A Mechanism for Eliciting Second-Order Beliefs and the Inclination to Choose

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Abstract

This paper describes a direct revelation mechanism for eliciting decision makers' introspective beliefs on sets of subjective prior or posterior probabilities. The proposed scheme constitutes a revealed-preference procedure for measuring the inclination of decision makers to choose one alternative over another modeled by Minardi and Savochkin (2015).

Keywords: Knightian uncertainty; second-order beliefs; probability elicitation; random choice; introspective beliefs; graded preferences.

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1 Introduction

Consider a decision maker who contemplates a choice that she will have to make sometime in the future between two courses of action whose consequences are uncertain. It is conceivable and arguably quite likely that the decision maker entertains subjective beliefs about the likely realization of the relevant consequences that cannot be represented by a unique subjective probability measure. Instead, the decision maker's beliefs are represented by a set of prior subjective probability distributions on a state space. This situation, dubbed Knightian uncertainty, was modeled in the seminal work of Bewley (2002).[1]

While being unsure about what she believes, the decision maker might expect that her beliefs will become clear by the time the choice must be made, as a result of conscious effort or subconscious impulse. Being aware of this, the decision maker may also entertain "second-order belief" regarding the likelihoods that different "first-order beliefs," are realized. In other words, the decision maker may entertain introspective belief over the likely selection of different priors that, in turn, govern her choice.

Second-order belief on the set of priors is a special case of the model of Karni and Safra (2016), according to whom actual choice behavior is governed by a random selection from a set of preference relations representing the decision maker's states of mind. The selection process is depicted by a probability measure representing the decision maker's introspective beliefs about her likely state of mind at the time the decision must be made. Knightian uncertainty is the special case in which the decision maker's states of mind are depicted by her subjective prior beliefs.

Formally, let S be a finite state space, $\Delta(X)$ the set of lotteries on a finite set, X, of outcomes, and H the set of all mapping on S to $\Delta(X)$, representing alternative courses of action and referred to as Anscombe-Aumann acts.[2] Menus are subsets of H. Let Π denote the set of priors that figure in the Knightian uncertainty model, then according to Karni and Safra (2016), given any doubleton menu $\{f, g\}$ the likelihood that f will be chosen is given

[1] For a more recent take, see Galaabaatar and Karni (2013).

[2] See Anscombe and Aumann (1963).

by

$$\Pr(f \mid \{f,g\} = \mu \left(\pi \in \Pi \mid \sum_{s\in S} u\left(f\left(s\right)\right)\pi\left(s\right) \geq \sum_{s\in S} u\left(g\left(s\right)\right)\pi\left(s\right) \right), \tag{1}$$

where u is an affine real-valued function on $\Delta\left(X\right)$ and μ represents the decision maker's second order beliefs.

Invoking Bewley's Knightian uncertainty, Minardi and Savochkin (2015) axiomatized a measure describing decision makers' inclination to choose one uncertain course of action over another and the level of confidence they have in the superiority of the preferred alternative. Departing from the standard revealed-preference methodology, Minardi and Savochkin characterize decision makers by binary relations on *pairs* of Anscombe-Aumann acts (that is, a binary relation $\succsim$ on $H \times H$). Given two pairs of such acts, (f,g) and (f',g'), then $(f,g) \succsim (f',g')$ is interpreted to mean that the decision maker is more confident in the superiority of f over g than in that of f' over g'. Minardi and Savochkin proceed to provide an axiomatic characterization of the relation $\succsim$, dubbed *graded preferences*, that allows its representation by a functional η in the sense that, for all $(f,g),(f',g') \in H \times H$, $(f,g) \succsim (f',g')$ if and only if $\eta\left(f,g\right) \geq \eta\left(f',g'\right)$. Moreover, they show that

$$\eta\left(f,g\right) = \Phi\left(\left\{\pi \in \Pi \mid \sum_{s\in S} u\left(f\left(s\right)\right)\pi\left(s\right) \geq \sum_{s\in S} u\left(f\left(s\right)\right)\pi\left(s\right)\right\}\right), \tag{2}$$

where u and Π are as above and Φ is a capacity measure on the subsets of the set of priors.[3]

