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Irresolute choice behavior

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Abstract

This paper proposes a model of irresolute choice rationalizing random choice behavior and examines its applications to decision making under certainty, uncertainty, and risk. Depending on the context, the representations feature canonical signal spaces. Decisions are governed by random draws of signals generating stochastic choice functions. Application to portfolio selection and experimental testing are discussed.

KEYWORDS

incomplete preferences, Knightian uncertainty, multiprior expected multiutility representations, multiutility representations, random choice

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1 | INTRODUCTION

Random choice behavior is the phenomenon of decision makers, facing repeated decisions under similar conditions, displaying choice patterns that are best depicted by probability distributions over the set of feasible alternatives. Such behavior may reflect the decision makers incomplete preferences due to lack of ability to compare the alternatives. According to von Neumann and Morgenstern (1947), "It is conceivable—and even in a way more realistic—to allow for cases where the individual is neither able to state which of two alternatives he prefers nor that they are equally desirable."[1]

[1]Leonard Savage broached the appropriateness of the postulate that all alternatives are readily comparable in a letter, dated March 25, 1958, to Karl Popper in which he discusses his work on the choice-based foundations of subjective probabilities. "There is, though," Savage wrote "a postulate that insists that economic situations can be ranked in a

Depending on the context, this noncomparablity may be due to the complexity of the feasible alternatives or, for lack of experience, the inability to assess their, potentially long-run, consequences. A topical example is the decision of whether or not to vaccinate against COVID-19, and whether, when, and how often to accept a booster shot.

Another source of incompleteness is variations in the decision maker's tastes or beliefs. Consider, for instance, a decision maker who faces repeated choices between dining in Indian and Chinese restaurants. It is conceivable that, even if in every instance the alternatives are comparable and the decision maker exhibits decisiveness, the observed pattern is that an Indian restaurant was chosen 70% of the times and the Chinese restaurant 30% of the times. This choice pattern may be accounted for by factors such as variations of taste for food that are not privy to an outside observer. If asked to express preferences for dining on Indian or Chinese food a week away, a decision maker may find it difficult to respond except probabilistically, as the choice depends on the taste for food when the time comes.

When the feasible alternatives are noncomparable, decision makers display indecisiveness (e.g., procrastination, hesitation, and irresolute choice). Alternative ideas have been proposed regarding the resolution of the indecisiveness. Bewley (2002) suggests that if among the noncomparable alternatives there is one that may be regarded as the status quo, or default, alternative, it is chosen.[2] Danan (2010) analyzes the implications of choice behavior that invokes deliberate randomization.[3] Evren et al. (2019) model choice behavior based on the secondary criterion of the top cycle among all undominated alternatives in the feasible set relative to a complete and transitive binary relation.

This paper addresses the same issue by proposing a new approach, dubbed *irresolute choice model* (henceforth ICM). Taking preference relations on choice sets as a primitive concept and departing from the completeness postulate, the model characterizes random choice behavior between noncomparable alternatives by a collection of nested partial orders each depicting different choice probabilities. The idea that stochastic choice is related to incomplete preferences may be traced to Luce (1959). However, the ICM is very different from, and may be regarded as an alternative to, Luce's model.[4]

The literature offers a variety of axiomatic models characterizing the representations of incomplete preferences under certainty (Evren & Ok, 2011; Ok, 2002), risk (Baucells & Shapley, 2008; Dubra et al., 2004), and uncertainty (Bewley, 2002; Galaabaatar & Karni, 2013; Nau, 2006; Ok et al., 2012; Riella, 2015; Seidenfeld et al., 1995). Unlike the case of complete preferences, in which the representation characterizes choice behavior (i.e., the alternative that commands the highest representation value is chosen), in the case of incomplete preferences the representations do not, in general, characterize the choice behavior. The main objective of this paper is to formally connect the representations of incomplete preferences to choice behavior.

linear order by the subject, and I freely admit that this seems to me to be a source of much difficulty in my theory. This stringent postulate is in conflict with the common experience of vagueness and indecision, and if I knew a good way to make a mathematical model of those phenomena, I would adopt it, but I despair of finding one." (see Zappia, 2020). Danan and Ziegelmeyer (2006), Sautua (2017), Cettolin and Riedl (2019), and Halevy et al. (2023) provide evidence of the prevalence of incomplete preferences in experimental settings.

[2]See also Masatlioglu and Ok (2005).

[3]Further discussion of Danan's work appears in Section 5.

[4]Further discussion of the relation between the model of this paper and Luce's model and its extensions will be easier to follow after the exposition of the ICM and is therefore discussed in Section 5.

The underlying premise of this work is that when facing a choice among noncomparable alternatives, a decision maker's actions are triggered by impulses, or signals, that are inherently random, or appear to be random to an observer who is not privy to the workings of the decision maker's mind. In either case, insofar as the observer is concerned, the decision maker's choices appear to be random. Therefore, as in Luce's (1959) model, the primitive data to be rationalized are probability distributions over the sets of feasible alternatives. I propose a general framework within which representations of *probabilistic choice behavior* are obtained that depend on the context (i.e., whether the decision problem is under certainty, risk, or uncertainty).

The main novelty of this work is the approach to modeling random choice behavior, which is more conceptual than technical. The incompleteness of the preference relation is modeled as a continuum of strict partial orders on the relevant choice sets depicting the binary relations "one alternative is strictly preferred over another with probability that is at most $\alpha \in [0, 1]$." These strict partial orders are linked by a monotonicity requirement. The results are characterizations of probabilistic choice representations.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 applies the model to decision making under certainty, emphasizing the methodological approach, and highlights the implication of applying the same methodology to decision making under uncertainty. Section 4 discusses some behavioral implications of the model. Section 5 provides concluding remarks and a brief review of the relevant literature.

2 | IRRESOLUTE CHOICE

2.1 | Preliminaries

Let A denote a *choice set*. Elements of A are *alternatives*. Denote by $\succ$ irreflexive and transitive binary relation on A, dubbed *strict preference relation*. For any alternatives $a, a' \in A$, $a \succ a'$ is the proposition that, facing a choice between these two alternatives, a decision maker characterized by $\succ$ chooses the alternative a, always. This behavior has the usual interpretation that a is *strictly preferred* over a' or, equivalently, that a' is *dominated* by a. I assume throughout that $\succ$ on A is nonempty.

