David Schmeidler's Contributions to Decision Theory

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1 Early work

David Schmeidler's interest in decision theory began early in his career. In his first publication on the subject, a one page article in *Econometrica* in 1971, he considered a transitive binary relation $\succsim$ on a connected topological space X, with nontrivial asymmetric part $\succ$. He proved that if for each x in X the upper and lower contour sets $\{y : y \succ x\}$ and $\{y : x \succ y\}$ are open and $\{y : y \succsim x\}$ and $\{y : x \succsim y\}$ are closed, then the relation $\succsim$ is complete.

Interpreting $\succsim$ as a (weak) preference relation, this remarkable observation is puzzling in that a technical, behaviorally unfalsifiable condition, continuity, has behaviorally falsifiable implications. Based on David's proof, analogous results were derived in the context of decision making under risk and under uncertainty. It also lead to a new understanding of the structure of incomplete preference relations. Specifically, let $\succ$ be a transitive and irreflexive binary relation on X and define $\succsim^*$ by $x \succsim^* y$ if $z \succsim^* x$ implies $z \succsim^* y$. Then $\succ$ may be incomplete even if, for all x in X, the upper and lower contour sets of x defined by $\succ$ are open and those defined by $\succsim^*$ are closed.

2 Decision making under uncertainty

In the early 1980s, David's research took a decisive turn toward the theory of decision making under uncertainty. Much of this research in the years that followed was motivated by the need to address unsatisfactory aspects of the existing theories, which built on the seminal work of Leonard Savage.

2.1 A first big bang

Decision theory under uncertainty had a spectacular start in the early 1950s with the work of Savage. He built upon Bruno de Finetti's theory of subjective probability, developed in the 1930s, and John von Neumann and Oskar Morgenstern's axiomatization of expected utility under risk in the 1947 second edition of their epoch-making game theory book. In an impressive tour de force, both conceptual and mathematical, Savage was able to integrate these two approaches in his classic 1954 book by axiomatizing the subjective expected utility (SEU) criterion when neither probabilities nor utilities are given.

Savage's parsimonious setting considers a space S of *states of the world*, whose subsets are called *events*, a space X of *consequences* and a collection $\mathcal{F}$ of maps $f : S \to X$ from states to consequences, called *acts*. Each act represents a course of action, succinctly described as a map that associates a consequence to each state. In this description, which presupposes some form of consequentialism, states are viewed as exclusive and exhaustive descriptions of all payoff-relevant contingencies that are unknown to the decision maker and outside his control.

The decision maker ranks acts via a binary *preference* relation $\succsim$ defined on $\mathcal{F}$. A numerical criterion $V : \mathcal{F} \to \mathbb{R}$ represents preference $\succsim$ when

$$f \succsim g \Longleftrightarrow V(f) \geq V(g)$$

The SEU criterion axiomatized by Savage is based on a *utility function* $u : X \to \mathbb{R}$ that ranks consequences according to the decision maker's tastes and on a *subjective probability measure* $P : 2^S \to [0,1]$ that quantifies the decision maker's degrees of belief over the likelihood of the different events. These two ingredients, meant to capture separately tastes and beliefs, are combined in the SEU criterion V via the integral

$$V(f) = \int_S u(f(s))\, dP(s)$$

Savage's axiomatization imposes a few axioms on the preference $\succsim$. The most famous among them is the so-called *sure-thing principle*, an independence axiom at the heart of Savage's derivation. Given an event E and a pair of acts f and g, the act

$$f_E g = \begin{cases} f(s) & \text{if } s \in E \\ g(s) & \text{else} \end{cases}$$

is equal to act f on event E and to act g otherwise.

SURE-THING PRINCIPLE Given any acts f, g, h, h' and any event E,

$$f_E h \succsim g_E h \Longleftrightarrow f_E h' \succsim g_E h'$$

The two comparisons in this axiom involve pairs of acts that only differ in their common parts h and h', as diagrammed next:

E	E^c
f	h
g	h

;

E	E^c
f	h'
g	h'

The sure-thing principle requires that, if act $f_E h$ is preferred to act $g_E h$, no reversal in preference can occur if their common part h is replaced by a different, but still common, part h'. In other words, ranking of acts should be independent of common parts.

Another aspect of Savage's approach is the requirement that the preferences be state independent, captured by what may be described as the comparative probability axiom. As first argued by Frank Ramsey and Bruno de Finetti, the natural way to elicit subjective beliefs is through bets on them. Given any two consequences $x \succ x'$, the binary act

$$x_E x' = \begin{cases} x & \text{if } s \in E \\ x' & \text{else} \end{cases}$$

that pays off the preferred consequence when event E is interpreted as a bet on E. Using bets, the preference $\succsim$ then induces a likelihood relation $\succsim^*$ on events as follows:

$$E \succsim^* F \Longleftrightarrow x_E x' \succsim x_F x'$$

In words, the decision makers regards event E as more likely than event F when he prefers to bet on E than on F. The next axiom ensures that the likelihood relation is well defined in that independent of the posited consequences x and x'.