According to Minardi and Savochkin, a decision maker's "inclination to choose" can be expressed verbally only (e.g., through responses to a consumer survey). The Karni-Safra introspective probability of the subset of priors according to which f is preferred to g is a natural instance of the Minardi-Savochkin representation in which $\Phi = \mu$, and $\eta\left(f,g\right) = \mu\left(f,g\right)$.

In this paper, I propose an incentive-compatible procedure for eliciting the inclination of a decision maker to select one act over another, thereby lending the Minardi-Savochkin measure revealed-preference meaning. The proposed

[3] A capacity measure on a measurable space (Ω,Σ) has the properties $\Phi\left(\varnothing\right) = 0, \Phi\left(\Omega\right) = 1$ and, for all $A,B \in \Sigma$, $\Phi\left(A\right) \leq \Phi\left(B\right)$ whenever $A \subseteq B$.

scheme combines a quadratic scoring rule with menu choice. It offers the decision maker the opportunity to choose from a menus of acts designed to induce her to reveal her introspective assessment of how likely she is to choose one act over another. The proposed mechanism invokes the revealed-preference approach to identify theoretical ingredients of models that depart from the revealed preference methodology.

The elicitation problem addressed in this paper is fundamentally different from that dealt with in the literature on the elicitation of subjective probabilities. Two aspects of the problem render it distinct and difficult.

First, with two exceptions, the literature dealing with probability elicitation is concerned with observable events in an objective state space.[4] These events are used to condition the subject's payoffs. By contrast, this paper deals with the elicitation of the probabilities of events in a *subjective state space*.[5] These events are subsets of *unobservable priors*. Consequently, the events of interest are private information and cannot be used to condition the decision maker's payoffs.

Second, the utility function is inherently state dependent,[6] and there is no known probability elicitation scheme that yields unbiased estimate of the subjective probabilities in the presence of state-dependent preferences. The proposed mechanism overcomes both difficulties by embedding the scoring rule in menu choice and exploiting the particular (linear) form in which the subjective states (that is, the subjective prior or posterior probabilities) affect the utility function.

2 The Elicitation Mechanism

2.1 The analytical framework

Let S be a finite state space and suppose that the decision maker believes that the probability distribution on S is a random variable, $\widetilde{\pi}$, taking values

[4]See discussion in section 4.2 below.

[5]The set of beliefs constitute a subjective state space à la Kreps (1979).

[6]To grasp the this observation, consider acts that are real-valued functions on S, representing the monetary payoffs contingent on the states in S. Let π denote a subjective probability distribution on S, representing the decision maker's belief. Then the utility of an act f depends on the subjective state π. Formally, $U(f,\pi) = \Sigma_{s\in S} u(f(s))\pi(s)$, where u is a real-valued function on the reals.

in the subset Π of the simplex $\Delta(S)$. The set Π is interpreted as the set of prior probability measures that figure in Bewley's (2002) representation of Knightian uncertainty. Let $F := \{h : S \to \mathbb{R}\}$ denote the subset of Anscombe-Aumann acts whose payoffs are sums of money. For any acts f and g in F, let $\eta(f,g)$ denote the Minardi and Savochkin's (2015) measure of the inclination of a decision maker to choose f over g.