The strict preference relation, $\succ$, induces the following derived binary relations on A. For all $a, a' \in A$,

(a) The *weak preference relation*, $\succcurlyeq$, is defined by: $a \succcurlyeq a'$ if, for all $a'' \in A$, $a'' \succ a$ implies that $a'' \succ a'$.[5] This is interpreted to mean that a' is *weakly dominated* by a.
(b) The *indifference relation*, $\sim$, is defined by $a \sim a'$ if $a \succcurlyeq a'$ and $a' \succcurlyeq a$. This is interpreted to mean that a' is weakly dominated by a, and a is weakly dominated by a'.
(c) The *noncomparability relation* $\bowtie$, is defined by: $a \bowtie a'$ if $\neg(a \succcurlyeq a')$ and $\neg(a' \succcurlyeq a)$. This is interpreted to mean that neither a' is weakly dominated by a nor a is weakly dominated by a'.
(d) The *negation of* $\succ$, denoted $\succeq$, is defined by $a \succeq a'$ if $\neg(a' \succ a)$.[6] This is interpreted to mean that a is not dominated by a'.

[5]Clearly, $a \succ a'$ implies that $a \succcurlyeq a'$.

It is natural to suppose that if presented with a choice between two alternatives, a and a', a decision maker would choose the former act if $a \succcurlyeq a'$ and $\neg(a' \succcurlyeq a)$. However, if $a \bowtie a'$ or $a' \sim a$, then the preference relation does not indicate which of the two alternatives will be chosen. Moreover, since $\succcurlyeq \supseteq \bowtie \cup \sim$, $a \succcurlyeq a'$ does not imply that a will be chosen from the subset $\{a, a'\}$.

2.2 | Irresolute choice model

The basic premise of this work is that, facing a choice between noncomparable or indifferent alternatives, the decision maker behaves *as if* he is awaiting a signal that would resolve his indecision and, thereby, determine his choice. The signal is presumed to be generated by a stochastic process whose nature is not specified extraneously but has the following behavioral expression. The decision maker procrastinates while waiting for the arrival of a signal and then chooses in a manner that reflects the underlying randomness of the signal-generating process.

For example, the underlying process may have the structure of the drift-diffusion model, in which procrastination is measured by the response time.[7] Another example, that I refer to as mental decoy, maintains that facing a choice between noncomparable alternatives, say a and a', the decision maker behaves "as if" a third alternative, $a'' \in A$, is randomly selected and serves as a reference point to resolve his indecision. If the third alternative is weakly inferior to a and is noncomparable or weakly preferred to a' then the decision maker chooses the alternative a and if it is inferior to a' and noncomparable or weakly preferred to a then the decision maker chooses a'. Otherwise, the decision maker procrastinates while waiting for another mental decoy to be randomly selected that would allow him to resolve his indecision. The intuition behind the mental decoy idea is provided by the well-known "decoy effect" in consumer decisions. The decoy effect pertains to a pattern of choice behavior according to which, when facing a choice between two products that have multiple attributes, but are noncomparable in the sense that neither product has more of all the desirable attributes than the other, the introduction of a third product that is dominated (in the sense of having less of the desirable attributes) by one of the existing products but not by another, tilts the consumer choice towards the dominating product. A third alternative dominated by both products does not affect the choice behavior and does not produce a significant shift in market share.[8] The mental decoy captures the same idea with strict preference instead of attributewise domination and random selection of the decoy alternative. Whichever is the signal-generating process, to the outside observer, the decision maker displays stochastic choice behavior.

To formalize this idea, let $K := \{\succ^{\alpha} | \, \alpha \in [0, 1]\}$ be a set of irreflexive and transitive binary relations on A, dubbed *stochastic choice relations*. Assume that the binary relations in K are *nested*. Formally, for all $\alpha' < \alpha$, $\succ^{\alpha} \subset \succ^{\alpha'}$. Equivalently, for all $\alpha' < \alpha$ and $a, a' \in A$, $a \succ^{\alpha} a'$ implies that $a \succ^{\alpha'} a'$. For each $\alpha \in [0, 1]$, the derived relations $\succcurlyeq^{\alpha}$, $\sim^{\alpha}$, $\bowtie^{\alpha}$, and $\succsim^{\alpha}$ are defined

[6]Note that $\succsim$ is reflexive but not necessarily transitive. The weak preference relation defined here was introduced in Galaabaatar and Karni (2013). Its significance and implications were investigated and discussed in Karni (2011), who showed that the relations $\succsim$ and $\succcurlyeq$ agree if and only if $\succ$ is negatively transitive and $\succcurlyeq$ is complete. The relation $\succ$ is not asymmetric part of $\succcurlyeq$. The indifference relation as is defined here, introduced in Galaabaatar and Karni (2013), is equivalent to that of Eliaz and Ok (2006).
[7]See, for example, Krajbich et al. (2014) and Baldassi et al. (2020).
[8]See Huber et al. (1982) and Ok et al. (2015).

as follows: $a \succcurlyeq^{\alpha} a'$ if, for all $a'' \in A$, $a'' \succ^{\alpha} a$ implies that $a'' \succ^{\alpha} a'$; $a \sim^{\alpha} a'$ if $a \succcurlyeq^{\alpha} a'$ and $a' \succcurlyeq^{\alpha} a$; $a \bowtie^{\alpha} a'$ if and only if $\neg(a \succcurlyeq^{\alpha} a')$ and $\neg(a' \succcurlyeq^{\alpha} a)$; $a \geqslant^{\alpha} a'$ if $\neg(a' \succ^{\alpha} a)$.

Let $I = (0, 1]$ and let $\mathcal{I}$ be the collection of half-open intervals in I. Formally, $\mathcal{I} = \{(\beta, \gamma] | 0 < \beta < \gamma \leq 1\}$. Denote by $\mathcal{B}$ the Borel field generated by $\mathcal{I}$ and let η be a Borel probability measure on $\mathcal{B}$. Then $(\mathcal{I}, \mathcal{B}, \eta)$ is a probability space. Corresponding to $\mathcal{I}$ define $\mathcal{K} = \{\succ^{\alpha} \in K | \alpha \in (\beta, \gamma]\}$.