COMPARATIVE PROBABILITIES For all pairs of disjoint events, E and F, outcomes $x \succ x'$ and $y \succ y'$ and acts f and g in $\mathcal{F}$,

$$x_E x' \succsim x_F x' \Longrightarrow y_E y' \succsim y_F y'$$

Implicit in comparative probabilities is the assumption that the ordinal preference-ranking (valuations) of the outcomes is state-independent. If a sure outcome is strictly preferred over another, then it must also be preferred conditional on any event. To grasp why this condition implies state-independence, suppose that $x = \$100$, $x' = \$10$, y is a fur coat and y' is a short-sleeve shirt. It is reasonable to assume that the first part of the implication above holds and that y is strictly preferred over y' if one lives in a place where the temperature is usually below freezing. The rank-order is reversed in the event that the temperature soars. This reversal is not allowed by comparative probabilities.

Savage's derivation did not make any structural assumption on the consequence space X, but, for technical reasons inherent to the numerical representation of subjective probabilities, had to impose a divisibility requirement on the state space that, for example, required it to be infinite.

In 1963 Frank Anscombe and Robert Aumann proposed an alternative formulation by adding a lottery structure to the consequence space and, with this, were able to axiomatize a SEU criterion with a finite state space. Fishburn (1970) streamlined this lottery setting and developed a SEU axiomatization for acts $f : S \to \Delta X$, where ΔX is a space of lotteries defined on an underlying prize space, X, and S is any state space.

Analogously to the Savage's sure thing principle and comparative probability the preference relation postulated by Anscombe and Aumann included the independence axiom of expected utility theory and is state independence, or monotonicity, axiom that requires the ranking of lotteries to be independent of the underlaying states.

2.2 State-dependent preferences and the representation of beliefs by probabilities

The requirement that the preference relations display state-independence renders it inapplicable to the analysis of decisions such as the choice of health, life or disability insurance, in which the states and consequences are confounded, a point that was forcefully underscored by Aumann in exchange of letters with Savage (published in Dreze, 1971, and in Aumann's collected papers). In his letter, Aumann describes a man who loves his wife very much and without whom his life is "less 'worth living.'" The wife falls ill; if she is to survive, she must undergo a routine yet dangerous operation. The husband is offered a choice between betting \$100 on his wife's survival or on the outcome of a coin flip. Even if the husband believes that his wife has an even chance of surviving the operation, he may still rather bet on her survival, because winning \$100 if she does not survive is "somehow worthless." If he bets on the outcome of a coin flip he might win but not be able to enjoy his winnings because his wife dies. In this situation, Aumann argues, Savage's notion of states (that is, whether, following the surgery, the wife is dead or alive) and consequences are confounded to the point that there is nothing that one may call a consequence (that is, something whose value is state independent).

Departing from the confines of the revealed preference methodology, Edi Karni and David proposed a model of subjective expected utility theory that admits state-dependent preferences. In this model decision makers are characterized by a pair of preference relations, $\succsim$ on the set $\Delta(X)^S$ of Anscombe-Aumann acts, and $\succsim^*$ on the set

$$L = \{\ell(x,s) \in \mathbb{R}_{++}^{X \times S} : \Sigma_{(s,x) \in S \times X} \ell(x,s) = 1\}$$

of hypothetical prize-state lotteries. Elements of L induce acts as follows: Let $H : L \rightarrow \Delta(X)^S$ be a function defined by $H(\ell(x,s)) = \ell(x,s) / \Sigma_{y \in X} \ell(x,s)$ for all $(x,s) \in X \times S$. The preference relations $\succsim$ on $\Delta(X)^S$ and $\succsim^*$ on L are consistent if they rank lotteries on nonnull states identically.[1]

CONSISTENCY For all s in S and ℓ, ℓ' in L, such that ℓ equals ℓ' outside s,

$$H(\ell) \succ H(\ell') \Longrightarrow \ell \succ^* \ell'$$

Let p be a probability distribution on S and L_p the subset of prize-state lotteries whose marginal distribution on S is p. Karni, Schmeidler and Vind (1983) showed that $\succsim$ on $\Delta(X)^S$ and $\succsim^*$ on L_p are consistent, Archimedean, weak orders satisfying the independence axiom of expected utility theory if and only if there is a utility function u on $X \times S$ and a probability distribution π on S such that $\succsim$ is represented by

$$f \mapsto \Sigma_{s \in S} \pi(s) \Sigma_{x \in X} u(x,s) f(x,s)$$

and $\succsim^*$ by

$$\ell \mapsto \Sigma_{(x,s) \in X \times S} u(x,s) \ell(x,s)$$

Moreover, u is unique up to positive linear transformation and the probability π conditional on the set of $\succsim^*$- nonnull states is unique. This model admits state-dependent preferences.

Savage sought to provide behavioral foundation of the Bayesian prior by inferring from choice behavior unique subjective probabilities that represent the decision maker's beliefs about the likely realization of events. However, in both Savage's and Anscombe and Aumann's models the utility and probability are jointly unique. In other words, the uniqueness of the subjective probabilities depends on the choice of the utility function. In particular, both models assumed that the utility function is state-independent. This assumption is not implied by the axiomatic structures of these models which renders the subjective probabilities that figure in the representation depend on the (arbitrary) choice of the utility function and may not represent the decision maker's beliefs. To grasp this point suffices it to observe that, if $\int_S u(f(s)) d\pi(s)$ represents the a preference relation, so does $\int_S \hat{u}(f(s),s) d\hat{\pi}(s)$, where $\hat{\pi}(s) = \pi(s)\gamma(s) / \int_S \gamma(s) d\pi(s)$ and $\hat{u}(f(s),s) = u(f(s)) / \gamma(s)$ for all $s \in S$ and $\gamma : S \rightarrow (0,\infty)$. This understanding undermines the foundations of Savage's program.