Given $f, g \in F$, let $\Pi(f,g)$ denote the event in Π consisting of subjective beliefs that favor f over g. Formally, let u be a real-valued function on the set of real numbers representing a subjective expected utility maximizing decision maker's risk attitudes. For all $f \in F$ and $\pi \in \Delta(S)$, define $U(f,\pi) = \Sigma_{s \in S} u(f(s))\pi(s)$, then

$$\Pi(f,g) := \{\pi \in \Pi \mid U(f,\pi) \geq U(g,\pi)\}. \tag{3}$$

Let μ denote the probability measure on $\Delta(S)$ representing the decision maker introspective second-order beliefs of Karni and Safra (2016) in (1).[7] If $f \succ g$ then, by Knightian uncertainty, $\Pi(f,g) = \Pi$. Hence, the introspective beliefs $\mu(\Pi(f,g)) = 1$, and $\eta(f,g) = 1$. If $g \succ f$ then $\Pi(g,f) = \Pi$, $\mu(\Pi(f,g)) = 0$ and $\eta(f,g) = 0$. If f and g are non-comparable then $\Pi(f,g) \subset \Pi$, $\mu(\Pi(f,g)) \in (0,1)$ and $\eta(f,g) \in (0,1)$.

2.2 The mechanism described

The objective of the scheme described below is the elicitation of the probability, $\mu(\Pi(f,g))$, that represents the decision maker's introspective beliefs that the true distribution π on S is in $\Pi(f,g)$. The procedure embeds quadratic scoring rules in menu choice in a way that makes the truthful revelation of $\mu(\Pi(f,g))$ incentive compatible.[8] It is assumed throughout that the decision maker's preferences exhibit Knightian uncertainty over acts and their random choice behavior is depicted by second-order, introspective, belief.

Because the event of interest is not observable, the elicitation scheme cannot involve payoffs contingent on the event of interest. To overcome this difficulty, the proposed mechanism offers the decision maker a choice from a menu that consists of the acts f and g that are modified to incorporate the

[7] Let $\mathcal{B}$ denote the Borel sigma algebra on $\Delta(S)$, then $(\Delta(S), \mathcal{B}, \mu)$ is the probability space representing the decision makers introspective beliefs à la Karni and Safra (2016).

[8] Section 4.2 provides some references to the literature on scoring rules.

payoffs associated with a quadratic scoring rule. This elicitation procedure harnesses the incentives built into the scoring rule to induce the decision maker to choose her responses in a way that reveals her disposition to choose f over g.

To describe the proposed mechanism formally, let $\mathcal{U}$ be the set of real-valued, twice continuously differentiable, strictly monotonic increasing concave utility functions on the set of real numbers, $\mathbb{R}$. For every given $c > 0$ and $u \in \mathcal{U}$, define a function $y_{u,c} : \mathbb{R} \to \mathbb{R}$ by $y_{u,c}(x) = c/u'(x)$, for all $x \in \mathbb{R}$. Let $Y := \{y_{u,c} \mid c > 0, u \in \mathcal{U}\}$.

Given $f, g \in F$, the mechanism requires the decision maker to report a number, $\alpha \in [0, 1]$. The decision maker is awarded the right to choose, at a later date, before that state $s \in S$ becomes known, from the menu $\{f(\alpha), g(\alpha)\}$, where

$$f(\alpha) := (f(s) - r(1-\alpha)^2 y_{u,c}(f(s))_{s \in S} \text{ and } g(\alpha) := (g(s) - r\alpha^2 y_{u,c}(g(s)))_{s \in S},$$

and $r > 0$ is a parameter selected by the designer to allow him to control the proximity of the acts $f(\alpha)$ and $g(\alpha)$ to the acts of interest, f and g, respectively. The role of the function $y_{u,c}$ is to counteract the estimation biases that are due to the curvature of the utility function.

2.3 The mechanism analyzed

The following theorem asserts that if a decision maker's preference relation over acts exhibits Knightian uncertainty, her random choice behavior is depicted by second-order belief and her risk attitudes are represented by a utility function $u \in \mathcal{U}$ then, for every given $f, g \in F$, in the limit as r tends to zero, the optimal choice of $\alpha^*(r; f, g)$ under the mechanism reveals the her probability $\mu(\Pi(f, g))$.