Let $\mathcal{B}_0$ be the collection of subsets of $\mathcal{K}$ each of which is the union of a finite number of members of $\mathcal{K}$.[9] Then, $\mathcal{B}_0$ is a field generated by $\mathcal{K}$ and, in turn, it generates $\mathcal{B}$. If $I = [0, 1]$ then $\mathcal{B}_0$ is no longer a field since $I \notin \mathcal{B}_0$. Yet, $\mathcal{B}$ and η can be defined as before. The new $\mathcal{B}$ is generated by the old $\mathcal{B}$ and the singleton $\{0\}$.[10]

The *ICM* is the triplet $(K, \mathcal{K}, \eta)$. Given any $a, a' \in A$, the interpretation of $a \succ^{\alpha} a'$ is as follows: Facing a choice between the alternatives a and a', alternative a is strictly preferred and, hence, chosen, over a' with probability that is at most $\eta([0, \alpha])$ and a' is chosen with probability at least $1 - \eta([0, \alpha])$. For example, if η is the Borel–Lebesgue measure then $a \succ^{\alpha} a'$ means that alternative a is chosen, over a' with probability that is at most α.[11]

If $a \succcurlyeq^{\alpha} a'$ then $a \succ^{\alpha'} a'$ for all $\alpha' < \alpha$. Given any $a, a' \in A$, let $\bar{\alpha}(a, a') := \sup\{\alpha \in [0, 1] | a \succ^{\alpha} a'\}$.[12] Then $a \succcurlyeq^{\bar{\alpha}(a,a')} a'$ and $\eta([0, \bar{\alpha}(a, a')])$ is the exact probability that a is chosen from the set $\{a, a'\}$, and a' is chosen with probability $\bar{\alpha}(a', a) = 1 - \eta([0, \bar{\alpha}(a, a')])$. Clearly, $a \succ a'$ implies that $a \succcurlyeq^{1} a'$. Henceforth, to maintain consistency, I use the symbol $\succ^{1}$ instead of $\succ$ to denote the strict preference relation. Consistently with the interpretation of the probabilistic choice relations, $a \succcurlyeq^{1} a'$ implies that a is chosen from the set $\{a, a'\}$ with probability that is at least, and therefore equal to, one. If $a \sim^{1} a'$ then $a \sim^{\alpha} a'$, for all $\alpha \in [0, 1]$. Consequently, insofar as the probability of a chosen over a' is concerned, the model is silent.

The binary relations $\succ^{\alpha}$ are a new concept whose interpretation merits further elaboration. To begin with let us revisit the familiar notion of strict preference relation $\succ$. This relation may be interpreted as a parsimonious way of representing data of a decision maker's actual choices as seen by an outside observer.

Consider observing a decision maker facing a choice between two alternatives, say a and a', choosing the alternative a. Can an observer conclude that the decision maker prefers the alternative a over a'? Such conclusion, based on one data point, is not warranted. However, if facing the same choice repeatedly, the decision maker is observed to choose a over a' consistently, the confidence that his choices do, in fact, reflect a preference of a over a' increases. It also increases the confidence in predicting that the next time around, facing the same choice, the decision maker will choose the alternative a. The binary relation $\succ$ is a formal way of representing this choice pattern.

By the same logic, observing a decision maker who faces the same repeated choice, choosing the alternative a $\eta([0, \bar{\alpha}(a, a')])$ percent of the time and chooses the alternative a' $(1 - \eta([0, \bar{\alpha}(a, a')]))$ percent of the times, warrants the conclusion that this choice pattern

[9]A typical set $B \in \mathcal{B}_0$ is

$$B = \cup_{i=1}^{n} \{\succ^{\alpha} \in \mathcal{K} | \alpha \in (\beta_i, \gamma_i]\}, \text{ where } \beta_1 < \gamma_1 < \beta_2 < \gamma_2 < \cdots < \beta_n < \gamma_n.$$

[10]See Chung (1974, pp. 22–23).

[11]Equivalently, alternative a' is strictly preferred over a with probability that is at most $1 - \eta([0, \alpha])$.

[12]That the supremum exists follows from the fact that the set is bounded and that $\neg(a' \sim a)$ implies that there is $\alpha' \in [0, 1]$ such that $a \succ^{\alpha'} a'$. Hence, the set is nonempty.

is a manifestation of some underlying (unspecified) stochastic process. The interpretation that a is preferred over a' with probability $\eta([0, \bar{\alpha}(a, a')])$ is an efficient way of summarizing the data and predicting the probability that the next time around facing the choice from the set $\{a, a'\}$ the decision maker will choose the alternative a. The binary relation $\succ^{\bar{\alpha}(a,a')}$ is a formal way of representing this pattern of (random) choice behavior. A concrete example may help clarify the meaning of the ICM. Consider the binary set $\{a, a'\}$ and suppose that η is the Borel–Lebesgue probability measure (i.e., $\eta([0, \bar{\alpha}(a, a')]) = \bar{\alpha}(a, a')$) then $\bar{\alpha}(a, a') = 2/3$ (i.e., the probability that a is chosen from the set $\{a, a'\}$ is 2/3). According to the ICM the implication is that $a \succ^{\alpha} a'$ according to all the probabilistic choice relations in the set $\{\succ^{\alpha} | \alpha \in [0, 2/3]\}$ and $a' \succ^{\alpha} a$ according to all the probabilistic choice relations in the complementary set $\{\succ^{\alpha} | \alpha \in (2/3, 1]\}$.[13]

By definition, $a \succcurlyeq^{\alpha} a'$ if and only if $\neg(a' \succ^{\alpha} a)$. Hence, the ICM may be equivalently stated as a set of binary relations $\bar{K} := \{\succcurlyeq^{\alpha} | \alpha \in [0, 1]\}$ on A with the corresponding Borel field and probability measure, $\bar{\eta}$. Note that $\neg(a' \succ^{\alpha} a)$ means that the statement "a' is chosen over a with probability at least $\bar{\eta}([0, \alpha])$" is false. Hence, by the preceding argument, $\alpha \leq 1 - \bar{\eta}((0, \bar{\alpha}(a, a'))$. The interpretation of $a \succcurlyeq^{\bar{\alpha}(a,a')} a'$ is as follows: Facing the choice between alternatives a and a' such that $\neg(a \sim a')$, a is chosen with probability $\bar{\eta}((0, \bar{\alpha}(a, a'))$. Hence, $a \succcurlyeq^{\bar{\alpha}(a,a')} a'$ if and only if $a \succ^{\bar{\alpha}(a,a')} a'$ and the two statements of the model are in agreement. Note also that, since $\succ^{\alpha} \subset \succ^{\alpha'}$ for all $\alpha' < \alpha$, we have $\succcurlyeq^{\alpha} \subset \succcurlyeq^{\alpha'}$.