The state-dependent expected utility model of Karni, Schmeidler and Vind (1983), suffers from the same problem because the subjective probabilities in that model depend on the arbitrary choice of the marginal probabilities, p, that define the subset L_p of L. In retrospect, it was realized that the original paper of Karni and Schmeidler (published in 2016) in which the domain of the preference relation $\succsim^*$ is the entire set L implies the existence of a unique subjective probability that represents the decision maker's beliefs and state-dependent utility function, unique up to positive linear transformation, representing the decision maker's tastes.

2.3 Probabilistically sophisticated choice

Subjective expected utility theory amalgamates two distinct and unrelated ideas, the representation of beliefs by probabilities, as revealed by choice behavior, and the representation of the preference relations that govern this behavior by a linear (in the probabilities) functional. Motivated by experimental evidence and theoretical considerations that challenged the expected utility representation, Machina and Schmeidler (1992, 1995) proposed models depicting probabilistically sophisticated choice behavior that severs the link between the representation of beliefs by probabilities and the linearity

[1] A state (event) is null if the preference relation is non-trivial and the decision maker is indifferent between any acts that agree outside that state (event).

of the representation in these probabilities. According to these models, decision makers entertain beliefs on the likely realization of events, represented by subjective probabilities, which they invoke to transform acts into lotteries on outcomes and evaluate these lotteries by a representation functional that is not necessarily linear in the probabilities. Formally, let $\mathcal{F}$ be the set of Savage's acts and π a probability measure on S. Define a function $\Upsilon : \mathcal{F} \rightarrow \Delta(X)$ by

$$\Upsilon(f) = \left(x_1, \pi(f^{-1}(x_1)), ..., x_n, \pi\left(f^{-1}(x_n)\right)\right)$$

for all $f \in \mathcal{F}$. Then, f is represented by $f \mapsto V(\Upsilon(f))$, where V is a mixture-continuous,[2] real-valued function on $\Delta(X)$ that is monotone increasing with respect to first-order stochastic dominance.[3]

To obtain this representation with a unique probability measure on S, Machina and Schmeidler (1992) departed from Savage's sure-thing principle and strengthened the comparative probability axiom.

STRONG COMPARATIVE PROBABILITIES For all pairs of disjoint events, A and B, outcomes $x^* \succ x$ and $y^* \succ y$ and acts f and g in $\mathcal{F}$,

$$x^*AxBf \succsim xAx^*Bf \Longrightarrow y^*AyBg \succsim y^*AyBg$$

where x^*AxBf is the act that pays off x^* if $s \in A$, x if $s \in B$, and $f(s)$ otherwise.

The intuition behind this axiom is that if a decision maker choices are guided by subjective beliefs represented by probabilities, then the first expression of preference reveals her belief that, conditional on the event $A \cup B$ and given the payoffs of f on the complementary event, the event A is at least as likely to obtain as B. The axiom asserts that this revelation is independent of the payoffs in the events A and B, provided that they are ranked in the same way by the preference relation, and the payoffs on the complementary event. Together with other postulates of Savage (1954) the independence of subacts that agree on the complementary event is invoked to obtain unconditional, finitely-additive, nonatomic, probability measure on the entire state space, and probabilistically sophisticated representation of choice behavior.

Machina and Schmeidler (1995) reiterated the same idea in the Anscombe-Aumann setting. There they replaced the independence axiom with monotonicity of the preference relation with respect to first-order stochastic dominance and a replacement axiom that captures the intuition of the strong comparative probability.

Probabilistically sophistication theory is consistent with the pattern of choices that violate expected utility theory depicted by the experiments of Allais (1953) and others. However, because the probability measures that figure in the probabilistically sophisticated representations are additive, the model is inconsistent with ambiguity aversion.

2.4 Normative doubts

The sure-thing principle is a beautiful axiom. It has a transparent, compelling, interpretation and in Savage's hands turned out to have a considerable traction in his derivation. Its intuitive appeal greatly contributed to making the SEU criterion the standard criterion in rational decision making under uncertainty. It ruled unchallenged for decades, essentially up to the early 1980s. The only significant SEU advance in the first 30 years after Savage's opus was the discovery by the late Peter Fishburn

[2] A real-valued function V on $\Delta(X)$ is mixture continuous if $V(\alpha p + (1-\alpha) q)$ is continuous in α for all $p, q \in \Delta(X)$.

[3] Because X is arbitrary and is not linearly ordered, the definition of first-order stochastic dominance is as follows: For all $p, q \in \Delta(X)$, p first-order stochastically dominates q if $\sum_{\{i \mid x_i \succcurlyeq x\}} p_i \geq \sum_{\{i \mid x_i \succcurlyeq x\}} q_i$ for all $x \in X$.

that Savage's axiomatization forced the utility function to be bounded, a nontrivial restriction in some applications.

Yet, in the seminal de Finetti's quantification of subjective probability, so heartily endorsed and adopted by Savage, a problem lingers. This probability quantifies a degree of belief based on some information that the decision maker has. Intuitively, the quality of this underlying information should affect the decision maker confidence in the subjective probabilities that quantify his degree of belief. Savage himself was aware that the information quality may affect the decision makers' confidence in their own probability assessments. As he wrote in his book "... there seem to be probability relations about which we feel relatively 'sure' as compared with others". This has nothing to do with bounded rationality: a perfectly rational decision maker might well share Savage's feeling.

