Stochastic Choice Functions and Irresolute Choice Behavior

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Abstract

This paper provides axiomatic characterizations of stochastic choice functions that are rationalizable by the irresolute choice model of Karni (2023) and examines its applications to demand theory and portfolio selection.

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Keywords: Stochastic choice functions**,** irresolute choice, random choice behavior, incomplete preferences, stochastic demand theory.

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"In interpreting human behavior there is a need to substitute "stochastic consistency of choices" for "absolute consistency of choices." The latter is usually assumed in economic theory but is not well supported by experience," Block and Marschak (1960).

1 Introduction

It is standard practice in economics and decision theory to depict individual choice behavior by rational (i.e., transitive) preference relations on sets of alternatives whose interpretations are context dependent.[1] Formally, a preference relation is a transitive and irreflexive binary relation, denoted by $\succ$, on a set A of alternatives, where $a \succ a'$ means that the alternative a is strictly preferred to a'.

The exact meaning of the last statement is open to interpretation. One interpretation is that the preference relation have substantive meaning, capturing intrinsic characteristics of the decision maker, that make him choose the alternative a whenever facing a choice between a and a'. An alternative, behavioral, interpretation takes the same statement to parsimoniously summarize the decision maker's revealed choices. According to this interpretation $a \succ a'$ means that, other things being the same, facing the need to choose, repeatedly, between the alternatives a and a' the decision maker consistently chooses a.

Underlying both interpretations is the notion described by Block and Marschak (1960) as "absolute consistency of choices." Absolute consistency of choices may depict accurately choice behavior in some situations (e.g., when the choice is between bets ranked by pointwise first-order stochastic dominance). However, in many situations (e.g., choice between dining in Chinese or Indian restaurants), repeated choices reveal that different alternatives are chosen on occasion, producing a pattern depicted by stable frequency distribution. Using the terminology of by Block and Marschak (1960), such behavior may be described as "stochastic consistency of choices." The stochastic pattern may be the manifestation of the effects of factors not captured by the primitives A and $\succ$. The neglected factors include unobserved

[1] Sometimes included in the definition of rational preference relation the condition of completeness (e.g., Mas-Collel, Whinston, and Green [1995]). However, there is nothing irrational in finding some alternatives noncomparable and exhibiting incomplete preferences.

psychological processes, such as boredom, variations in mood, physical or biological processes such as changing needs, or inability to compare the alternatives due to their complexity or lack of familiarity that is resolved by deliberate randomization; by exogenous stimuli (e.g., imitation of others); or by subconscious neurological process (e.g., drift diffusion). Whatever might be the underlying reasons, to an observer, the decision maker's choice behavior appears to be stochastic. Put differently, from an observer point of view, there are inputs (i.e., sets of feasible alternatives) and the outputs (i.e., the alternatives chosen). Lacking the ability to discern what is going on in the decision maker's mind, one must, provisionally, settle on models that make sense of the observed choice patterns and derive their implications.[2]

A stochastic choice function assigns to every element in every given feasible set of alternatives a probability of being selected. Thus, stochastic choice functions are formal summaries of the relationships between the inputs and outputs and have been a subject of extensive studies of stochastic choice behavior.[3]

In Karni (2023), I proposed a theory, dubbed irresolute choice model, in which stochastic choice is expressed by a set of transitive and irreflexive binary relations $\succ^{\alpha}$ on A, where $a \succ^{\alpha} a'$ is interpreted to mean that facing repeated choices from a binary set, $\{a, a'\}$, of feasible alternatives, ceteris paribus, the relative frequency with which a decision maker chooses alternative a is α and that of choosing a' is $1 - \alpha$.

The main purpose of this paper is to study the relation between stochastic choice function and irresolute choice behavior depicted by the irresolute choice model. In particular, characterize the stochastic choice functions that are rationalizable by irresolute choice models.[4] In addition, this paper discusses the representations of stochastic choices as random utility process and its application to the theories of individual and market demand as well as portfolio selection.

The contribution of this work to the literature dealing with the modelling and analysis of stochastic choice behavior is better understood after the ideas

[2]Improved understanding of the way the brain works may one day allow researchers to model the decision-making process at the neurological level.

[3]Some of these studies are discussed in section 6.1 below.

[4]The study of the relationship between preference relations depicted by the irresolute choice model and stochastic choice rules is a contribution to the research agenda that explores the relationship between preference relations and choice functions (see Mas-Collell, Whinston and Green (1995) Ch. 1).

and results of this work are presented. I therefore relegate the discussion of the related literature to the concluding section.

The paper is organized as follows. The next section introduces the stochastic choice functions and a model of irresolute choice. Section 3 analyzes the relationships between stochastic choice functions and irresolute choice behavior depicted by this model. Section 4 discusses the representations of stochastic choice functions. Section 5 applies the irresolute choice model to the theories of individual and market demands and portfolio selection. Section 6 discusses the related literature and offers some concluding remarks.

2 Stochastic Choice Functions and the Irresolute Choice Model

2.1 Stochastic choice functions

Let A denote an arbitrary set with $\mid A \mid \geq 2$, referred to as the *choice set.* Elements of A are *alternatives,* depending on the context may be interpreted as courses of action, bundles of goods, or outcomes that the decision maker cares about. Denote by $\mathcal{A}$ the set of all non-empty finite subsets of A. Elements of $\mathcal{A}$, dubbed *menus,* represent feasible sets of mutually exclusive alternatives that a decision maker may have to choose form.

A *stochastic choice function* (SCF) is a mapping $P : A \times \mathcal{A} \rightarrow [0, 1]$ such that

$$\Sigma_{a \in M} P(a, M) = 1, \text{ for every } M \in \mathcal{A}$$

and

$$P(a', M) = 0, \text{ for every } a' \in A \backslash M.$$

For all binary menus, $\{a, a'\} \in \mathcal{A}$, $P(a, \{a, a'\}) = 1$ means that facing the choice between a and a' the decision maker always chooses a, and $P(a, \{a, a\}) = 1/2$ by definition.

I consider SCFs that feature two attributes. The first attribute, *regularity*, asserts that the probability of choosing an alternative from a menu is (weakly) smaller the more inclusive the menu.[5] This property restricts the structure of the SCF across menus in the spirit of the weak axiom of revealed preference. In particular, it asserts that if an alternative a is revealed to be chosen from

[5] See Block and Marschak (1960).

a menu M' with certain frequency, then it is revealed to be chosen from a submenu $M \subset M'$ at lease as frequently. Formally,

(A.1) **Regularity:** For all M, $M' \in \mathcal{A}$ such that $M \subset M'$ and $a \in M$, $P(a, M') \leq P(a, M)$.[6]

The second attribute requires that the restriction of the probabilistic choice relation depicted by SCF to binary menus be stochastically transitive. Formally,

(A.2) **Stochastic Transitivity (ST):** For all $a, a', a'' \in A$ and $\lambda \in [1/2, 1)$, $P(a, \{a, a'\}) > \lambda$ and $P(a', \{a', a''\}) > \lambda$ imply $P(a, \{a, a''\}) > \lambda$.

The literature dealing with stochastic choice behavior contains distinct conceptions of stochastic choice transitivity.[7] One such concept is Partial Stochastic Transitivity (PST). Formally, for all a, $a', a'' \in A$, $P(a, \{a, a'\}) > 1/2$ and $P(a', \{a', a''\}) > 1/2$ implies $P(a, \{a, a''\}) \geq \min\{P(a, \{a, a'\}), P(a', \{a', a''\})\}$.[8]

Proposition: A stochastic choice function satisfies Stochastic Transitivity if and only if it satisfies Partial Stochastic Transitivity.

