special institutions.

Funding No funding was provided for this article.

Data availability There is no data in this submission.

Declarations

Conflict of interest The author declares that he has no funding, conflict of interest, competing interests or personal relationships that could have appeared to influence the work reported in this paper.

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