David first thought of this issue by considering the toss of a coin. Heads and tails are judged to be equally likely when dealing with a well tested coin that has been flipped a number of times with approximately equal instances of heads and tails. When dealing with an untested coin, however, a decision maker might well again judge – out of symmetry – heads and tails to be equally likely. In both cases, the decision maker's judgements are quantified by a subjective probability with value 1/2. Yet, the evidence behind such judgements, so the confidence in them, is dramatically different. It is then natural to expect that a decision maker would be much more confident in his 1/2 probability for the tested coin than for the untested one.

This "confidence" problem is bypassed in the Savage derivation that, by building on the von Neumann-Morgenstern derivation, at a key junction presupposes that the decision maker just behaves as if his subjective probability were correct or, in purely subjective terms, as if he had a full confidence in them, with no *ambiguity*. In so doing, Savage's derivation reduces uncertainty to risk and thus gets rid, by *fiat*, of any ambiguity concern. To see how this reductionism affects the SEU criterion, let us go back to coins. In light of the previous discussion, it is natural to expect that a decision maker may well prefer to bet on a well-tested coin. This is how his ambiguity concerns should translate into choice behavior. The SEU criterion is unable to account for this natural choice. To see why, call I the tested coin and II the untested one. The state space is

$$S = \{H, T\} \times \{H, T\} = \{HH, HT, TH, TT\}$$

Acts 1_I and 1_{II} are, respectively, bets of 1 euro on coin I and on coin II. The next table summarizes the decision problem:

	HH	HT	TH	TT
1_I	1	1	0	0
1_{II}	1	0	1	0

The subjective probability P that quantifies the beliefs previously discussed is such that

$$P(HH \cup HT) = P(HH \cup TH) = \frac{1}{2}$$

By setting $u(1) = 1$ and $u(0) = 0$, a standard normalization, under this P for a SEU decision maker we have $u(1_I) = u(1_{II}) = 1/2$ and so

$$1_I \sim 1_{II}$$

a counter-intuitive pattern.

In a similar spirit, with urns rather than coins, already in 1961 Daniel Ellsberg had come up with some ingenious thought experiments that have cast doubt on the normative appeal of the sure-thing principle when the information upon which subjective probabilities beliefs rely is made explicit and its quality is varied. Specifically, following Ellsberg (1961) let us consider a single urn, with 90 balls. The decision maker is told that:

(i) in the urn balls are either red, yellow or green;

(ii) there are 30 red balls.

No information is given on the proportion of yellow and green balls in the 60 balls that are not red. The decision maker has to choose among the following 1 euro bets on the colors of a ball drawn from the urn:

1. bet 1_R: it pays 1 euro if the ball drawn from the urn is red;
2. bet 1_Y: it pays 1 euro if the ball drawn from the urn is yellow;
3. bet $1_{R\cup G}$: it pays 1 euro if the ball drawn from the urn is either red or green;
4. bet $1_{Y\cup G}$: it pays 1 euro if the ball drawn from the urn is either yellow or green.

If we model the decision maker's choice among these bets in a Savage framework, the state space is $S = \{R, Y, G\}$, with the following three possible states

1. R: a red ball is drawn;
2. Y: a yellow ball is drawn;
3. G: a green ball is drawn.

The next table summarizes the decision maker's decision problem:

	R	Y	G
1_R	1	0	0
1_Y	0	1	0
$1_{R\cup G}$	1	0	1
$1_{Y\cup G}$	0	1	1

The decision maker has a much better information on the event R and its complement $Y \cup G$ than on the events Y and G and their complements. As a result, it seems reasonable to expect that a decision maker would regard 1_R as a "safer" bet than 1_Y and, therefore, would prefer to bet on R rather than on Y. That is,

$$1_R \succ 1_Y$$

For the same reason, when comparing bets on $R \cup G$ and on $Y \cup G$ it seems reasonable to expect that a decision maker would prefer to bet on the latter event because of the much better information that he has on it. That is,

$$1_{Y\cup G} \succ 1_{R\cup G}$$

Summing up, the quality of the information on which the decision maker's beliefs are based should lead to the following preference pattern

$$1_R \succ 1_Y \quad \text{and} \quad 1_{Y\cup G} \succ 1_{R\cup G} \tag{1}$$

The decision maker consistently prefers to bet on events on which he has superior information. Pattern (1) has been indeed confirmed in a number of actual experiments that carried out Ellsberg's thought experiment.

But, the common preference pattern (1) is not compatible with the sure-thing principle. In fact, consider the event $E = \{R, Y\}$. Bets 1_R and 1_Y are identical on E^c. According to the sure-thing principle, changing their common value in E^c from 0 to 1 should not alter their ranking. But, these modified acts are the bets $1_{R\cup G}$ and $1_{Y\cup G}$, respectively. Hence, by the sure-thing principle we have

$$1_R \succsim 1_Y \Longleftrightarrow 1_{R\cup G} \succsim 1_{Y\cup G}$$

This relation is violated by the pattern (1).