Proof. (Sufficiency) Let $a, a', a'' \in A$, be such that $P(a, \{a, a'\}) > 1/2$ and $P(a', \{a', a''\}) > 1/2$ and define

$$L := \{\lambda \in [1/2, 1) \mid P(a, \{a, a'\}) > \lambda\} \cap \{\lambda \in [1/2, 1) \mid P(a', \{a', a''\}) > \lambda\}.$$

By ST, for all $\lambda \in L$ it holds that $P(a, \{a, a''\}) > \lambda$. Hence, $P(a, \{a, a''\}) \geq \sup L = \min\{P(a, \{a, a'\}), P(a', \{a', a''\})\}$.

(Necessity) Suppose that $P(a, \{a, a'\}) > \lambda$ and $P(a', \{a', a''\}) > \lambda$. Since $\lambda \geq 1/2$, PST implies that $P(a, \{a, a''\}) \geq \min\{P(a, \{a, a'\}), P(a', \{a', a''\})\} > \lambda$. □

2.2 Irresolute choice model

Let $([1/2, 1], \mathcal{B}_{[1/2,1]}, \eta)$ be a probability space, where $\mathcal{B}_{[1/2,1]}$ is the Borel $\sigma-$field on $[1/2, 1]$ and η a Borel probability measure on $\mathcal{B}_{[1/2,1]}$. Let $K :=$

[6] The axiom may be stated as follow: For all M, $M' \in \mathcal{A}$ and $a \in M \cap M'$, $\max\{P(a, M), P(a, M')\} \leq P(a, M \cap M')$.

[7] See Fishburn (1973) for a discussion of different notions of stochastic transitivities.

[8] Another concept, Moderate Stochastic Transitivity, is obtained from PST by replacing the strict inequalities in the hypothesis with weak inequalities. He and Natenzon (2022) show that a version of Moderate Stochastic Transitivity is necessary and sufficient for a binary stochastic choice rule, ρ, to have a moderate utility representations proposed by Halff (1976).

$\{\succ^{\alpha} \mid \alpha \in [1/2,1]\}$ be an indexed set of irreflexive and transitive binary relations on A, satisfying set-inclusion monotonicity (i.e., for all $\alpha, \alpha' \in [1/2,1]$, $\alpha' \leq \alpha$ if and only if $\succ^{\alpha} \subseteq \succ^{\alpha'}$).[9]

For each $\alpha \in [1/2,1]$, the derived relations $\succcurlyeq^{\alpha}$, $\sim^{\alpha}$, $\bowtie^{\alpha}$ and $\succsim^{\alpha}$ are defined as follows: $a \succcurlyeq^{\alpha} a'$ if, for all $a'' \in A$, $a'' \succ^{\alpha} a$ implies that $a'' \succ^{\alpha} a'$; $a \sim^{\alpha} a'$ if $a \succcurlyeq^{\alpha} a'$ and $a' \succcurlyeq^{\alpha} a$; $a \bowtie^{\alpha} a'$ if $\neg (a \succcurlyeq^{\alpha} a')$ and $\neg (a' \succcurlyeq^{\alpha} a)$; $a \succsim^{\alpha} a'$ if $\neg (a' \succ^{\alpha} a)$. By transitivity, $a \succ^{\alpha} a'$ implies that $a \succcurlyeq^{\alpha} a'$. In particular, $a \succcurlyeq^{1/2} a'$ and $\neg \left(a \succ^{1/2} a'\right)$ if and only if $a' \succcurlyeq^{1/2} a$ and $\neg \left(a' \succ^{1/2} a\right)$. Furthermore, if $a \succcurlyeq^{1/2} a'$ and $a' \succcurlyeq^{1/2} a$ then, by definition, $a \sim^{1/2} a'$ and $\bowtie^{1/2} = \varnothing$, (that is, the derived binary relations $\succcurlyeq^{1/2}$ and $\succsim^{1/2}$ are complete).

Given any $a, a' \in A$ define $\Lambda(a,a') = \{\alpha \in [1/2,1] \mid a \succ^{\alpha} a'\}$, then $\Lambda(a,a') \in \mathcal{B}_{[1/2,1]}$. *Probabilistic choice relations* are $\succ^{\alpha} \in K$ such that, for all $a, a' \in A$, $a \succ^{\alpha} a'$, where $\alpha = \eta(\Lambda(a,a'))$. The interpretation of $a \succ^{\eta(\Lambda(a,a'))}$ a' is that alternative a is strictly preferred over a' with probability of, at least, $\eta(\Lambda(a,a'))$. Moreover, the interpretation of $a \succcurlyeq^{\eta(\Lambda(a,a'))} a'$ is that given a choice from the binary menu $\{a,a'\}$, a is chosen with probability $\eta(\Lambda(a,a'))$ and a' is chosen with probability $1 - \eta(\Lambda(a,a'))$.[10] For example, if η is the Borel-Lebesgue measure and $a, a' \in A$ such that $a \succcurlyeq^{\eta(\Lambda(a,a'))} a'$ and $\neg \left(a' \succcurlyeq^{\eta(\Lambda(a,a'))} a\right)$ then the alternative a is chosen, over a' with probability $\eta(\Lambda(a,a')) = \sup \Lambda(a,a') - \inf \Lambda(a,a')$.[11]

Let $\mathcal{I} \subset \mathcal{B}_{[1/2,1]}$ be the subset if intervals in $\mathcal{B}_{[1/2,1]}$. Clearly, $\Lambda(a,a') \in \mathcal{I}$. The *irresolute choice model* (ICM) is a triplet $(K, \mathcal{I}, \eta)$. Define an equivalence relation, $\approx$, on the set of ICMs as follows: $(K, \mathcal{I}, \eta) \approx (K, \mathcal{I}, \eta')$ if η agrees with η' on $\mathcal{I}$. Equivalent ICMs are said to belong to the same *equivalence class*.

Consistent with the interpretation of the probabilistic choice relations, $a \succcurlyeq^{1} a'$ and $\neg (a' \succcurlyeq^{1} a)$ imply that $\Lambda(a,a') = [1/2,1]$ and a is chosen from the set $\{a,a'\}$ with a probability $\eta([1/2,1]) = 1$. If $a \sim^{1} a'$ then, insofar as the probability of a chosen over a' is concerned, the model is silent. By definition, for all $a \in A$ and $\alpha \in [1/2,1]$, $a \succcurlyeq^{\alpha} a$ and $a \succcurlyeq^{\alpha} a$, hence $a \sim^{\alpha} a$.

[9] Robert (1971) studied the relations between nested semiorders and the family of binary relations $\{\succ^{\alpha} \mid \alpha \in [1/2,1)\}$ induced by P (i.e., $\forall a, a' \in A$, $a \succ^{\alpha} a'$ if and only if $P(a, \{a,a'\}) > \alpha$).

[10] Denote by $\Lambda^{c}(a,a')$ the complement of $\Lambda(a,a')$ in $[1/2,1]$ then $1 - \eta(\Lambda(a,a')) = \eta(\Lambda^{c}(a,a'))$.

[11] That the supremum and infimum exist follows from the fact that the set $\Lambda(a,a')$ is bounded and that $\neg \left(a' \sim^{\eta\left(\Lambda(a,a')\right)} a\right)$ implies that there is $\alpha' \in [0,1]$ such that $a \succ^{\alpha'} a'$. Hence, $\Lambda(a,a')$ is nonempty.

3 The Relationship between ICM and SCF

3.1 Two questions

The depictions of the input (i.e., $M \in \mathcal{A}$) - output (i.e., $a \in M$) patterns by ICM and SCF raises two questions about the relationship between them:

(a) If a decision maker's choice behavior is described by an ICM, do his choices from menus necessarily generate an SCF that satisfies regularity and stochastic transitivity?

(b) If a decision maker's choice behavior is depicted by an SCF that satisfies stochastic transitivity and regularity is there an ICM that generates his choices?