Theorem: *If the preference relation exhibits random choice behavior over acts depicted by a set of priors,* Π, *second-order belief,* μ, *and a utility function* $u \in \mathcal{U}$ *then* $\lim_{r \to 0} \alpha^*(r; f, g) = \mu(\Pi(f, g))$, *for all* $f, g \in F$.

The proof is in the Appendix.

2.4 Examples

Consider a risk-neutral decision maker contemplating a choice between two bets on the outcome of the next US presidential election. The bet f pays off \$100 if the candidate of the Democratic party wins and nothing otherwise. The bet g pays off \$85 if the candidate of the Democratic party wins and \$25 if the candidate of the Republican party wins. The decision maker currently believes that the probability of the candidate of the Democratic party winning the election is no greater than 0.75 and no less than 0.5. He expects to have a clear idea closer to election day, when he will have to choose between the two bets. The objective is to elicit the current inclination to decision maker to favor the bet f over the bet g.

Given the menu $M = \{f, g\}$ and the decision maker's risk attitudes, under the proposed scheme, the decision maker's problem is to choose $\alpha \in [0,1]$ so as to maximize[9]

$$\int_{\Pi(f(\alpha),g(\alpha))} \left[\Sigma_{s\in S}(f(s) - r(1-\alpha)^2)\pi(s)\right] d\mu(\pi) \tag{4}$$

$$+ \int_{\Pi\setminus\Pi(f(\alpha),g(\alpha))} \left[\Sigma_{s\in S}\left(g(s) - r\alpha^2\right)\pi(s)\right] d\mu(\pi).$$

Let

$$F(\alpha';\alpha) = \int_{\Pi(f(\alpha'),g(\alpha'))} \left[\Sigma_{s\in S}(f(s) - r(1-\alpha)^2)\pi(s)\right] d\mu(\pi) \tag{5}$$

and

$$G(\alpha';\alpha) = \int_{\Pi\setminus\Pi(f(\alpha'),g(\alpha'))} \left[\Sigma_{s\in S}(g(s) - r\alpha^2)\pi(s)\right] d\mu(\pi). \tag{6}$$

Then the necessary and sufficient condition is:

$$(1-\alpha^*) \int_{\Pi(f(\alpha),g(\alpha))} \pi(s)\, d\mu(\pi) + \frac{dF(\alpha';\alpha^*)}{d\alpha'}\Big|_{\alpha'=\alpha^*} = \alpha^* \int_{\Pi\setminus\Pi(f(\alpha'),g(\alpha'))} \pi(s)\, d\mu(\pi) + \frac{G(\alpha';\alpha^*)}{d\alpha'}\Big|_{\alpha'=\alpha^*}. \tag{7}$$

[9]Risk neutrality implies that $y_{u,c}(x) = 1$ for all x.

But

$$\frac{dF(\alpha';\alpha^*)}{d\alpha'}\mid_{\alpha'=\alpha^*}=\Sigma_{s\in S}(f(s)-r(1-\alpha)^2)\hat{\pi}(s)=\Sigma_{s\in S}(g(s)-r\alpha^2)\hat{\pi}(s)=\frac{G(\alpha';\alpha^*)}{d\alpha'}\mid_{\alpha'=\alpha^*},$$

where $\hat{\pi}$ is in the intersection of the boundaries of $\Pi(f,g)$ and $\Pi\backslash\Pi(f,g)$. Hence, $(1-\alpha^*)\mu(\Pi(f,g)) = \alpha^*\mu(\Pi\backslash\Pi(f,g))$, and, consequently, $\alpha^* = \mu(\Pi(f,g))$. In this case, because the marginal utility is constant, the elicitation procedure yields an unbiased estimate of the disposition to choose f over g for all values of r, and not just in the limit as r tends to zero.

Applied to the example of betting on the outcome of the election result, $\alpha^* = \mu\{\pi \in [0.5, 0.75] \mid \pi 100 > \pi 85 + (1-\pi) 25\}$ is the the decision maker current probabilistic belief that, at the time when he will have to choose between f and g, the he will choose the bet f.