3 | IRRESOLUTE CHOICE BEHAVIOR

The ICM is a refinement of decision models that admit incomplete preferences; as such, it may be superimposed on the models of decision making under certainty, risk, or uncertainty with incomplete preferences. To analyze the behavioral implications of the ICM in these contexts, I superimpose the ICM on the relevant decision models. Specifically, this section provides a detailed analysis of the application of the ICM to decision making under certainty emphasizing the method and deriving the stochastic choice function it entails. Following that it outlines the results of applying the ICM to decision making under uncertainty.

3.1 | Axiomatic characterization of irresolute choice behavior under certainty

Let the choice set A be a nonempty topological space, and denote by $\succcurlyeq$ a preorder on A. For every $a \in A$, the upper and lower $\succcurlyeq$-contour sets of a are defined, respectively, by $\mathbb{U}_{\succcurlyeq}(a) = \{a' \in A | a' \succcurlyeq a\}$ and $\mathbb{L}_{\succcurlyeq}(a) = \{a' \in A | a \succcurlyeq a'\}$. The preorder $\succcurlyeq$ is *continuous* if $\mathbb{U}_{\succcurlyeq}(a)$ and $\mathbb{L}_{\succcurlyeq}(a)$ are closed, for all $a \in A$. A nonempty set $\mathcal{U}$ of real-valued functions on A is said to *represent* $\succcurlyeq$ if, for all $a, a' \in A$, $a \succcurlyeq a'$ if and only if $u(a) \geq u(a')$, for all $u \in \mathcal{U}$.

Let $\{\succ^{\alpha} | \alpha \in [0, 1]\}$ be a set of probabilistic choice relations on A, and $\{\succcurlyeq^{\alpha} | \alpha \in [0, 1]\}$ the corresponding model expressed in terms of the negations of $\succ^{\alpha}$. For each $\alpha \in [0, 1]$ the structure of $\succcurlyeq^{\alpha}$ is depicted, axiomatically, as follows:

[13]Another way of putting it, $a \succ^{\alpha} a'$ according to all the probabilistic choice relations in the set $\{\succ^{\alpha} | \alpha \in [0, 2/3]\}$ implies that $a' \succ^{1-\alpha} a$ according to all the probabilistic choice relations in the set $\{\succ^{1-\alpha} | \alpha \in [0, 1/3)\}$.

(P1) (*Partial preorder*). For each $\alpha \in [0, 1] \succcurlyeq^{\alpha}$ is transitive and reflexive.
(P2) (*Continuity*). For every $a \in A$ and $\alpha \in [0, 1]$, $\mathbb{U}_{\succcurlyeq^{\alpha}}(a)$ and $\mathbb{L}_{\succcurlyeq^{\alpha}}(a)$ are closed.

The representation of irresolute choice behavior requires that the stochastic choice relations in the set $\{\succcurlyeq^{\alpha} | \, \alpha \in [0, 1]\}$ be linked. The next axiom provides this link.

(P3) (*Monotonicity*). For all $\alpha, \alpha' \in [0, 1]$, $\succcurlyeq^{\alpha} \subseteq \succcurlyeq^{\alpha'}$ if and only if $\alpha' \leq \alpha$.

Lemma 1. *The irresolute choice model* $\{\succcurlyeq^{\alpha} | \, \alpha \in [0, 1]\}$ *satisfies monotonicity if and only if, for every* $a \in A$, $\mathbb{U}_{\succcurlyeq^{\alpha}}(a) \subseteq \mathbb{U}_{\succcurlyeq^{\alpha'}}(a)$ *if and only if* $\alpha' \leq \alpha$.

Proof. Monotonicity is equivalent to the proposition, for all $a, a' \in A$, $a' \succcurlyeq^{\alpha} a$ implies that $a' \succcurlyeq^{\alpha'} a$ if and only if $\alpha' \leq \alpha$. The last statement is equivalent to the proposition, for all $a \in A$, $\mathbb{U}_{\succcurlyeq^{\alpha}}(a) \subseteq \mathbb{U}_{\succcurlyeq^{\alpha'}}(a)$ if and only if $\alpha' \leq \alpha$. □

The following theorem extends Evren and Ok (2011) Corollary 1, to include irresolute choice behavior.[14]

Theorem 1. *Let* A *be a locally compact separable metric space and* $\{\succcurlyeq^{\alpha} | \alpha \in [0, 1]\}$ *be binary relations on* A. *Then, the following conditions are equivalent:*

(i) *For every* $\alpha \in [0, 1]$, $\succcurlyeq^{\alpha}$ *satisfies* (*P*1) *and* (*P*2) *and jointly* $\succcurlyeq^{\alpha}$, $\alpha \in [0, 1]$, *satisfy* (*P*3).
(ii) *There exists a collection* $\{\mathcal{U}^{\alpha} | \alpha \in [0, 1]\}$ *of real-valued, continuous, strictly* $\succcurlyeq^{\alpha}$-*increasing, functions such that, for every* $\alpha \in [0, 1]$, $\mathcal{U}^{\alpha}$ *represents* $\succcurlyeq^{\alpha}$, *and* $\alpha \geq \alpha'$ *if and only if* $\mathcal{U}^{\alpha} \supseteq \mathcal{U}^{\alpha'}$.

Proof. (Sufficiency). Suppose that A is a locally compact separable metric space and $\{\succcurlyeq^{\alpha} | \, \alpha \in [0, 1]\}$ be binary relations on A satisfying (P1) and (P2) then, by Evren and Ok (2011) Corollary 1, for each $\alpha \in [0, 1]$, there exists a set $\mathcal{U}^{\alpha}$ of real-valued, continuous, functions representing $\succcurlyeq^{\alpha}$ and every $u \in \mathcal{U}^{\alpha}$ is strictly $\succcurlyeq^{\alpha}$-increasing. Let $\mathcal{U}^{\alpha}$ be the set of all (continuous) real functions u such that $a \succcurlyeq^{\alpha} a'$ implies $u(a) \geq u(a')$ and $\mathcal{U}^{\alpha'}$ be the set of all continuous real functions u such that $a \succcurlyeq^{\alpha'} a'$ implies $u(a) \geq u(a')$. Then $\succcurlyeq^{\alpha} \subseteq \succcurlyeq^{\alpha'}$ if and only if $u \in \mathcal{U}^{\alpha'}$ then $u \in \mathcal{U}^{\alpha}$. Thus, $\mathcal{U}^{\alpha'} \subseteq \mathcal{U}^{\alpha}$. By the representation, $\mathbb{U}_{\succcurlyeq^{\alpha'}}(a) \supseteq \mathbb{U}_{\succcurlyeq^{\alpha}}(a)$ if and only if $\mathcal{U}^{\alpha} \supseteq \mathcal{U}^{\alpha'}$. By (P3) and Lemma 1, $\alpha \geq \alpha'$ if and only if $\mathbb{U}_{\succcurlyeq^{\alpha'}}(a) \supseteq \mathbb{U}_{\succcurlyeq^{\alpha}}(a)$. Hence, $\alpha \geq \alpha'$ if and only if $\mathcal{U}^{\alpha} \supseteq \mathcal{U}^{\alpha'}$.