To answer these questions, I introduce the following additional definitions and notations. For each $M \in \mathcal{A}$ and $a \in M$ define $\Lambda(a, M) = \cap_{a' \in M}\{\alpha \in [1/2, 1] \mid a \succcurlyeq^{\alpha} a'\}$. In words, $\Lambda(a, M)$ is the set (interval) of indices designating the random choice relations that rank the alternative a (weakly) higher than any other alternative in the menu M. An alternative $a \in M$ is said to be *dominated* if $\Lambda(a, M)$ is a set of η-measure zero. Let $D(M)$ denote the subset of dominated alternatives in M and let $UD(M) = M \backslash D(M)$ denote the subset of *undominated* alternatives in M. Formally, $UD(M) = \{a \in M \mid \eta(\Lambda(a, M)) > 0\}$. Note that $UD(M)$ is not empty. For each $M \in \mathcal{A}$ and $a \in M$, I write $a \succcurlyeq^{\alpha} M$ if and only if $a \succcurlyeq^{\alpha} a'$, for all $a' \in M$. A mapping $C : \mathcal{A} \to \mathcal{A}$ is a *choice function induced by ICM* if $C(M) = UD(M)$, for all $M \in \mathcal{A}$.Let $UD(M) = \{a_1, ..., a_m\}$ then $\mathcal{J}(M) := \{\Lambda(a_1, M), ..., \Lambda(a_m, M)\}$ is a partition of the unit interval whose cells are elements of $\mathcal{B}_{[1/2,1]}$.

3.2 SCFs generated by ICM

Given an ICM, $(K, \mathcal{I}, \eta)$, define a stochastic choice function $P : A \times \mathcal{A} \to [0, 1]$ by

$$P(a, M) = \begin{bmatrix} \eta(\Lambda(a, M)) & \text{if } a \in UD(M) \\ 0 & \text{if } a \notin UD(M) \end{bmatrix}. \tag{1}$$

The SCF $P(a, M)$ so defined is said to be *generated* by the ICM $(K, \mathcal{I}, \eta)$. Note that, for binary menus, $M = \{a, a'\}$, by definition, $P(a', \{a, a'\}) = \eta(\Lambda(a', \{a, a'\}))$ and $P(a, \{a, a'\}) = \eta(\Lambda(a, \{a, a'\})) = 1 - \eta(\Lambda(a', \{a, a'\}))$.

The following theorem asserts that the answer to the first question posed in the preceding section is affirmative.

Theorem 1. *A stochastic choice function P on $A \times \mathcal{A}$ generated by an irresolute choice model, $(K, \mathcal{I}, \eta)$, satisfies regularity and stochastic transitivity.*

Proof. Given an ICM $(K, \mathcal{I}, \eta)$ let P on $A \times \mathcal{A}$ be the SCF generated by it (i.e., P is defined in (1)). Let $M \subset M'$ and denote by $\mathcal{J}(M)$ and $\mathcal{J}(M')$ the corresponding induced partitions of the unit interval.

If $UD(M') = UD(M)$ then $\mathcal{J}(M) = \mathcal{J}(M')$ and $P(a, M') = P(a, M)$, for all $a \in M$. If $UD(M') \neq UD(M)$ then either $a \in UD(M) \cap D(M')$, or $a \in UD(M) \cap UD(M')$. In the former case $P(a, M') = 0 \leq P(a, M)$ and, in the latter case, $a \not\succ^{\alpha} M'$ for all $\alpha \in \Lambda(a, M')$. But $\Lambda(a, M') \subseteq \Lambda(a, M)$. Hence

$$P(a, M) = \eta(\Lambda(a, M)) \geq \eta(\Lambda(a, M')) = P(a, M'). \tag{2}$$

Thus, P satisfies regularity.

Let $a, a', a'' \in A$ and consider the binary menus $\{a, a'\}$, $\{a', a''\}$, $\{a, a''\}$. Suppose that $a \succ^{\alpha} a'$ and $a' \succ^{\alpha} a''$ then by definition $\alpha \in \Lambda(a, \{a, a'\}) \cap \Lambda(a', \{a', a''\})$. But $\Lambda(a, \{a, a'\}) = [1/2, \sup \Lambda(a, \{a, a'\})]$ and $\Lambda(a', \{a', a''\}) = [1/2, \sup \Lambda(a', \{a', a''\})]$. Without loss of generality, assume that $\sup \Lambda(a, \{a, a'\}) \leq \Lambda(a', \{a', a''\})$, then $\Lambda(a, \{a, a'\}) \cap \Lambda(a', \{a', a''\}) = \Lambda(a, \{a, a'\})$. By transitivity of the relations in K, $\Lambda(a, \{a, a'\}) \subseteq \Lambda(a, \{a, a''\})$. Hence, $\eta(\Lambda(a, \{a, a''\})) \geq \eta(\Lambda(a, \{a, a'\})) = \min\{\eta(\Lambda(a, \{a, a'\}), \eta(\Lambda(a', \{a', a''\})))$. Suppose that $P(a, \{a, a'\}) > \lambda$ and $P(a', \{a', a''\}) > \lambda$ then, by definition, $\eta(\Lambda(a, \{a, a'\})) > \lambda$ and $\eta(\Lambda(a', \{a', a''\}))\} > \lambda$. By the argument above, $\eta(\Lambda(a, \{a, a''\})) \geq \min\{\eta(\Lambda(a, \{a, a'\})), \eta(\Lambda(a', \{a', a''\}))\} > \lambda$. Hence, by definition, $P(a, \{a, a''\}) > \lambda$. Thus, P is stochastically transitive. ∎

3.3 Rationalizable SCF

The next theorem asserts that the answer to the second question posed in the preceding section is affirmative, and that the generating ICM is unique up to equivalence class. An SCF P^* on $A \times \mathcal{A}$ is said to be *rationalized* by an ICM if $P^*(a, M) = P(a, M)$, for all $(a, M) \in A \times \mathcal{A}$, where P is generated by an ICM.

Theorem 2. *If $P^* : A \times \mathcal{A} \to [0, 1]$ is an SCF satisfying regularity and stochastic transitivity then there is a unique equivalence class of ICMs that rationalizes it.*

Proof. Let P^* on $A \times \mathcal{A}$ be an SCF satisfying stochastic transitivity and regularity. We need to show that there exists an ICM $(K, \mathcal{I}, \eta)$ such that all

$\succ^{\alpha} \in K$ are transitive, irreflexive and satisfy set-inclusion monotonicity, and that the SCF P the ICM generates satisfies $P(a, M) = P^*(a, M)$, for all $(a, M) \in A \times \mathcal{A}$.

For all $\alpha \in [1/2, 1)$ define binary relations $\succ^{\alpha}$ on A by $a \succ^{\alpha} a'$ if $P^*(a, \{a, a'\}) > \alpha$ and $a \succcurlyeq^1 a'$ if $P^*(a, \{a, a'\}) = 1$.[12]

If $P^*(a, \{a, a'\}) > \alpha$ then $P^*(a, \{a, a'\}) > \alpha'$, for all $\alpha' \leq \alpha$, Thus, by definition, $a \succ^{\alpha} a'$ implies that $a \succ^{\alpha'} a'$. Consequently, $\succ^{\alpha} \subseteq \succ^{\alpha'}$. If $P^*(a, \{a, a'\}) = 1$ then $P^*(a, \{a, a'\}) > \alpha'$, for all $\alpha' < 1$. Hence, $a \succcurlyeq^1 a'$ implies $a \succ^{\alpha'} a'$ and $\succ^1 \subseteq \succcurlyeq^1 \subseteq \succ^{\alpha'}$, for all $\alpha' \leq 1$. Thus, $\succ^{\alpha}$ satisfies set-inclusion monotonicity for all $\alpha \in [1/2, 1]$.