The examples below illustrate how the elicitation mechanism is applied to utility functions representing risk attitudes that are often invoked in economic analysis.

Constant Relative Risk Aversion**:** Risk neutrality is a special case of the class of utility functions, often used in applications in economics and finance, displaying constant relative risk aversion (CRRA). For utility functions displaying CRRA, $y_{u,c}(x) = x^\alpha$ for all $x \in \mathbb{R}$, where $\alpha \in (0,1)$ is the measure relative risk aversion. Hence, the menu defined by the elicitation mechanism is:

$$\{f(\alpha) := (f(s) - r(1-\alpha)^2 f(s)^\alpha)_{s\in S},\ g(\alpha) := (g(s) - r\alpha^2 g(s)^\alpha)_{s\in S}\}.$$

Constant Absolute Risk Aversion**:** In the case of constant absolute risk aversion (CARA), it is easy to verify that $y_{u,c}(x) = ce^{kx}/k$, where k is the Arrow-Pratt measure of absolute risk aversion. The corresponding menu is:

$$\{f(\alpha) := (f(s) - r(1-\alpha)^2 ce^{kf(s)}/k)_{s\in S},\ g(\alpha) := (g(s) - r\alpha^2 ce^{kg(s)}/k)_{s\in S}\}.$$

Expo-Power Utility Function**:** The expo-power family of utility function was first proposed by Saha (1993). The two parameter variation, used by Holt and Laury (2002), is given by

$$u(x) = \frac{1-\exp\left(-\alpha x^{1-\rho}\right)}{\alpha}, \tag{8}$$

where x denotes the decision maker's wealth and $1 \geq \rho \geq 0$. In this case, $y_{u,c}(x) = c(1-\rho)\, e^{-\alpha x^{1-\rho}} x^{-\rho}$ and the corresponding menu is:

$$\{f(\alpha) := (f(s) - r(1-\alpha)^2 c(1-\rho)\, e^{-\alpha x^{1-\rho}} x^{-\rho})_{s\in S},\ g(\alpha) := (g(s) - r\alpha^2 c(1-\rho)\, e^{-\alpha x^{1-\rho}} x^{-\rho})_{s\in S}\}.$$

The one-parameter variation, used in Abdellaoui, Barrios, and Wakker (2007), is given by

$$u(x) = -e^{\frac{-x^{\rho}}{\rho}},\ \text{for } \rho \neq 0 \text{ and } u(x) = -1/x \text{ for } \rho = 0. \tag{9}$$

For $\rho \in (0, 1]$, the one parameter expo-power utility function displays decreasing absolute and increasing relative risk aversion.[10] In this case, for $\rho \in (0, 1)$, $y_{u,c}(x) = ce^{\frac{x^{\rho}}{\rho}} x^{1-\rho}$ and the corresponding menu is:

$$\{f(\alpha) := (f(s) - r(1-\alpha)^2 cx^{1-\rho} e^{\frac{x^{\rho}}{\rho}})_{s\in S},\ g(\alpha) := (g(s) - r\alpha^2 cx^{1-\rho} e^{\frac{x^{\rho}}{\rho}})_{s\in S}\}.$$

For $\rho = 0$, $y_{u,c}(x) = x^2$ and the corresponding menu is:

$$\{f(\alpha) := (f(s) - r(1-\alpha)^2 x^2)_{s\in S},\ g(\alpha) := (g(s) - r\alpha^2 x^2)_{s\in S}\}.$$

3 Discussion

3.1 An Alternative Interpretation

The indecisiveness of a decision maker concerning a choice between courses of action, that she will have to make sometime in the future may be due to ambiguity regarding her prior beliefs. It may also be due to anticipation of receiving new information that may affect her posterior beliefs. In the latter instance, the decision maker may have a unique prior and her hesitations reflect the uncertainty surrounding the anticipated new information that might influence her choice. The decision maker's posterior beliefs can be described by a set of signal-contingent, posterior subjective probability distributions on a state space. In this case, the decision maker may entertain a "second-order belief" regarding the likelihoods of the signals whose canonical representation is the set of "first-order posterior beliefs." Under this interpretation the

[10] Note that $-\frac{u''(x)}{u'(x)} = x^{-(1-r)} + (1-r)\, x^{-1}$.