2.5 A second big bang

Ellsberg's experiments helped to clarify and popularize the problem of ambiguity. They were not followed, however, by important theoretical advances able to address it: nobody knew how to model properly this intuitively important aspect of decision making under uncertainty. The SEU reign continued, still essentially unchallenged on the normative side, for another twenty years until David, inspired by the coin example, entered the scene in the early 1980s. His fundamental contributions gave a new start to the theoretical study of rational decision making under uncertainty, which had remained essentially dormant after Savage's big bang start. His contributions are both in contents and methods. They shaped the field and engaged a generation of scholars in economic theory and, more generally, in any realm where decision making under uncertainty is central.

David understood that to extend the SEU to cope with ambiguity in decision problems there were at least two possibilities.

(i) To relax the assumption that the representing probability P be additive by assuming that it is only a capacity, that is, a normalized monotone set function. In this case, the lack of additivity would reflect the presence of ambiguity, whose importance is somehow measured by the nonadditivity of P. For example, in the previous urn Ellsberg experiment a possible capacity P would be such that $P(R) = 1/3$ and $P(Y) = P(G) = 0$. The equality $P(Y) = P(G)$ reflects the symmetry of ignorance, but, since P is no longer additive, we can set $P(Y) = P(G) = 0$. The nonadditivity gap $P(Y \cup G) - P(Y) - P(G) = 2/3$ models the presence of ambiguity.

(ii) To relax the assumption that the representing probability P be unique by assuming that beliefs may be represented by sets $\mathcal{C}$ of probability measures. In this case, ambiguity would be reflected by the nonsingleton nature of these sets, whose sizes would somehow reflect the importance of ambiguity in the decision problem that the DMs face. For example, in the single urn Ellsberg experiment a possible set $\mathcal{C}$ consists of all probabilities that give probability 1/3 to R.

David pursued approach (i) in his seminal 1989 paper, first drafted in 1982. It axiomatized the so-called *Choquet Expected Utility* criterion, which extends the SEU criterion by allowing for capacities in place of additive probability measures. Besides its conceptual novelty, the derivation is mathematically nontrivial because it had to rely on a nonadditive theory of integration that David reinvented by himself, to later learn that an earlier theory had been developed in the early 1950s by Gustave Choquet (so the name "Choquet" for the integral), one of the most prominent mathematicians of his time.

It is hard to overestimate the importance of this paper, which also set the agenda methodologically for the field, as we discuss shortly. Before, however, we turn to the approach (ii) that David developed along with his young student, then lifetime collaborator, Itzhak Gilboa. Their classic 1989 paper, first drafted in 1986, axiomatized the Maximin Expected Utility (MEU) criterion, which played a central role in both the theoretical developments and empirical applications of modern decision under uncertainty.

The basic idea of the MEU criterion is, at the same time, simple and appealing: since decision makers have not enough information to form a meaningful single subjective probability, they use a set of them, consisting of all subjective probabilities compatible with their limited information. Using this set $\mathcal{C}$, acts f are then ranked via the criterion

$$\min_{P\in\mathcal{C}}\int_S u\left(f\left(s\right)\right)\mathrm{d}P\left(s\right)$$

in which the minimum reflects a cautious attitude of the decision makers that results from a negative attitude toward ambiguity.

The theoretical problem that Gilboa and Schmeidler (1989) successfully addressed was to provide a behavioral underpinning of this criterion, as Schmeidler (1989) did for the Choquet Expected Utility criterion. This brings us to a key methodological contribution of David. His 1989 piece invokes a version of the Anscombe and Aumann (1963) setting developed by Fishburn, in his 1970 classic utility theory book. Savage's derivation did not make any structural assumption on the consequence space X but, for technical reasons inherent to the numerical representation of subjective probabilities, had to impose a divisibility requirement on the state space that, for example, required it to be infinite. In contrast, Anscombe and Aumann (1963) added a lottery structure to the consequence space and, with this, were able to axiomatize a SEU criterion with a finite state space. In the hands of David, this setting became a powerful functional-analytic setting, with ΔX just assumed to be a convex subset of a vector space, an assumption that renders also the space $\mathcal{F}=(\Delta X)^S$ of acts a convex subset of a function space. In this abstract Anscombe-Aumann setup, representation theorems could then build upon linear and nonlinear Riesz-like theorems, as one can learn from the classic linear analysis opus of Nelson Dunford and Jacob Schwartz of 1957 (a favorite of David, who had a first-rate mathematical knowledge) and from subsequent nonlinear analyses, as developed for instance in Convex Analysis since the 1960s.

This methodological innovation made possible a systematic investigation of decision making under uncertainty. After Savage's one, David caused a second big bang in the field, even more fruitful at a theoretical level. A number of axiomatic analyses for different decision criteria under uncertainty were (and are) developed within this functional-analytic framework, which is so germane to this topic. Indeed, uncertainty adds, by its nature, a functional dimension to the decision problem as it deals with maps f from states into consequences. Savage's most brilliant measure-theoretic approach was not so easily extended beyond its original domain and this was a main reason why so little happened in the field for decades after his 1954 masterpiece. David's functional-analytic approach, instead, paved the way to the development of modern decision theory under uncertainty. In this regard, the functional-analytic way in which he framed, on p. 577 of his 1989 piece, the von Neumann-Morgenstern Representation Theorem was most revealing.