Let $a, a', a'' \in A$ and suppose that $a \succ^{\alpha} a'$ and $a' \succ^{\alpha} a''$. By definition $P^*(a, \{a, a'\}) > \alpha$ and $P^*(a', \{a', a''\}) > \alpha$. By stochastic transitivity $P^*(a, \{a, a''\}) > \alpha$. Hence, by definition, $a \succ^{\alpha} a''$. Thus, $\succ^{\alpha}$ is transitive.

By definition, for all binary menus, $\{a, a'\}$, $P^*(a, \{a, a'\}) + P^*(a', \{a, a'\}) = 1$. Hence, $P^*(a, \{a, a\}) = 1/2$ and, by definition, for all $\alpha \in [1/2, 1]$, $\neg (P^*(a, \{a, a\}) > \alpha)$. Hence, $\neg (a \succ^{\alpha} a)$. Thus, $\succ^{\alpha}$ is irreflexive.

Let $K = \{\succ^{\alpha} \mid \alpha \in [1/2, 1]\}$ be the set of binary relations defined above. By definition, for all $(a, M) \in A \times \mathcal{A}$,

$$\Lambda(a, M) = \cap_{a' \in M} \{\alpha \in [1/2, 1] \mid a \succcurlyeq^{\alpha} a'\} = \cap_{a' \in M} \Lambda(a, \{a, a'\}) \in \mathcal{I}.$$

Let η^* be a Borel probability measure on $\mathcal{B}_{[1/2,1]}$ agrees with P^* on $\{\Lambda(a, M) \mid (a, M) \in A \times \mathcal{A}\} \subseteq \mathcal{I}$ (i.e., $\eta^*(\Lambda(a, M)) = P^*(a, M)$, for all $(a, M) \in A \times \mathcal{A}$.

Denote by P be an SCF generated by the ICM $(K, \mathcal{I}, \eta^*)$. Then P is given by (1) and, by definition, $P(a, M) = P^*(a, M)$, for all $(a, M) \in A \times \mathcal{A}$. By Theorem 1, P is stochastically transitive and satisfies regularity. Moreover, because $\mathcal{A}$ contains all the binary menus $\{a, a'\}$, the SCF P^* fully characterizes the binary relations in the index set K of the rationalizing ICM. By (1), the same P is induced by all the ICMs that belong to the same equivalence class. ■

[12] The first part of this definition is the same as that of Roberts (1971).

4 Representations and the Canonical Signal Spaces

4.1 Representations

In Karni (2023), I showed that the ICM in conjunction with the existing models of decision making under certainty, under risk, and under uncertainty are represented by sets of utility functions (in the cases of decision making under certainty and under risk) and sets of utility-probability pairs (in the case of decision making under uncertainty). To grasp this point, consider the case of decision making under certainty.

Let the choice set A be a nonempty topological space. A nonempty set $\mathcal{U}$ of real-valued functions on A is said to *represent* a transitive and irreflexive binary relation $\rhd$ on A if, for all $a, a' \in A$, $a \rhd a'$ if and only if $u(a) > u(a')$, for all $u \in \mathcal{U}$. The following is a corollary of Theorem 1 in Karni (2023).

Corollary: *Let A be a locally compact separable metric space and $(K, \mathcal{I}, \eta)$ an ICM, where $\succ^\alpha \in K$ are continuous, then there exists a collection $\{\mathcal{U}^\alpha \mid \alpha \in [1/2, 1]\}$ of sets of real-valued, continuous, strictly $\succ^\alpha -$increasing, functions such that, for every $\alpha \in [1/2, 1]$, $\mathcal{U}^\alpha$ represents $\succ^\alpha$, and $\alpha \geq \alpha'$ if and only if $\mathcal{U}^\alpha \supseteq \mathcal{U}^{\alpha'}$.*

The uniqueness of the representation is as follows: Given any nonempty subset $\mathcal{U}^\alpha$ of $\mathbb{R}^A$, define the map $\Upsilon_{\mathcal{U}^\alpha} : A \to \mathbb{R}^{\mathcal{U}^\alpha}$ by $\Upsilon_{\mathcal{U}^\alpha}(a)(u) := u(a)$. Two nonempty subsets $\mathcal{U}^\alpha$ and $\mathcal{V}^\alpha$ of continuous real-valued functions on A represent the same preorder if, and only if, there exists an $f : \Upsilon_{\mathcal{U}^\alpha}(A) \to \Upsilon_{\mathcal{V}^\alpha}$ such that (i) $\Upsilon_{\mathcal{V}^\alpha} = f(\Upsilon_{\mathcal{U}^\alpha})$; and (ii) for every $b, c \in \Upsilon_{\mathcal{U}^\alpha}(A)$, $b > c$ if and only if $f(b) > f(c)$.[13]

4.2 Canonical signal spaces

The premise underlying the stochastic choice behavior depicted by ICM is that choices are governed by unspecified, random, signal–generating process. Consider the choice between two alternatives, say a and a', such that $\neg(a \sim$

[13] See Evren and Ok (2011). Note that, in general, for arbitrary multi-utility representations, $\mathcal{V}^\alpha$ and $\mathcal{V}^{\alpha'}$, of two preorders, $\succcurlyeq^\alpha$ and $\succcurlyeq^{\alpha'}$, such that $\succcurlyeq^\alpha \subset \succcurlyeq^{\alpha'}$ does not imply that $\mathcal{V}^\alpha \supset \mathcal{V}^{\alpha'}$. Given $\succcurlyeq^\alpha$ and facing a choice from a binary set $\{a, a'\}$, the probability that the decision maker chooses the alternative a is independent of the representation. In other words, if $\mathcal{U}^\alpha$ and $\mathcal{V}^\alpha$ are two representations of $\succcurlyeq^\alpha$, then the functions in $\mathcal{V}^\alpha$ are given by the uniqueness of the representation.

a'). Then $\eta(\Lambda(a,\{a,a'\}))$ may be interpreted as the probability of a signal that would resolve the indecision in favor of a. By Theorem 1 and the Corollary, this is the case if and only if $u(a) > u(a')$, for all $u \in \cup_{\alpha\in\Lambda(a,\{a,a'\})}\mathcal{U}^{\alpha}$. By set-inclusion monotonicity, this is equivalent to $u \in \mathcal{U}^{\sup\Lambda(a,\{a,a'\})}$.

Given SCF P^{*}, let $(K,\mathcal{I},\eta)$ be the ICM that rationalizes it. Let $F : 2^{\mathcal{U}}\backslash\varnothing \to [0,1]$ be a probability measure such that, for $\alpha \in [0,1]$, $F(\mathcal{U}^{\alpha}) = \eta([1/2,\alpha])$. Then $P^{*}(a,\{a,a'\}) = F\left(\mathcal{U}^{\sup\Lambda(a,\{a,a'\})}\right)$, for all $a,a' \in A$. In other words, facing a choice between two alternatives, a and a' that are not indifferent to one another, the decision maker behaves *as if* a function u is selected from $\mathcal{U}^{1}$ according to a probability measure F and a is chosen if $u \in \mathcal{U}^{\sup\Lambda(a,\{a,a'\})}$ and a' is chosen if $u \in \mathcal{U}^{1}\backslash\mathcal{U}^{\sup\Lambda(a,\{a,a'\})}$. Therefore, the set $\mathcal{U}^{1}$ may be taken to be the *canonical signal space*.

Corresponding to the partition $\mathcal{J}(M)$, define a partition of $\mathcal{U}^{1}$ as follows: For each $a_i \in M = \{a_1,...,a_m\}$ let

$$Q(a_i,M) := \{u \in \mathcal{U}^{1} \mid u(a_i) > u(a'), \forall a' \in M\backslash\{a_i\}\}. \tag{3}$$

Then, $\alpha \in \Lambda(a_i,M)$ if and only if $u \in Q(a_i,M)$. Since $\mathcal{U}^{1}$ is the canonical signal space, the probability of the signal $u \in Q(a_i,M)$ is $F(Q(a_i,M)) = \eta(\Lambda(a_i,M))$. By Theorem 2, given an SCF P^{*} rationalized by an ICM we have, every M and $a_i \in M$,

$$P^{*}(a_i,M) = F(Q(a_i,M,)),\ \forall a_i \in M. \tag{4}$$

Thus, the random choice behavior depicted by an SCF P^{*} may be interpreted as follows: When facing a choice from a menu M, the decision maker behaves *as if* a utility function $u \in \mathcal{U}^{1}$ is randomly selected from the measure F and $a_i \in M$ is chosen if $u \in Q(a_i,M)$.