"second-order belief" constitutes the decision maker's subjective information structure.

Lu (2016) and Dillenberger, Lleras, Sadowski and Takeoka (2014) modeled second-order belief on the set of posteriors. These models describe decision makers who anticipate receiving private signals before choosing an act from a menu of acts. In Lu's model, decision makers are subjective expected utility maximizers who receive private signals and whose choice from menu of acts is governed by a posterior distribution on the states. A decision maker's information structure (that is, her beliefs on the set of signals) is depicted by a distribution, μ, on the set of posteriors (the canonical signal space) $\Delta(S)$. This second-order belief is revealed by a random choice rule describing the decision maker's actual choice behavior. In the special case of doubleton menus the probability, $\rho_{\{f,g\}}(f)$ that f is chosen form the menu $\{f,g\}$ according to the random choice rule ρ, is given by:

$$\rho_{\{f,g\}}(f)=\mu\left(\pi\in\Delta(S)\mid\sum_{s\in S}u(f(s))\pi(s)\geq\sum_{s\in S}u(g(s))\pi(s)\right). \qquad (10)$$

Dillenberger et al. model a decision maker who chooses among menus of acts as if she has unique probability distribution over the set of posterior distributions over the state space that she might face at the time of choosing from the menu. The scenario envisioned is that before choosing an act from a menu, the decision maker receives a signal that allows her to update her prior probability distribution over the states. Given the posterior the decision maker chooses the act from the menu that maximizes her expected utility. A decision maker's representation of preference relation on acts involve a unique probability measure on a canonical signal space (that is, the set of distributions on the state space) representing her subjective belief on the set of posteriors. Presumably the choice of acts from the menu, which is not part of their formal model, is random and corresponds to the rule depicted in (10)

The mechanism of this paper is applicable to the elicitation of the measure μ in (10).

3.2 Related literature

The study of incentive-compatible mechanisms designed to elicit experts or decision makers beliefs, dubbed proper scoring rules, initiated by Brier (1950) and Good (1952) has been extended (e.g., Savage [1971], Kadane and Winkler [1988], Bickel (2007), Fang, Stinchcombe and Whinston [2010]) and applied in experimental work (e.g., Nyarko and Schotter [2002]).[11] These schemes entail payoffs contingent on the observed event of interest. Because the subject matter of this work concerns events of interest, subsets of beliefs under which some act preferable over another, is not observable, the aforementioned schemes are not directly applicable.

Chambers and Lambert (2014, 2016) and Karni (2016) introduce elicitations schemes designed to elicit decision makers' subjective probability of an event in the objective state space as well as their second-order beliefs. The former work involves the elicitation of subjective information structures of Bayesian decision makers and their second-order beliefs on posterior subjective probabilities of the event of interest. The latter work proposes a mechanism for elicitating of the set of priors of the event of interest and second-order beliefs on this set and, in the case of Bayesian decision makers, also the subjective information structures and second-order beliefs on posterior subjective probabilities. The event in the support of the second-order beliefs is a set of subjective probabilities that, by definition, are not publicly observable.

Despite this similarity, the mechanisms of Chambers and Lambert and Karni are quite different both in form in substance. Karni (2016) provides a detailed discussion of the differences. The mechanism proposed by Chambers and Lambert is designed to elicit the information structure in the choice-based models of Dillenberger, Lleras, Sadowski and Takeoka (2014) and Lu (2016). Since these models are anchored in the revealed preference methodology, it is not surprising that the information can be elicited using revealed preference methods. By contrast, the mechanism of this paper and the one proposed in Karni (2016) invoke a revealed-preference approach to elicit introspective beliefs that are articulated in models that depart from the revealed preference methodology.