(Necessity) Assume that (ii) holds. Corollary 1 of Evren and Ok (2011) implies that, for every $\alpha \in [0, 1]$, $\succcurlyeq^{\alpha}$ satisfies (P1) and (P2). Suppose that $\alpha' \leq \alpha$ if and only if $\mathcal{U}^{\alpha} \supseteq \mathcal{U}^{\alpha'}$.

[14]Other results of Evren and Ok (2011), including their Theorem 1 and Corollaries 2 and 3, may be extended in the same way.

By the representation, $\mathcal{U}^{\alpha} \supseteq \mathcal{U}^{\alpha'}$ if and only if $\mathbb{U}_{\succcurlyeq^{\alpha'}}(a) \supseteq \mathbb{U}_{\succcurlyeq^{\alpha}}(a)$. Hence, $\alpha' \leq \alpha$ if and only if $\mathbb{U}_{\succcurlyeq^{\alpha'}}(a) \supseteq \mathbb{U}_{\succcurlyeq^{\alpha}}(a)$, for all $a \in A$, which, by Lemma 1 is equivalent to (P3). □

The uniqueness of the representation is as follows: Given any nonempty subset $\mathcal{U}^{\alpha}$ of $\mathbb{R}^{A}$, define the map $\Upsilon_{\mathcal{U}^{\alpha}} : A \to \mathbb{R}^{\mathcal{U}^{\alpha}}$ by $\Upsilon_{\mathcal{U}^{\alpha}}(a)(u) := u(a)$. Two nonempty subsets $\mathcal{U}^{\alpha}$ and $\mathcal{V}^{\alpha}$ of continuous real-valued functions on A represent the same preorder if, and only if, there exists an $f : \Upsilon_{\mathcal{U}^{\alpha}}(A) \to \Upsilon_{\mathcal{V}^{\alpha}}$ such that $(i)\Upsilon_{\mathcal{V}^{\alpha}} = f(\Upsilon_{\mathcal{U}^{\alpha}})$; and (ii) for every $b, c \in \Upsilon_{\mathcal{U}^{\alpha}}(A)$, $b > c$ if and only if $f(b) > f(c)$.[15]

3.2 | The indifference relation

The case in which the alternatives under consideration belong to the same indifference class requires special attention. By definition, $a \sim^{1} a'$ if and only if $a \succcurlyeq^{1} a'$ and $a' \succcurlyeq^{1} a$.

Lemma 2. *For all $a, a' \in A$, $a \succcurlyeq^{1} a'$ if and only if $u(a) \geq u(a')$, for all $u \in \mathcal{U}^{1}$.*

Proof. By definition $a \succcurlyeq^{1} a'$ if $\hat{a} \succ^{1} a$ then $\hat{a} \succ^{1} a'$, for all $\hat{a} \in A$. Hence, by definition, $a' \succcurlyeq^{1} a''$ implies that $a \succcurlyeq^{1} a''$. By Theorem 1, this is equivalent to $u(a') \geq u(a'')$ implying that $u(a) \geq u(a'')$, for all $u \in \mathcal{U}^{1}$. Consider a sequence $\left(a''_n\right) \subset A$ such that $a' \succcurlyeq^{1} a''_n$ for $n = 1, 2, \ldots$ and $a' = \lim_{n\to\infty} a''_n$. This is equivalent to $u(a') \geq u\left(a''_n\right)$ for $n = 1, 2, \ldots$ and, by the continuity of u, $u(a') = \lim_{n\to\infty} u\left(a''_n\right)$, for all $u \in \mathcal{U}^{1}$. Moreover, $a \succcurlyeq^{1} a''_n$, $n = 1, 2, \ldots$, which is equivalent to $u(a) \geq u\left(a''_n\right)$, $n = 1, 2, \ldots$ and $u(a) \geq \lim_{n\to\infty} u\left(a''_n\right) = u(a')$, for all $u \in \mathcal{U}^{1}$. Hence, by Theorem 1, $a \succcurlyeq^{1} a'$ if and only if $u(a) \geq u(a')$, for all $u \in \mathcal{U}^{1}$. □

By definition of $\sim^{1}$ and Lemma 2, $a \sim^{1} a'$ if and only if $u(a) = u(a')$, for all $u \in \mathcal{U}^{1}$. By Theorem 1, $\mathcal{U}^{\alpha} \subseteq \mathcal{U}^{1}$ for all $\alpha \in [0, 1]$. Thus, $a \sim^{1} a'$ implies that $a \sim^{\alpha} a', a \sim^{\alpha'} a'$, for all $\alpha, \alpha' \in [0, 1]$. Consequently, the ICM is silent with regard to the probability of selection of any alternatives belonging to the same indifference class.

3.3 | Canonical signal space and random choice

The premise underlying the stochastic choice behavior depicted by the ICM is that choices between noncomparable or indifferent alternatives are governed by unspecified random signal-generating process. For example, if the process is drift-diffusion the signal corresponds to the cumulative process reaching the threshold of confidence.

[15]See Evren and Ok (2011). Note that, in general, for arbitrary multiutility representations, $\mathcal{V}^{\alpha}$ and $\mathcal{V}^{\alpha'}$, of two preorders, $\succcurlyeq^{\alpha}$ and $\succcurlyeq^{\alpha'}$, such that $\succcurlyeq^{\alpha} \subset \succcurlyeq^{\alpha'}$ does not imply that $\mathcal{V}^{\alpha} \subset \mathcal{V}^{\alpha'}$.