5 Stochastic Demand and Portfolio Choice

5.1 Stochastic demand functions

The application of the ICM to the theory of market demand is based on the following idea. When a consumer faces a menu consisting of commodity bundles, a utility function is selected at random from the canonical signal space according to some implicit probability measure and the commodity bundle that maximizes this utility function is chosen. In this context the two questions of section 3.1 correspond to two issues concerning stochastic demand. First, what is the nature of the stochastic individual and market demands induced by irresolute choice behavior? Second, can the data induced by stochastic individual and market demands, be rationalized by irresolute choice behavior?

To model market demand, let $Z = \{1, ..., Z\}$ be the set of individuals in the market, and let $\mathbb{R}^n_+$ denote the set of alternatives representing commodity bundles. Menus are feasible budget sets, $B(p, Y_z) = \{x \in \mathbb{R}^n_+ \mid x \cdot p \leq Y_z\}$, where $p = (p_1, ..., p_n) \in \mathbb{R}^n_{++}$ denotes the commodity price vector and Y_z the income of individual z. Denote by $\bar{B}$ the set of budget sets. Assuming non-satiation, the undominated subset of $B(p, Y_z) \in \bar{B}$ is $UB(p, Y_z) = \{x \in \mathbb{R}^n_+ \mid x \cdot p = Y_z\}$.

To answer the first question, let $\mathcal{U}^1_z$ denotes the canonical signal space corresponding to an ICM depicting the behavior of individual z. Then, given a budget set $B(p, Y_z)$ the realization of the random demands $\widetilde{x}^z(p, Y_z)$ may be described as follows: For each $u \in \mathcal{U}^1_z$, let $x^*(p, Y_z, u)$ be the solution to the program

$$\max u(x) \text{ subject to } x \in B(p, Y_z),$$

and denote by $x^*_i(p, Y_z, u)$ its $i-th$ entry. Then the stochastic commodity demands are driven by the random selection of a function $u \in \mathcal{U}^1_z$. Let $\widetilde{u}$ be random utility function, then, $\widetilde{x}^z(p, Y_z) = x^*(p, Y_z, \widetilde{u})$ is the observed random demands.

For every $B(p, Y_z) \in \bar{B}$ and $x \in B(p, Y_z)$, let $U_z(x) = \{u \in \mathcal{U}^1_z \mid u(x) \geq u(x'), \forall x' \in B(p, Y_z)\}$. The revealed stochastic demand is an SCF $P : \mathbb{R}^n_+ \times \bar{B} \to [0, 1]$ given by

$$P(x, B(p, Y_z)) = F_z(U_z(x)). \tag{5}$$

The random demand for commodity i by individual z is $\widetilde{x}^k_i(p, Y_z) = x^*_i(p, Y_z, \widetilde{u})$, whose support is $[0, Y_z/p_i]$. Thus, given the budget set $B(p, Y_z)$ the probability that the individual z chooses x^k_i is $P(x^k, B(p, Y_k))$. Given an income profile $Y = (Y_1, ..., Y_Z)$ and a price vector p, the market *stochastic demand function for commodity* i is: $\widetilde{X}_i(p, Y) = \Sigma^Z_{z=1}\widetilde{x}^z_i(p, Y_z)$.

It is standard practice in economics to treat individual demands as independent variables.[14] The analogous assumption in the present context

[14] This assumption is reasonable when applied to commodities such as milk and gas; it is much less compelling when applied to other commodities.

maintains that individual demands are stochastically independent random variables.[15] If $\widetilde{x}_i^z(p, Y_z)$, $z \in Z$, are stochastically independent, then the distribution, μ, of the market demand for commodity i, $\widetilde{X}_i(p, Y)$, is given by the convolution $\mu_i^Z = P(x_i^1, B(p, Y_1)) * P(x_i^2, B(p, Y_2)) * ... * P(x_i^Z, B(p, Y_K))$. Expected demand is given by

$$E\left[\widetilde{X}_i(p, Y)\right] = \Sigma_{z=1}^Z \int_{U_z\left(x_i^*(p, Y_z, u)\right)} x_i^*(p, Y_z, u)\, dF_z(u) . \tag{6}$$

Its variance is

$$Var\left(\widetilde{X}_i(p, Y)\right) = \Sigma_{z=1}^Z \int_{U_k\left(x_i^*(p, Y_z, u)\right)} [x_i^*(p, Y_z, u) - E\left[\widetilde{X}_i(p, Y)\right]]^2 dF_z(u) . \tag{7}$$

Standard practice notwithstanding, in many markets individual demands are correlated, possibly because of implicit social effects such as conformism and status seeking. For instance, the demand for clothes is affected by fashion, the demand for vacation spots may be affected by the anticipated composition of the clientele, and demand for stocks may respond to information shared by many investors that respond to it in similar way. In these cases, the linearity of expectations implies that $E_k\left[\widetilde{X}_i(p, Y)\right] = \Sigma_{z=1}^Z E_z\left(\widetilde{x}_i^k(p, Y_z)\right)$. The variance of market demand, however, depends on the correlations among the individual demands and takes the form

$$Var\left(\widetilde{X}_i(p, Y)\right) = \Sigma_{z=1}^Z Var_z\left(\widetilde{x}_i^z(p, Y_z)\right) + 2\Sigma_{j<z} Cov_z\left(\widetilde{x}_i^j(p, Y_j), \widetilde{x}_i^k(p, Y_z)\right) . \tag{8}$$

In commodity markets in which individual demands are positively correlated, the individual stochastic choice behavior implied by the ICM induces greater demand fluctuation.

If the data summarizing individual demand behavior constitute SCFs satisfying regularity and stochastic transitivity, then by Theorem 2, it is ratioanlizable by ICMs.

5.2 Comparative statics

Consider next the consequences of income and price variations on market demands. Suppose that, ceteris paribus, the income of individual z

[15] A collection of random avriables is said to be independent if every finite subcollection is independent.

increases form Y_z to Y_z'. The supports of the random demands increase to $[0, Y_z'/p_i]$, $i = 1, ..., n$. For each $u \in \mathcal{U}$, the optimal bundle changes from $x^*(p, Y_z, u)$ to $x^*(p, Y_z', u)$, and the corresponding change in the demand for commodity i is from $x_i^*(p, Y_z, u)$ to $x_i^*(p, Y_z', u)$. For, each $u \in \mathcal{U}$, $x^*(p, Y_z, u) \in \arg\max_{x \in B(p, Y_z)} u(x)$ and $x^*(p, Y_z', u) \in \arg\max_{x \in B(p, Y_z')} u(x)$, (5) implies that

$$\Pr(x_i^*(p, Y_z', u)) = \Pr(x_i^*(p, Y_z, u)) = F_z(u).$$

The change in the demand distribution of commodity i depends on the income effects implies by the utility functions in the canonical signal space.

Similar considerations apply to relative price variations. Suppose that the price of commodity i increases from p_i to p_i'. Denote the new price vector by p'. Let $x^*(p', Y_z, u)$ denote the optimal bundle given the budget set $B(p', Y_z)$ corresponding to $u \in \mathcal{U}_k$ and let $x_i^*(p', Y_z, u)$ denote its i entry. Then by the same argument as above,

$$\Pr(x_i^*(p, Y_z, u)) = \Pr(x_i^*(p', Y_z, u)) = F_z(u).$$

The change in the market demand for commodity i is a random variable given by $\widetilde{X}_i(p', Y) - \widetilde{X}_i(p, Y) = \Sigma_{z=1}^{Z}\left[\widetilde{x}_i^k(p', Y_z) - \widetilde{x}_i^k(p, Y_z)\right]$.