[11]For a survey of proper scoring rules and their application see Gneiting and Raftery (2004).

3.3 Incentives and biases

The analysis of the proposed mechanism makes it clear that, except in the case of risk-neutral agents, the elicitation of the exact probability of the set of priors that favor one act, say f, over another act, say g, requires that the deviations from the payoffs of acts of interest $f(\alpha) - f$ and $g(\alpha) - g$ introduced by the mechanism's reward structure vanish in the limit. This is the implication of the proper scoring rules; it is not specific to the proposed scheme. Elicitation mechanisms are measurement tools whose accuracy depends on their incentive structure. Proper scoring rules present the elicitor with a trade-off between the power of the incentives and the accuracy of measurement. In general, the more powerful the incentive, the less accurate the measurement. The mechanism is still useful for obtaining an approximation of the sought after value. Moreover, if the utility function is approximately linear in the relevant range, application of the proposed mechanism with strong incentives can yield good approximations. Conceptually, the novelty of mechanism introduced in this paper is that it constitutes a revealed preference approach to the elicitation of information about belief that is otherwise only gleaned by introspection and verbal testimony.

Another concern regarding the application of the mechanism is that the utility function of the decision maker may not be known to the mechanism designer. Suppose the decision maker is risk averse whose utility function is not known. The mechanism can still be applied, assuming that the decision maker is risk neutral.

Suppose that the decision maker's utility function is u then, given $f, g \in F$ and applying the mechanism with $y_{u,c}(x) = 1$ for all x, the decision-maker's problem is to choose $\alpha \in [0,1]$ so as to maximize

$$\int\limits_{\Pi(f(\alpha),g(\alpha))} \left[\Sigma_{s\in S} u(f(s) - r(1-\alpha)^2)\pi(s)\right] d\mu(\pi) + \int\limits_{\Pi\setminus\Pi(f(\alpha),g(\alpha))} \left[\Sigma_{s\in S} u(g(s) - r\alpha^2)\pi(s)\right] d\mu(\pi).$$

Following the analysis in the proof of the theorem it is easy to verify that, in the limit as r tend to 0, the necessary and sufficient condition is:

$$(1-\alpha^*) \int\limits_{\Pi(f(\alpha),g(\alpha))} \left[\Sigma_{s\in S} u'(f(s))\pi(s)\right] d\mu(\pi) = \alpha^* \int\limits_{\Pi\setminus\Pi(f(\alpha'),g(\alpha'))} \left[\Sigma_{s\in S} u'(g(s))\pi(s)\right] d\mu(\pi).$$

Equivalently,

$$[\Sigma_{s\in S}u'(f(s))\pi(s)](1-\alpha^{*})\mu(\Pi(f,g))=\alpha^{*}\mu(\Pi\backslash\Pi(f,g))[\Sigma_{s\in S}u'(g(s))\pi(s)].$$

Hence, the mechanism yields a biased estimate of $\mu(\Pi(f,g))$ whose magnitude depends on expected marginal utilities under f and g.[12] If these values are close to one another (e.g., if the utility function is approximately linear in the relevant range) then the mechanism elicit a good approximation of the probability of the event of interest.

3.4 Concluding remark

Interest in a decision maker's inclination to choose among alternative courses of action stems from the realistic presumption that in many situations the preference relation over the choice set might not be complete, giving rise to indecisiveness. The level of confidence a decision maker feels regarding her disposition to choose may be articulated verbally. Minardi and Savochkin (2015) mention marketing surveys as an example of a tool intended to elicit decision makers' level of confidence in their preferences for one alternative over another. They also note that "using this type of data in economics will probably require implementing some sort of an incentive scheme." (Minardi and Savochkin (2015) p. 301). The mechanism described in this paper is such a scheme.