To give a flavor of the axiomatic work made possible by this approach, we present couple of axioms. To axiomatize the MEU criterion, Gilboa and Schmeidler (1989) adopted a weak form of the independence axiom.

CERTAINTY INDEPENDENCE: for any $f,g\in\mathcal{F}$, any $x\in X$ and any $0<\alpha<1$,

$$f\succ g\Longrightarrow\alpha f+(1-\alpha)x\succ\alpha g+(1-\alpha)x \tag{2}$$

This weak independence axiom only requires that the decision maker's preference over two acts f and g is not to be affected by mixing them with a common state-constant act x. Since state-constant acts are, obviously, not subject to state uncertainty, ambiguity – which is about the likelihood of events – should not matter when acts are mixed with them. It is, indeed, easy to see that this weak independence axiom is not violated in the earlier coin and urn examples.

The next axiom, introduced by Schmeidler (1989), models a negative attitude toward ambiguity.

UNCERTAINTY AVERSION: for any $f, g \in \mathcal{F}$ and any $0 < \alpha < 1$,

$$f \sim g \Longrightarrow \alpha f + (1-\alpha) g \succsim f$$

According to this axiom, the decision maker always prefers to mix indifferent acts, that is, to randomize over them when mixing is interpreted in terms of randomization. To fix ideas, think of $\alpha = 1/2$ and of the acts 1_Y and 1_G in the single-urn Ellsberg experiment. It is reasonable to assume that

$$1_G \sim 1_Y \quad \text{and} \quad \frac{1}{2}1_G + \frac{1}{2}1_Y \succsim 1_Y$$

that is, a decision maker who does not like the presence of ambiguity would prefer to toss a fair coin and, according to whether heads or tails come up, to select either 1_G or 1_Y. Indeed, this randomization provides some hedging toward the presence of ambiguity.

These two axioms, combined with standard order and continuity ones, underlie the axiomatic analysis of Gilboa and Schmeidler (1989). The vector structure of the abstract Anscombe-Aumann setting permits to prove the theorem by first showing that the axioms deliver a superlinear functional I defined over the utility-valued acts $u \circ f$, and then showing, for instance via an application of the Hahn-Banach Theorem, that this functional admits a maxmin representation through a set $\mathcal{C}$ of probability measures. Later, Savagean versions of their theorem have been developed, but they heavily built on the seminal derivation of Itzhak and David.

2.6 Case based decision theory

The possibility of applying the classical theory of expected utility basically relies on two assumptions:

1. decision makers know (are able to describe) the available *actions* $a \in A$, the possible *states of the world* $s \in S$, the relevant *outcomes* $r \in R$, and the *outcome function*

$$\begin{array}{rccc} \rho : & A \times S & \rightarrow & R \\ & (a, s) & \mapsto & \rho(a, s) \end{array}$$

   attaching outcomes to action-state pairs.

2. decision makers are able to form a belief μ on S describing the likelihood of the different states and to assess the utility u of outcomes.

Armed with this, expected utility yields the choice criterion

$$\max_{a \in A} \int u(\rho(a, s))\, d\mu(s).$$

In many decision problems the cognitive task of steps 1 and 2 above seem daunting (and sometimes arbitrary) to say the least. Think of the possibility of a NATO intervention in the current Ukraine crisis. The alternative actions are relatively clear: one may do nothing, impose economic sanctions, use limited military force (say, air strikes only), or opt for a full-blown military intervention. The problems relative to step 1 concern the analysis of the short-run and long-run outcomes of each action in relation to the possible eventualities (e.g. the reactions of Russia and China), which are themselves difficult to list in an exclusive and exhaustive manner. As to to step 2, assume these difficulties have been overcome. What is the probability that military intervention develops into a third World War? What is the utility of such a war?

Case based decision theory is a reaction to these concerns. It builds on the idea that, when steps 1 and 2 are not practicable (or even advisable) decisions can be made by means of analogies to past cases. A rational decision maker will choose actions that are similar to those that performed well (produced good outcomes) in similar past situations, and try to avoid those that performed poorly (lead to bad ones). It is important to note that the outcomes of past actions have been observed, hence their *utilities* have already been experienced. In order to obtain a decision criterion, one needs a (most often subjective) assessment of the *similarities* between past problems and actions, and the ones at hand.

In case based decision theory, a *case* is a triple (q, b, r), where q is a (past) decision problem, b is the action taken in the face of q, and r is the obtained outcome. A *memory* M is the set of such cases (as recollected by the decision maker). A decision maker is characterized by a utility function u, which assigns a numerical value to outcomes, and a similarity function s, which assigns nonnegative values to pairs of problems and actions.

Armed with this, case based decision theory yields the choice criterion

$$\max_{a \in A} \sum_{(q,b,r) \in M} s\left(\left(q, b\right), \left(p, a\right)\right) u\left(r\right)$$

in which the action that provides the highest *similarity weighted utility* is selected in the (new) problem p.

This criterion can be seen as driven by the statistical decision rule that maps each memory (set of past observations) M into the maximizer of the resulting expected utility

$$M \mapsto \arg\max_{a \in A} \sum_{r} u\left(r\right) \pi_M\left(r \mid a\right)$$

where

$$\pi_M\left(r \mid a\right) = \frac{\sum_{(q,b,t) \in M : t = r} s\left(\left(q, b\right), \left(p, a\right)\right)}{\sum_{(q,b,t) \in M} s\left(\left(q, b\right), \left(p, a\right)\right)}$$

is a similarity based empirical probability, which is proportional to the support that past evidence provides in favor of a yielding outcome r in decision problem p.