Example: Consider the case in which the set of utility functions of individual k consists of Cobb-Douglas utility functions (i.e., $\mathcal{U}_z = \{x_1^{\beta_1} x_2^{\beta_2} ... x_n^{\beta_n} \mid \beta \in \left[\underline{\beta}_z, \bar{\beta}_z\right]^n, \underline{\beta}_z \geq 0, \Sigma_{i=1}^n \beta_i = 1\}$). Let $u_\beta := x_1^{\beta_1} x_2^{\beta_2} ... x_n^{\beta_n}$ and denote by g_z the joint probability distribution function on $\left[\underline{\beta}_z, \bar{\beta}_k\right]^n$. Then, $x_i^*(p_i, Y_z, u_\beta) = \beta_i Y_z / p_i : Y_z, i = 1, ..., n$. The stochastic demand for commodity i by individual z, $\widetilde{x}_i^k(p, Y_z)$ is depicted by g_z. Formally, let $\bar{g}_z(\beta_i)$ denote the marginal distribution of β_i then

$$\Pr\{\widetilde{x}_i^k(p, Y_z) = x_i^*(p_i, Y_z, u_\beta)\} = \bar{g}_z(\beta_i). \tag{9}$$

If the income of individual k increases from Y_z to Y_z', then the demand increases proportionally, (i.e.,for all $i = 1, ..., n$, $x_i^*(p_i, Y_z', u_\beta) = (Y_z'/Y_z) x_i^*(p_i, Y_z; u_\beta)$) and $\Pr((Y_z'/Y_z) x_i^*(p_i, Y_z, u_\beta)) = \Pr(x_i^*(p_i, Y_z, u_\beta)) = \bar{g}_z(\beta_i)$. Similarly, if the price of commodity i increase to p_i', then the demand decreases proportionally (i.e., $x_i^*(p_i', Y_z; u_\beta) = (p_i/p_i') x_i^*(p_i, Y_z; u_\beta)$) and $\Pr((p_i'/Y_z) x_i^*(p_i, Y_z; u_\beta)) = \Pr(x_i^*(p_i, Y_z; u_\beta)) = \bar{g}_z(\beta_i)$.

If the utility functions of all individuals are Cobb-Douglas functions, then their demands are independent random variables. Consequently, given an income profile I and price vector p, the distribution of the market demand $\widetilde{X}_i(p,I)$ is the convolution of the distributions $\bar{g}_z$, $z=1,...,Z$.

Let I' be another income profile. Then the change in the expected market demand is

$$\Sigma_{z=1}^{Z}\left[E_z\left(\widetilde{x}_i^z\left(p_i,Y_z'\right)\right)-E_z\left(\widetilde{x}_i^z\left(p_i,Y_z\right)\right)\right]=\Sigma_{z=1}^{Z}\left[\left(Y_z'/Y_z\right)-1\right]E_z\left(\widetilde{x}_i^z\left(p_i,Y_z\right)\right), \tag{10}$$

where $E_z\left(\widetilde{x}_i^z\left(p_i,Y_z\right)\right)=\int_{\underline{\beta}_z}^{\bar{\beta}_z}x_i^*\left(p_i,Y_z;u_\beta\right)\bar{g}_z\left(\beta_i\right)d\beta_i$. The variance of individual demands increases by a factor $\left(I_k'/I_k\right)^2$ (i.e., $Var\left(\widetilde{x}_i^z\left(p,Y_z'\right)\right)=\left(Y_z'/Y_z\right)^2 Var\left(\widetilde{x}_i^z\left(p,Y_z\right)\right)$).

5.3 Stochastic portfolio choice

Consider next the application of the ICM to the theories of portfolio choice and financial markets. Let $S=\{s_1,...,s_n\}$ be a finite state space, and denote by $\{e^1,...,e^n\}$ the corresponding set of Arrow securities.[16] The set of alternatives, $\mathbb{R}^n$, are portfolios of Arrow securities (i.e., portfolio is $y\in\mathbb{R}^n$, where y_i denotes the number of Arrow securities of type e^i in the portfolio). Denote by $\bar{y}=(1,...,1)$ the portfolio that consists of one Arrow security of each state. Then $\bar{y}$ is a unit of a risk-free asset. Let $q=(q_1,...,q_n)$ denote the vector of prices of the Arrow securities then the price of $\bar{y}$ is $\bar{q}=\Sigma_{i=1}^n q_i$.

Let $\bar{y}_z$ denote the initial endowment of risk-free asset of individual z whose value is $w_z=\bar{y}_z\cdot\bar{q}$. Then the budget set of individual z is $B(q,w_z)=\{y\in\mathbb{R}^n \mid y\cdot q^\tau=w_z\}$, where q^τ is the transposed of q.

Denote by Π_z a set of subjective probability distributions on S representing the possible beliefs of individual z about the likely realizations of the states, and let u_z be a real-valued function on $\mathbb{R}$, representing the individual's risk-attitudes. A preference relation, $\succ_z$, of individual z is said to exhibit *Knightian uncertainty* if, for all $y,y'\in\mathbb{R}^n$, $y\succ_z y'$ if and only if $\Sigma_{i=1}^n u_z(y_i)\pi(s_i)>\Sigma_{i=1}^n u_z(y_i')\pi(s_i)$, for all $\pi\in\Pi_z$.[17] Note that, in this instance, Π_z constitutes individual z's canonical signal space corresponding to the ICM. Let $\mathcal{V}_z:=\{\Sigma_{i=1}^n u_z(y_i)\pi(s_i)\mid\pi\in\Pi_z\}$ with generic element v_k.

Let G_k denote a probability measure on Π_z induced by a ICM. Define

$$\Pi_z(y)=\{\pi\in\Pi_z \mid u_z(y)\cdot\pi\geq u_z(y')\cdot\pi,\forall y'\in B(q,w_z)\}. \tag{11}$$

[16] An Arrow security e^i pays off \$1 in the state s_i and nothing otherwise.

[17] See Bewley (2002) and Galaabaatar and Karni (2013).

The the optimal portfolios of Arrow securities of individual k corresponding to $\pi \in \Pi_k$ is

$$\widetilde{y}^z (q, w_z) (\pi) = \arg \max_{y \in B(q,w_z)} u_z (y) \cdot \pi. \tag{12}$$

Then $\widetilde{y}^z (q, w_z)$ is a random variable whose distribution (and that of $\widetilde{y}_i^z (q, w_z)$) induced by G_z. Then the SCF generated by ICM that represent the random portfolio choices of individual k is:

$$P_z (y, B (q, w_z)) = G_z\{\Pi_z (y)\}. \tag{13}$$

The market demand for Arrow security e^i is the sum of individual demands, whose distribution is the convolution of the distributions of the individual demands.

6 Related Literature and Concluding Remarks

6.1 Related literature

Luce (1959) pioneered the study of random choice behavior. A primitive of Luce's model is a stochastic choice function summarizing the observed frequencies of choice of alternatives in the feasible sets in a variety of situations encountered in psychology and economics. Luce explored (sufficient) conditions on the choice probabilities that admit a numerical scale that represents individual stochastic choice behavior. In the notations of this paper, for a finite set of alternatives, say $A = \{a_1, ..., a_n\}$, Luce's proposed structure of the stochastic choice function is represented by (strictly positive) utility vector, unique up to positive scalar multiplication, such that $P (a, M) = u (a) / \Sigma_{a' \in M} u (a')$, for all nonempty $M \subseteq A$ and $a \in M$.