Given $\succcurlyeq^{\alpha}$ the chance with which the subject will choose a over a' when facing the choice from the set $\{a, a'\}$ does not depend on the representation. In other words, if $\mathcal{U}^{\alpha}$ and $\mathcal{V}^{\alpha}$ are two representations of $\succcurlyeq^{\alpha}$ then the functions in $\mathcal{V}^{\alpha}$ are given by the uniqueness of the representation.

Consider the choice between two alternatives, a and a' such that $\neg(a \sim a')$, then the probability of a signal that would resolve the indecision in favor of a is $\eta([0, \bar{\alpha}(a, a')])$. Let $p(a, \{a, a'\})$ denote the probability of choosing the alternative a from the set $\{a, a'\}$. Then, $p(a, \{a, a'\}) = \eta([0, \bar{\alpha}(a, a')])$, for all $a, a' \in A$.[16] By the representation of the ICM this is the case if and only if $u(a) \geq u(a')$, for all $u \in \mathcal{U}^{\bar{\alpha}(a,a')}$.

Given an ICM $(K, \mathcal{K}, \eta)$, let $\mathcal{U}^1$ be the set of utility functions representing $\succcurlyeq^1$. Corresponding to the collection of half-open intervals $\mathcal{I}$ in the unit interval, define a collection of "intervals" in $\mathcal{U}^1$. Formally, $\mathcal{C} = \{\mathcal{U}^\beta \backslash \mathcal{U}^\gamma | (\beta, \gamma] \in \mathcal{I}\}$. Let $\mathcal{C}_0$ be the collection of subsets of $\mathcal{U}^1$ each of which is the union of a finite number of members of $\mathcal{C}$. Then, $\mathcal{C}_0$ is a field generated by $\mathcal{C}$ and, in turn, it generates $\mathcal{B}$, the Borel field generated by $\mathcal{I}$. Then $(\mathcal{U}^1, \mathcal{C}, \upsilon)$, where υ is the Borel probability measure on $\mathcal{B}$.

By definition, $\upsilon(\mathcal{U}^0) = 0, \upsilon(\mathcal{U}^1) = 1$, and $\upsilon(\mathcal{U}^\alpha) \geq \upsilon(\mathcal{U}^{\alpha'})$, for all $\alpha \geq \alpha'$. Then, for all $a, a' \in A, p(a, \{a, a'\}) = \upsilon(\mathcal{U}^{\bar{\alpha}(a,a')})$. In other words, if $\neg(a \sim a')$, the decision maker behaves *as if* a function u is selected from $\mathcal{U}^1$ according to a probability measure η and a is chosen if $u \in \mathcal{U}^{\bar{\alpha}(a,a')}$ and a' is chosen if $u \in \mathcal{U}^1 \backslash \mathcal{U}^{\bar{\alpha}(a,a')}$. Therefore, the set $\mathcal{U}^1$ may be taken to be the *canonical signal space*. Since $p(a, \{a, a'\}) = \eta([0, \bar{\alpha}(a, a')])$, for all $a, a' \in A$ we have that $\upsilon(\mathcal{U}^{\bar{\alpha}(a,a')}) = \eta([0, \bar{\alpha}(a, a')])$.

It is worth underscoring that if $\bar{\alpha}(a', a) = 0$ then there is no $u \in \mathcal{U}^0$ such that $u(a') > u(a)$. To grasp this, consider two alternatives, $a, a' \in A$ such that $\bar{\alpha}(a', a) = 0$. Since $\bar{\alpha}(a', a) = 0$ if and only if $\bar{\alpha}(a, a') = 1$, by Theorem 1, $u(a) \geq u(a')$, for all $u \in \mathcal{U}^1$. But $\mathcal{U}^0 \subseteq \mathcal{U}^1$ implies that for no $u \in \mathcal{U}^0$ it holds that $u(a') > u(a)$.

3.4 | Stochastic choice

Many decision problems require the decision maker to choose an alternative from a finite set of feasible alternatives that include more than two elements. To see how the ICM may be applied to choice from such sets, consider the following adaptation of the model.[17]

Let $M \subset A$ be a feasible set of alternatives and, to simplify the exposition, suppose that no two alternatives in M belong to the same indifference class. An alternative $a \in M$ is said to be *dominated* if for no $\alpha \in [0, 1]$ it holds that $a \succcurlyeq^\alpha a'$, for all $a' \in M \backslash \{a\}$. Let $D(M)$ denote the subset of dominated alternatives in M and let $UD(M) = M \backslash D(M)$ denote the subset of *undominated* alternatives in M.[18] Note that $UD(M)$ is nonempty.

Let $UD(M) = \{a_1, ..., a_m\}$ and, for each $a_i \in UD(M)$ define $\Lambda_i(M) = \{\alpha \in [0, 1] | \, a_i \succcurlyeq^\alpha a_j, \forall \, a_j \in UD(M) \backslash \{a_i\}\}$. In words, $\Lambda_i(M)$ is the set of indices designating the stochastic choice relations that rank the alternative a_i (weakly) higher than any other undominated alternative in the menu M. Define $\underline{\alpha}(a_i; M) = \inf \Lambda_i(M)$ and $\bar{\alpha}(a_i; M) = \sup \Lambda_i(M)$.[19] By definition, $\underline{\alpha}(a_i; M)$ and $\bar{\alpha}(a_i; M)$ are the indices of the probabilistic choice relations such that $\succcurlyeq^{\bar{\alpha}(a_i;M)} \subseteq \succcurlyeq^\alpha \subseteq \succcurlyeq^{\underline{\alpha}(a_i;M)}$, for all $\alpha \in \Lambda_i(M)$.

[16] In terms of the mental-decoy generating process $p(a, \{a, a'\})$ is the probability that the decoy alternative, a'', is weakly inferior to a and noncomparable or weakly preferred to a'.

[17] A full characterization of the relationships between irresolute choice behavior and stochastic choice functions is beyond the scope of this paper and is provided in Karni (2023).

[18] Formally, an alternative, $a \in M$ is undominated if, for some $\alpha \in [0, 1]$, $a \succcurlyeq^\alpha a'$, for all $a' \in M \backslash \{a\}$.

[19] That the infimum and supremum exist follows from the facts that the set $\Lambda_i(M)$ is bounded and, because a_i is undominated, $\Lambda_i(M)$ nonempty.