[12]The bias is a monotonic increasing function of the difference $[\Sigma_{s\in S}u'(f(s))\pi(s)]/[\Sigma_{s\in S}u'(g(s))\pi(s)]-1$.

APPENDIX

3.5 Proof of the Theorem:

Fix $f, g \in F$. Then under the scoring rule, the decision maker problem is: Choose $\alpha \in [0,1]$ so as to maximize

$$\int_{\Pi(f(\alpha),g(\alpha))} \left[\Sigma_{s\in S} u(f(s) - r(1-\alpha)^2 y_{u,c}(f(s)))\pi(s)\right] d\mu(\pi) \tag{11}$$

$$+ \int_{\Pi\backslash\Pi(f(\alpha),g(\alpha))} \left[\Sigma_{s\in S} u(g(s) - r\alpha^2 y_{u,c}(g(s)))\pi(s)\right] d\mu(\pi).$$

Denote the solution by α^*. Let

$$F(\alpha';\alpha) := \int_{\Pi(f(\alpha'),g(\alpha'))} \left[\Sigma_{s\in S} u(f(s) - r(1-\alpha)^2 y_{u,c}(f(s)))\pi(s)\right] d\mu(\pi) \tag{12}$$

and

$$G(\alpha';\alpha) := \int_{\Pi\backslash\Pi(f(\alpha'),g(\alpha'))} \left[\Sigma_{s\in S} u(g(s) - r\alpha^2 y_{u,c}(g(s)))\pi(s)\right] d\mu(\pi). \tag{13}$$

Then the necessary and sufficient condition is:

$$(1-\alpha^*) \int_{\Pi(f(\alpha),g(\alpha))} \left[\Sigma_{s\in S} u'\left((f(s) - r(1-\alpha^*)^2 y_{u,c}(f(s)))\right) y_{u,c}(f(s))\,\pi(s)\right] d\mu(\pi) + \frac{dF(\alpha';\alpha^*)}{d\alpha'}\Big|_{\alpha'=\alpha} \tag{14}$$

$$-\alpha^* \int_{\Pi\backslash\Pi(f(\alpha'),g(\alpha'))} \left[\Sigma_{s\in S} u'(g(s) - r\alpha^{*2} y_{u,c}(g(s))) y_{u,c}(g(s))\,\pi(s)\right] d\mu(\pi) + \frac{G(\alpha';\alpha^*)}{d\alpha'}\Big|_{\alpha'=\alpha^*} = 0.$$

In the limit, as $r \to 0$, $f(\alpha^*) = f$, $g(\alpha^*) = g$. But,

$$u'(f(s)) y_{u,c}(f(s)) = u'(g(s)) y_{u,c}(g(s)) = c, \ \forall s \in S, \tag{15}$$

and

$$\lim_{r\to 0}\frac{dF\left(\alpha';\alpha^{*}\right)}{d\alpha'}\mid_{\alpha'=\alpha^{*}}=\Sigma_{s\in S}u(f\left(s\right))\hat{\pi}\left(s\right)=\Sigma_{s\in S}u\left(g\left(s\right)\right)\hat{\pi}\left(s\right)=\lim_{r\to 0}\frac{G\left(\alpha';\alpha^{*}\right)}{d\alpha'}\mid_{\alpha'=\alpha^{*}}, \tag{16}$$

where $\hat{\pi}$ is in the intersection of the boundaries of $\Pi\left(f,g\right)$ and $\Pi\backslash\Pi\left(f,g\right)$.[13] Hence (14) implies that, in the limit as r tends to zero,

$$(1-\alpha^{*})\mu\left(\Pi\left(f,g\right)\right)=\alpha^{*}\mu\left(\Pi\backslash\Pi\left(f,g\right)\right). \tag{17}$$

Thus, $\lim_{r\to 0}\alpha^{*}\left(r;f,g\right):=\alpha^{*}=\mu\left(\Pi\left(f,g\right)\right).$ ■

[13] That such $\hat{\pi}$ exist follows from the non-comparability of f and g.

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