Luce's model requires that every alternative in every menu has strictly positive probability of being chosen and a constancy of probability ratios condition. Formally, $p (a; M) > 0$ for every $a \in M$ and, for every $a, a' \in A$, the ratio $P (a, M) / P (a', M)$ is constant over all menus $M \subseteq A$ that contain a and a'. Neither of these conditions seems natural, nor are they intuitively compelling.[18] Therefore it is worth underscoring that neither of these conditions is required by the ICM and the corresponding SCFs. According to the ICM $P (a; M) = 0$ for all $a \in D (M)$ and, while adding alternatives to

[18] Recent research including Ahumada and Ulku (2018), Echenique and Saito (2019), and Horan (2021) extend Luce's seminal work to address these weaknesses of the model.

a menu may decrease the probabilities of choosing existing alternatives, the decreases are not necessarily equiproportional.

The notion of random choice governed by random selection of utility functions was explored by Block and Marschak (1960). Specifically, Block and Marschak treat the (finite) set of utilities as primitive and postulate the existence of a random utility vector $U = (u(a_1), ..., u(a_n))$ (unique up to increasing monotone transformation) which induces random rankings of the elements of the alternatives such that, for all $a \in M$, $P(a, M)$ is equal to the probability of the set of rankings whose elements rank a above every other alternative in the menu M. They show that this condition requires that no two alternatives can be assigned the same rank. Formally, for all $a \neq a'$, $\Pr\{u(a) = u(a')\} = 0$. They also show that the existence of probability distribution on rankings consistent with the probabilities $P(a, M)$ implies regularity. Unlike Block and Marschak's model in which the utility functions are primitives, the utility functions that constitute the canonical signal space in the ICM are derived from the underlying set of probabilistic choice relations, and the regularity condition is derived from the set inclusion monotonicity condition. Moreover, the ICM admits infinite sets of utility functions and does not require that distinct alternatives are assigned the different utilities.

The problem of revealed stochastic preference deals with a similar question – namely – whether the distribution of observed choices from variety of feasible sets of alternatives is consistent with preference maximization. Applied to a population, the distributions of observed choices arise because of heterogeneity of tastes and/or beliefs. Applied to individuals, the distribution is a reflection of stochastic variables underlying individual preferences. McFadden and Richter (1971, 1990), Falmagne (1978), Fishburn (1978), Stoye (2019) addressed the question of consistency of the distribution of observed choices with optimizing behavior.[19] The ICM may be regarded as a contribution to the part of this literature that deals with individual stochastic choice behavior. However, unlike the literature on stochastic preference in which the set of utility functions is a primitive ingredient of the models, the primitive of the ICM is a set of incomplete probabilistic choice relations, that admit random utility representation. This difference is reflected in the axiomatic structures of the models. Recently, there has been a revival of interest in random utility models of choice and random choice behavior that

[19] McFadden (2005) synthesizes and extends the literature on stochastic preference. He also provides an extensive reference list to this literature.

reflects preference for deliberated randomization.[20]

Roberts (1971) studied the properties of homogeneous families of semiorders. His primitives consist a set A of alternatives and a binary probability function $P : A \times A \to [0,1]$ that satisfies $P(a,a') + P(a',a) = 1$, for all $a, a' \in A$. The choice probabilities, P, are assumed to satisfy Strong Stochastic Transitivity (in the notations of this paper, for all a, $a', a'' \in A$, $P(a,\{a,a'\}) \geq 1/2$ and $P(a',\{a',a''\}) \geq 1/2$ implies $P(a,\{a,a''\}) \geq \max\{P(a,\{a,a'\}), P(a',\{a',a''\})\}$).[21] Following Luce (1958, 1959) Roberts invokes the function P to define binary relations on A as follows $a \succ^{\lambda} a'$ if and only if $P(a,\{a,a'\}) > \lambda$.[22] Roberts (Theorem 4) shows that, for A finite, $\{\succ^{\lambda} \mid \lambda \in [1/2,1)\}$ is induced by binary choice probabilities if and only if it satisfies the following axioms: For all $a, a' \in A$ and $\lambda, \lambda' \in [1/2,1)$, (a) $a \succ^{\lambda} a'$ implies $\neg(a' \succ^{\lambda'} a)$ (b) Either $\succ^{\lambda} \subseteq \succ^{\lambda'}$ or $\succ^{\lambda'} \subseteq \succ^{\lambda}$. Roberts' main result is that each of the relations in $\{\succ^{\lambda} \mid \lambda \in [1/2,1)\}$ is a semiorder and the set itself must be homogeneous in the sense that a common weak order on A underlies (i.e., is compatible with) every semiorder. The main thrust of this paper is different from that of Roberts (1971) in several respects. First, Roberts' analysis is confined to stochastic choice form binary menus whereas the main focus of this paper is stochastic choice from feasible menus. Second, the axiomatic structure of this paper is different from that of Roberts and are not required to satisfy Strong Stochastic Transitivity. Consequently, the binary relations of ICM are not semiorders and are not homogeneous. Third, given the axiomatic structure, Roberts' main result is identifying a set of conditions that are equivalent to Strong Stochastic Transitivity, whereas the main objective of this paper is the study of the axiomatic foundations that rationalizes the revealed probabilistic choice behavior depicted by stochastic choice functions.

At the individual level, random choice behavior may reflect the decision maker's indifference among feasible alternatives or his inability to compare them because of their complexity or the lack of familiarity with their consequences. Ok and Tserenjigmid (2020) model these aspects of random choice behaviors by stochastic choice functions. They characterize stochastic choice

[20] Various aspects of random utility models of choice behavior have been studies by Gul, Natenzon, and Pesendorfer (2014), Fudenberg, Iijima, and Strzalecki (2015), and Frick, Iijima, and Strzalecki (2019). Danan (2010), Agranov and Ortoleva (2017), and Cettolin and Riedl (2019) examined random choice based on deliberate randomization.

[21] See also Block and Marschak (1960).

[22] I use the notations of this paper.

functions that assign positive probabilities solely to alternatives that constitute maximal elements of the feasible sets. They do not study the probability distributions on the sets of maximal elements.

Karni and Safra (2016) axiomatized the representation of decision makers' perceptions of the stochastic process underlying the selection of their state of mind which, in turn, govern their choice behavior. This work may be regarded as providing axiomatic foundations of a probability measure on the canonical signal space based on the decision makers' introspections.

Becker (1962) argued that some economic theorems, such as the law of demand, do not depend on agents in the market behaving rationally. He showed that even if consumers choose their consumption bundles without attempting to optimize of some objective function, the change in the budget set caused by the relative price changes will force them to respond in a way that, in the aggregate, produces a downward-slopping demand functions. In Becker's analysis, households' choices may be irrational but not stochastic. Consequently, unlike in the market demand theory implied by the ICM, market demand is non-stochastic.

Cerreia-Vioglio et al. (2022), adopt a purely behavioral approach to the study the law of demand in the presence of stochastic choice. Accordingly, in their analysis the primitive are stochastic choice functions depicting consumer's choices to which they attribute no preferential (mental) interpretation. They showed that, if these functions are consistent in the sense of Luce's (1959) choice axiom, and if strictly dominated alternatives are eliminated, then the law of demand for normal goods holds on average. In their concluding remarks Cerreia-Vioglio et. al mention the possibility of taking a complementary preferential approach to stochastic consumer theory. In this paper, I link the stochastic choice functions to random preferential relations and showed, in Theorem 2, that any SCF may be generated by a stochastic preference that was hinted to by Cerreia-Vioglio et al. It is also worth mentioning that, whereas Cerreia-Vioglio et al. (2022) analysis is focused on the comparative statics properties average demand, this paper analysis of the implications of random choice behavior for market demand is concerned with both the implied the average and variance of the demands.