Without loss of generality rearrange the elements of $UD(M)$ in an ascending order of set inclusion (i.e., $\succcurlyeq^{\underline{\alpha}(a_1;M)} \subseteq \succcurlyeq^{\underline{\alpha}(a_2;M)} \subseteq, ..., \subseteq \succcurlyeq^{\underline{\alpha}(a_m;M)}$). If $\underline{\alpha}(a_1; M) = 1$ then, by Theorem 1, a_1 is the only element of the undominated set and $D(M) = M \backslash \{a_1\}$. In general, we have $1 > \underline{\alpha}(a_1; M) > \underline{\alpha}(a_2; M) >, ..., > \underline{\alpha}(a_{m-1}; M) > \underline{\alpha}(a_m; M) = 0$.[20]

Let $J_1 := [1, \underline{\alpha}(a_1; M)]$ and $J_i := (\underline{\alpha}(a_i; M) - \underline{\alpha}(a_{i+1}; M)], i = 1, ..., m-1$. Then, $\mathcal{J} := \{J_1, ..., J_{m-1}\}$ is a partition of the unit interval. Corresponding to $\mathcal{J}$ define a partition of $\mathcal{U}^1$ as follows: Let $P_1(M) := \{u \in \mathcal{U}^1 | \, u \in \mathcal{U}^{\underline{\alpha}(a_1;M)}\}$, $P_i(M) := \{u \in \mathcal{U}^1 | \, u \in \mathcal{U}^{\underline{\alpha}(a_{i+1};M)} \backslash \mathcal{U}^{\underline{\alpha}(a_i;M)}\}$, $i = m-1, ..., 2$, and $P_m(M) := \{u \in \mathcal{U}^1 | u \in \mathcal{U}^1 \backslash \mathcal{U}^{\underline{\alpha}(a_{m-1};M)}\}$.[21] Then, $\alpha \in \Lambda_i(M)$ if and only if, for all $u \in P_i$, $u(a_i) \geq u(a_j)$, for all $\forall \, a_j \in M \backslash \{a_i\}$.

A *stochastic choice function* is a function p that, for every nonempty subset M of A, returns a probability distribution $p(M)$ over M. The probability of choosing a from the set M is denoted $p(a, M)$. Since $\mathcal{U}^1$ is the canonical signal space, the probability of receiving a signal $u \in P_i$ is: $\eta(\mathcal{U}^{\underline{\alpha}(a_{i+1};M)}) - \eta(\mathcal{U}^{\underline{\alpha}(a_i;M)})$, $i = 1, ..., m$. The stochastic choice function is said to be *induced by the ICM* if there is a function $c : 2^A \backslash \varnothing \to 2^A \backslash \varnothing$ given by $c(M) = UD(M)$ and, for all $M \subseteq A$,

$$p(a_i, M) = \begin{bmatrix} \eta\left(\mathcal{U}^{\underline{\alpha}(a_{i+1};M)}\right) - \eta\left(\mathcal{U}^{\underline{\alpha}(a_i;M)}\right) & \text{if } a_i \in UD(M), \\ 0 & \text{if } a_i \notin UD(M). \end{bmatrix}$$

Thus, when facing a choice form M, the decision maker behaves *as if* a utility function $u \in \mathcal{U}^1$ is selected according to the probability measure η and the *undominated alternative*, a_i is chosen if $u \in \mathcal{U}^{\underline{\alpha}(a_{i+1};M)} \backslash \mathcal{U}^{\underline{\alpha}(a_i;M)}$, $i = 1, ..., m-1$.

3.5 | Irresolute choice behavior under uncertainty

To explore the application of the ICM to subjective expected utility theory, I invoke the model of Galaabaatar and Karni (2013). This model admits incomplete beliefs and tastes. It includes Bewley's Knightian uncertainty model (i.e., complete tastes and incomplete beliefs) and the subjective expected multiutility model (i.e., complete beliefs and incomplete tastes) as special cases.

The analytical framework is that of Anscombe and Aumann (1963) in which the choice set, $H := \Delta(X)^S$, consists of all the mappings from a (finite) state space, S, to a set $\Delta(X)$ whose elements are probability distributions on a finite set, X, of prizes.

Assume that the choice set H is *bounded* (i.e., there exist $\bar{h}$ and $\underline{h}$ in H such that $\bar{h} \succ^1 h \succ^1 \underline{h}$, for all $h \in H - \{\bar{h}, \underline{h}\}$) and let $\{\succ^\alpha | \alpha \in [0, 1]\}$ be stochastic choice relations on H. For each $\alpha \in [0, 1]$, let $\mathcal{U}^\alpha$ be a nonempty closed set of real-valued functions on X and, for every $u \in \mathcal{U}^\alpha$, let $\Pi^\alpha(u)$ be a nonempty closed set in $\mathbb{R}^{|S|}$. Define $\Phi^\alpha = \{(\pi, u) | u \in \mathcal{U}^\alpha, \pi \in \Pi^\alpha(u)\}$. Then $\{\Phi^\alpha | \alpha \in [0, 1]\}$ is said to *represent* the ICM $\{\succ^\alpha | \alpha \in [0, 1]\}$ if the following conditions hold:

(a) For all $h \in H$ and $(\pi, u) \in \Phi^1$,

[20]By definition, $\bar{\alpha}(a_1; M) = 1$, and $\bar{\alpha}(a_i; M) = \underline{\alpha}(a_{i-1}; M)$, for all $i = 2, ..., m$.

[21]Since indifference is not allowed, there is no ambiguity with regard to which element of the partition each utility function belongs to.

$$\sum_{s\in S}\pi(s)\sum_{x\in X}\bar{h}(x,s)u(x) > \sum_{s\in S}\pi(s)\sum_{x\in X}h(x,s)u(x) > \sum_{s\in S}\pi(s)\sum_{x\in X}\underline{h}(x,s)u(x). \tag{1}$$

(b) For all $h, h' \in H$,

$$h \succ^{\alpha} h' \Leftrightarrow \sum_{s\in S}\pi(s)\sum_{x\in X}h(x,s)u(x) > \sum_{s\in S}\pi(s)\sum_{x\in X}h'(x,s)u(x), \forall\,(\pi,u)\in\Phi^{\alpha}. \tag{2}$$