Itzhak and David developed this theory and its multifaceted applications in a constant stream of papers and a book written in the last thirty years.

2.7 Summing up

David Schmeidler is, along with Leonard Savage, the pioneer of the theoretical study of rational decision making under uncertainty. His work has shaped the field, inspired a generation of scholars, and will continue to do so for the years to come.

A David Schmeidler's Publications in Decision Theory

A.1 Articles in journals

1. David Schmeidler, A condition for the completeness of partial preference relations, *Econometrica*, 39, 403-404, 1971.

2. Ehud Kalai and David Schmeidler, Aggregation procedure for cardinal preferences: A formulation and proof of Samuelson's Impossibility Conjecture, *Econometrica*, 45, 1431-1438, 1977.

3. David Schmeidler, A bibliographical note on a theorem of Hardy, Littlewood and Polya, *Journal of Economic Theory*, 20, 125-128, 1979.

4. Edi Karni, David Schmeidler and Karl Vind, On state dependent preferences and subjective probabilities, *Econometrica*, 51, 1021-1031, 1983.

5. Edi Karni and David Schmeidler, Self-preservation as a foundation of rational behavior under risk, *Journal of Economic Behavior and Organization*, 7, 71-81, 1986.

6. David Schmeidler, Integral representation without additivity, *Proceedings of the American Mathematical Society*, 97, 255-261, 1986.

7. David Schmeidler, Subjective probability and expected utility without additivity, *Econometrica*, 57, 571-87, 1989.

8. Itzhak Gilboa and David Schmeidler, Maximin expected utility with non-unique prior, Journal of Mathematical Economics, 18, 141-153, 1989.

9. Edi Karni and David Schmeidler, Fixed preferences and changing tastes, *American Economic Review P&P*, 80, 262-267, 1990.

10. Edi Karni and David Schmeidler, Atemporal dynamic consistency and expected utility theory, *Journal of Economic Theory*, 54, 401-408, 1991.

11. Mark Machina and David Schmeidler, A more robust definition of subjective probability, *Econometrica*, 60, 745-780, 1992.

12. Edi Karni and David Schmeidler, On the uniqueness of subjective probability, *Economic Theory*, 3, 267-277, 1993.

13. Itzhak Gilboa and David Schmeidler, Updating ambiguous beliefs, *Journal of Economic Theory*, 59, 33-49, 1993.

14. Itzhak Gilboa and David Schmeidler, Additive representations of non-additive measures and the Choquet integral, *Annals of Operations Research*, 51, 43-65, 1994.

15. Itzhak Gilboa and David Schmeidler, Canonical representations of set functions, *Mathematics of Operations Research*, 20, 197-212, 1995.

16. Itzhak Gilboa and David Schmeidler, Case-based decision theory, *Quarterly Journal of Economics*, 110, 605-639, 1995.

17. Mark Machina and David Schmeidler, Bayes without Bernoulli: Conditions for probabilistically sophisticated choice, *Journal of Economic Theory*, 67, 106-128, 1995.

18. Itzhak Gilboa and David Schmeidler, Case-based optimization, *Games and Economic Behavior*, Vol. 15, 1-26, 1996.

19. Itzhak Gilboa and David Schmeidler, Act-similarity in case-based decision theory, *Economic Theory*, 9, 47-61, 1997.

20. Itzhak Gilboa and David Schmeidler, Cumulative utility consumer theory, *International Economic Review*, 38, 737-61, 1997.

21. Itzhak Gilboa and David Schmeidler, Case-based knowledge and induction, *IEEE SMC Transactions Part A*, 30, 85-95, 2000.

22. Michèle Cohen, Itzhak Gilboa, Jean-Yves Jaffray and David Schmeidler, An experimental study of updating ambiguous beliefs, *Risk, Decision and Policy*, 5, 123-33, 2000.

23. Itzhak Gilboa and David Schmeidler, A cognitive model of individual wellbeing, *Social Choice and Welfare*, 18, 269-288, 2001.

24. Itzhak Gilboa and David Schmeidler, Reaction to price changes and aspiration level adjustments, *Review of Economic Design*, 6, 215-23, 2001.

25. Itzhak Gilboa and David Schmeidler, Cognitive foundations of probability, *Mathematics of Operations Research*, 27, 68-81, 2002.

26. Itzhak Gilboa, David Schmeidler and Peter Wakker, Utility in case-based decision theory, *Journal of Economic Theory*, 105, 483-502, 2002.

27. Itzhak Gilboa and David Schmeidler, Inductive inference: An axiomatic approach, *Econometrica*, 71, 1-26, 2003.

28. Itzhak Gilboa and David Schmeidler, A derivation of expected utility maximization in the context of a game, *Games and Economic Behavior*, 44, 172-82, 2003.

29. Dov Samet, Iddo Samet and David Schmeidler, One observation behind two envelope puzzles, *American Mathematical Monthly*, 111, 347-351, 2004.

30. Itzhak Gilboa, Dov Samet and David Schmeidler, Utilitarian aggregation of beliefs and tastes, *Journal of Political Economy*, 112, 932-938, 2004.