6.2 Consistency with violations of the weak axiom of revealed preference

According to the revealed preference approach, stochastic choice functions are empirical manifestations of random choice behavior that may be governed by decision makers' indifference among feasible alternatives, their inability to compare and rank the alternatives, variations in their moods, and/or changing needs. Whatever the underlaying motivations, the reasons for the observed stochastic choice may not be accessible to an outsider; if they are driven by subconscious impulses they may not be accessible even to the decision maker himself. It is necessary in such cases to build theories that make sense of observations that consist of feasible alternatives and actual choices. The irresolute choice model is a way of making sense of observed random choices in repeated decision situations involving the same feasible set of alternatives summarized by stochastic choice functions.

The model is also consistent with some violations of the weak axiom of revealed preference (WARP).[23] To see why, let $x^0 \in \mathbb{R}^n_+$ denote the initial endowment of a decision makers commodities. Let p and p' be two price vectors and consider the budget sets $B(p, x^0 \cdot p)$ and $B(p', x^0 \cdot p')$. The corresponding undominated sets are

$$UD\left(B\left(p, x^0 \cdot p\right)\right) = \{x \in B\left(p, x^0 \cdot p\right) \mid \exists u \in \mathcal{U} \text{ s.t. } u(x) \geq u(x'), \forall x' \in B\left(p, x^0 \cdot p\right)\}$$

and

$$UD\left(B\left(p', x^0 \cdot p'\right)\right) = \{x \in B\left(p', x^0 \cdot p'\right) \mid \exists u \in \mathcal{U} \text{ s.t. } u(x) \geq u(x'), \forall x' \in B\left(p', x^0 \cdot p'\right)\}.$$

If $x^0 \in UD(B(p, x^0 \cdot p)) \cap UD(B(p', x^0 \cdot p'))$ then $UD(B(p, x^0 \cdot p)) \cap intB(p', x^0 \cdot p')$ and $UD(B(p', x^0 \cdot p')) \cap intB(p, x^0 \cdot p)$ are nonempty.[24] Let $x^* \in UD(B(p, x^0 \cdot p)) \cap intB(p', x^0 \cdot p')$ and $x^{**} \in UD(B(p', x^0 \cdot p')) \cap intB(p, x^0 \cdot p)$ then the choices x^* from $B(p', x^0 \cdot p')$ and x^{**} from $B(p, x^0 \cdot p)$ constitute a violation of WARP.[25] Such choice is consistent with irresolute choice behavior.

[23] I am grateful to Yujian Chen for calling my attention to this point.

[24] $intB\left(p, x^0 \cdot p\right)$ and $intB\left(p', x^0 \cdot p'\right)$ are the interiors of the corresponding budget sets in the $\mathbb{R}^n$ topology.

[25] In the case of complete preferences, $\mathcal{U}$ is a singleton set.

References

[1] Ahumada, Alonso and Ulku, Levent (2018) "Luce Rule with Limited Consideration," *Mathematical Social Sciences*, 93, 52-56.

[2] Agranov,Marina and Ortoleva, Pietro (2017) "Stochastic Choice and Preferences for Randomization," *The Journal of Political Economy*, 125, 40-68.

[3] Becker, Gary S. (1962) "Irretional Behavior and Economic Theory," *The Journal of Political Economy,* LXX, 1-13.

[4] Bewley, Truman F. (2002) "Knightian Decision Theory: Part I," *Decision in Economics and Finance,* 25, 79-110.

[5] Block, H. D. and Marchak Jacob, (1960) Random Ordering and Stochastic Theories of Responses," in I. Olkin, S. Ghurye, W. Hoeffding, W. Madow and H. Mann (Eds.) *Contributions to Probability and Statistics. Stanford: Stanford University Press.* 97-122.

[6] Cerreia-Vioglio, Simone, Maccheroni Fabio, Marinacci, Massimo, Rustichini, Aldo (2022) "Law of Demand and Stochastic Choice." *Theory and Decision,* 92(3), 513-529.

[7] Cettolin, Elena and Riedl, Arno (2019) "Revealed Preferences under Uncertainty: Incomplete Preferences and Preferences for Randomization," *Journal of Economic Theory,* 181, 547-585.

[8] Danan, Eric (2010) "Randomization vs. Selection: How to Choose in the Absence of Preference?" *Management Science* 56, 503–518.

[9] Echenique, Federico and Saito, Kota (2019) "General Luce Model," *Economic Theory*, 68, 811-826.

[10] Eliaz, Kfir and Ok, Efe, A. (2006) "Indifference or indecisiveness? Choice-theoretic Foundations of Incomplete Preferences," *Games and Economic Behavior*, 56, 61–86.

[11] Evren, Özgür and Ok, Efe, A. (2011) "On the Multi-Utility Representation of Preference Relations," *Journal of Mathematical Economics*, 47, 554-563.

[12] Falmagne, J. (1978) "A Representation Theorem for Finite Random Scale Systems," *Journal of Mathematical Psychology* 18, 52-72.

[13] Fishburn, Peter (1973) "Binary Choice Probabilties: On the varieties of Stochastic Transitivity," *Jounal of Mathematical Psychology* 10, 327-352

[14] Fishburn, Peter (1978) "Choice probabilities and choice functions." *Jounal of Mathematical Psychology* 18, 205-219.

[15] Frick, Mira, Iijima, Ryota, and Strzalecki, Tomasz (2019) "Dynamic Random Utility," *Econometrica*, 87, 1941–2002.

[16] Fudenberg, Drew, Iijima, Ryota, and Strzalecki, Tomasz (2015) "Stochastic Choice and Revealed Perturbed Utility," *Econometrica*, 83, 2371–2409.

[17] Galaabaatar, Tsogbadral and Karni Edi (2013) "Subjective Expected Utility with Incomplete Preferences," *Econometrica,* 81, 255–284.

[18] Gul, Faruk, Natenzon, Paulo and Pesendorfer, Wolfgang (2014) "Random Choice as Behavioral Optimization," *Econometrica*, 82, 1873–1912.

[19] Hallff, Henry M (1976) "Choice Theories for Differentially Comparable Alternatives," *Journal of Mathematical Psychology* 14, 244-246.

[20] He, Junnan and Natenzon Paulo (2022) "Moderate Utility," Unpublished manuscript.

[21] Horan, Sean (2021) "Stochastic Semi-Orders," *Journal of Economic Theory*, 192, Article 105171.

[22] Karni, Edi (2023) "Irresolute Choice Behavior," *International Journal of Economic Theory* (2023).

[23] Karni, Edi and Safra, Zvi (2016) "A Theory of Stochastic Choice under Uncertainty," *Journal of Mathematical Economics* 63, 164-173.

[24] Luce, Duncan, R. (1958) "A Probabilistic Theroy of Utility," *Econometrica* 26, 193-224.

[25] Luce, Duncan, R. (1959) *Individual Choice Behavior: A Theoretical Analysis.* New York: Wiley.

[26] Mas-Colell, Andreu, Whinston, Michael D. and Green, Jerry, R. (1995) *Microeconokic Theory.* Oxford University Press, Oxford.

[27] McFadden, Daniel L., Richter, Marcel, K. (1991) "Stochastic Rationality and Revealed Stochastic Preference." Boulder: Westview Press In Chipman, J., McFadden, D., Richter, K. (eds.) *Preferences, uncertainty, and rationality*, 161-186.

[28] McFadden, Daniel L., (2005) "Revealed Stochastic Preference; A Synthesis." *Economic Theory*, 26, 245-264.

[29] Ok, Efe A. and Tserenjigmid, Gerelt (2020) "Indifference, Indecisiveness, Experimentation and Stochastic Choice," *Theoretical Economics* 17, 651-686.

[30] Roberts, Fred S. (1971) "Homogeneous Families of Semiorders and the Theory of Probabilistic Consistency," *Journal of Mathematical Psychology* 8, 248-263.

[31] Stoye, Jörg (2019) "Revealed Stochastic Preference: A One-Paragraph Proof and Generalization." *Economic Letters,* 177, 66-68.