31. Itzhak Gilboa and David Schmeidler, Subjective distributions, *Theory and Decision*, 56, 345-357, 2004.

32. Antoine Billot, Itzhak Gilboa, Dov Samet and David Schmeidler, Probabilities as similarity-weighted frequencies, *Econometrica*, 73, 1125-1136, 2005.

33. Enriqueta Aragones, Itzhak Gilboa, Andrew Postlewaite and David Schmeidler, Fact free learning, *American Economic Review*, 95, 1355-1368, 2005.

34. Itzhak Gilboa, Offer Lieberman and David Schmeidler, Empirical similarity, *Review of Economics and Statistics*, 88, 433-444, 2006.

35. Antoine Billot, Itzhak Gilboa and David Schmeidler, Axiomatization of an exponential similarity function, *Mathematical Social Sciences*, 55, 107-115, 2008.

36. Itzhak Gilboa, Andrew Postlewaite and David Schmeidler, Probability and uncertainty in economic modeling, *Journal of Economic Perspectives*, 22, 173-188, 2008.

37. Itzhak Gilboa, Andrew Postlewaite and David Schmeidler, Is it always rational to satisfy Savage's axioms?, *Economics and Philosophy*, 25, 285-96, 2009.

38. Itzhak Gilboa, Offer Lieberman and David Schmeidler, On the definition of objective probabilities by empirical similarity, *Synthese*, 172, 79-95, 2010.

39. Itzhak Gilboa, Fabio Maccheroni, Massimo Marinacci and David Schmeidler, Objective and subjective rationality in a multiple prior model, *Econometrica*, 78, 755-770, 2010.

40. Itzhak Gilboa and David Schmeidler, Likelihood and simplicity: An axiomatic approach, *Journal of Economic Theory*, 145, 1757-1775, 2010.

41. Itzhak Gilboa, Offer Lieberman and David Schmeidler, A similarity-based approach to prediction, *Journal of Econometrics*, 162, 124-131, 2011.

42. Itzhak Gilboa, Andrew Postlewaite and David Schmeidler, Rationality of belief or: Why Savage's axioms are neither necessary nor sufficient for rationality, *Synthese*, 187, 110-131, 2012.

43. Andrew Postlewaite and David Schmeidler, Rationality and uncertainty, *Rivista Internazionale di Scienze Sociali*, 102, 289-292, 2012.

44. Itzhak Gilboa, Larry Samuelson and David Schmeidler, Dynamics of inductive inference in a unified model, *Journal of Economic Theory*, 148, 1399-1432, 2013.

45. Enriqueta Aragones, Itzhak Gilboa, Andrew Postlewaite and David Schmeidler, Rhetoric and analogies, *Research in Economics*, 68, 1-10, 2014.

46. Shiri Alon and David Schmeidler, Purely subjective maxmin expected utility, *Journal of Economic Theory*, 152, 382-412, 2014.

47. Itzhak Gilboa, Larry Samuelson and David Schmeidler, No-betting-Pareto Dominance, *Econometrica*, 82, 1405-1442, 2014.

48. Gabrielle Gayer, Itzhak Gilboa, Larry Samuelson, and David Schmeidler, Pareto efficiency with different beliefs, *Journal of Legal Studies*, 43(S2), S151-S171, 2014.

49. Itzhak Gilboa, Andrew Postlewaite, Larry Samuelson and David Schmeidler, Economic models as analogies, *Economic Journal*, 124, F513-F533, 2014.

50. Edi Karni and David Schmeidler, An expected utility theory for state-dependent preferences. *Theory and Decision*, 81, 467-478, 2016.

51. Itzhak Gilboa, Andrew Postlewaite, Larry Samuelson and David Schmeidler, Economics: Between prediction and criticism, *International Economic Review*, 59, 367-390, 2018.

52. Itzhak Gilboa, Andrew Postlewaite, Larry Samuelson and David Schmeidler, What are axiomatizations good for?, *Theory and Decision*, 86, 339-359, 2019.

53. Itzhak Gilboa, Stefania Minardi, Larry Samuelson and David Schmeidler, States and contingencies: How to understand Savage without anyone being hanged, *La Revue Economique*, 71, 365-385, 2020.

54. Itzhak Gilboa, Larry Samuelson and David Schmeidler, Learning (to disagree?) in large worlds, *Journal of Economic Theory*, 199, 2020.

55. Itzhak Gilboa, Andrew Postlewaite and David Schmeidler, The complexity of the consumer problem, *Research in Economics*, 75, 96-103, 2021.

56. Itzhak Gilboa, Larry Samuelson and David Schmeidler, No-Betting Pareto under Ambiguity, *Theory and Decision*, 93, 625-645, 2022.

A.2 Books and chapters in books

1. Edi Karni and David Schmeidler, Utility theory with uncertainty, in *Handbook of mathematical economics* v. 4, (W. Hildenbrand and H. Sonnenschein, eds.), North Holland, Amsterdam, 1991.

2. Itzhak Gilboa and David Schmeidler, *A theory of case-based decisions*, Cambridge University Press, 2001.

3. Itzhak Gilboa and David Schmeidler, *Case-Based Predictions*, World Scientific Publishers, Economic Theory Series, 2012.

4. Itzhak Gilboa, Larry Samuelson and David Schmeidler, *Analogies and theories: Formal models of reasoning*, Oxford University Press, 2015.