By Theorem 1 of Galaabaatar and Karni (2013), the stochastic choice relations $\succ^{\alpha}$, $\alpha \in [0,1]$ are strict partial orders (i.e., transitive and irreflexive) satisfying the Archimedean (i.e., for all $f, g, h \in H$, if $f \succ^{\alpha} g$ and $g \succ^{\alpha} h$ then there exist $\beta, \gamma \in (0,1)$ such that $\beta f + (1-\beta)h \succ^{\alpha} g$ and $g \succ^{\alpha} \gamma f + (1-\gamma)h$), independence (i.e., for all $f, g, h \in H$ and $\alpha \in (0,1]$, $f \succ^{\alpha} g$ if and only if $\alpha f + (1-\alpha)h \succ^{\alpha} \alpha g + (1-\alpha)h$), dominance (i.e., for all $h, g \in H$, and $\alpha \in [0,1]$, if $g \succ^{\alpha} h^{s}$ for every $s \in S$, then $g \succ^{\alpha} h$, where h^{s} the constant act that pays off $h(s)$ in every state) and monotonicity (i.e., for all $\alpha, \alpha' \in [0,1]$, $\succ^{\alpha} \subseteq \succ^{\alpha'}$ if and only if $\alpha' \leq \alpha$) axioms if and only if $\succ^{\alpha}$ is represented by (1) and (2), for all $\alpha \in [0,1]$ and $\alpha \geq \alpha'$ if and only if $\Phi^{\alpha} \supseteq \Phi^{\alpha'}$.

3.5.1 | Knightian uncertainty

The theory of subjective expected utility with incomplete preferences includes two special cases: the case in which the incompleteness is due solely to incomplete beliefs and the case in which it is due solely to incomplete tastes.

The case of incomplete beliefs was axiomatized by Bewley (2002), who dubbed it "Knightian uncertainty." Tastes completeness (i.e., unambiguous risk attitudes) requires that the restriction of the preference relation to constant acts exhibits negative transitivity. Let $\Delta(X) \subset H$ be identified with the subset of constant acts. The stochastic choice relations $\succ^{\alpha}$, $\alpha \in [0,1]$ have a structure depicted by the aforementioned axioms and, in addition, the preference relation $\succ^{1}$ displays negative transitivity if and only if $\succ^{\alpha}$ is represented by (1) and (2) with $\Phi^{\alpha} = \{u\} \times \Pi^{\alpha}$, where for each $\alpha \in [0,1]$, Π^{α} is a set of probability measures on S and $\alpha \geq \alpha'$ if and only if $\Pi^{\alpha} \supseteq \Pi^{\alpha'}$. Moreover, u is unique up to positive affine transformation, the closed convex hull of Π^{α} is unique and, for each $\pi \in \Pi^{\alpha}$, $\pi(s) > 0$ for all $s \in S$.

3.5.2 | Complete beliefs

For each event E, denote by rEq the act whose payoff is $r \in \Delta(X)$ for all $s \in E$ and $q \in \Delta(X)$ for all $s \in S\backslash E$. Denote by $r\gamma q \in \Delta(X)$ the constant act whose payoff in every state is $\gamma r + (1-\gamma)q$. A *bet* on an event E is the act rEq, whose payoffs satisfy $r \succ^{1} q$.[22]

Suppose that the decision maker considers the constant act $r\gamma q$ preferable to the bet rEq. Because the payoffs are the same, this preference indicates that he believes that γ exceeds the likelihood of E. This belief is said to be *coherent* if it holds that $r'\gamma q'$ is preferable to the bet $r'Eq'$ for all constant acts r' and q' such that $r' \succ^{1} q'$. By the same logic a preference of a bet rEq over

[22]By monotonicity, $r \succ^{1} q$ implies that $r \succ^{\alpha} q$, for all $\alpha \in [0,1]$.

the constant act $r\gamma q$ means that the decision maker believes the probability of E to exceed γ. A binary relation $\succ^1$ on H is said to exhibit *coherent beliefs* if, for all events E and $r, q, r', q' \in \Delta(X)$ such that $r \succ^1 q$ and $r' \succ^1 q'$, $r\gamma q \succ^1 rEq$ if and only if $r'\gamma q' \succ^1 r'Eq'$, and $rEq \succ^1 r\gamma q$ if and only if $r'Eq' \succ^1 r'\gamma q'$.

Coherent beliefs are complete if when facing a choice between a constant act $r\gamma q$ and a bet rEq the decision maker acts decisively. This idea is captured by the following axiom, which is due to Galaabaatar and Karni (2013). For all events E and $\gamma \in [0, 1]$, and constant acts r and q such that $r \succ^1 q$, either $r\gamma q \succ^1 rEq$ or $rEq \succ^1 r\gamma' q$, for every $\gamma > \gamma'$. To grasp the completeness of beliefs idea it is useful to think of the constant act $r\gamma q$ as a two-stage compound lottery that assigns, in the first stage, a probability γ to the lottery r and probability $(1 - \gamma)$ to the lottery q. The preference relation displays complete beliefs if, for all such pairs of two-stage compound lotteries, either $r\gamma q$ is strictly preferred over the corresponding bet rEq, indicating that γ exceeds the subjective probability the decision maker assigns to the event E, or the bet is strictly preferred over any two-stage lottery that assigns r a first-stage probability smaller than γ, indicating that the subjective probability the decision maker assigns to the event E exceeds any probability strictly smaller than γ. This condition leaves no gap, or open subset, in the unit interval representing subjective probabilities of E.

If the decision maker's beliefs are complete, then the incompleteness of the stochastic choice relations $\succ^\alpha$, $\alpha \in [0, 1]$, on H is due entirely to the incompleteness of his tastes. In this case, Galaabaatar and Karni (2013) Theorem 4 implies that the representations of the stochastic choice relations $\succ^\alpha$, $\alpha \in [0, 1]$ in (1) and (2) entail that $\Phi^\alpha = \mathcal{U}^\alpha \times \{\pi\}$ and $\alpha \geq \alpha'$ if and only if $\mathcal{U}^\alpha \supseteq \mathcal{U}^{\alpha'}$. Moreover, the probability measure, π, is unique and $\pi(s) > 0$, for all $s \in S$, and if $\mathcal{V}^\alpha$ is another set of real-valued functions on X that represent $\succ^\alpha$ in the sense of (2) then $\langle \mathcal{V}^\alpha \rangle = \langle \mathcal{U}^\alpha \rangle$, where $\langle \mathcal{U}^\alpha \rangle := cl\{con(\mathcal{U}^\alpha) + \{\theta \mathbf{1}_X\}_{\theta \in \mathbb{R}}\}$.[23]

4 | BEHAVIORAL IMPLICATIONS

Any theory that purports to describe natural or social phenomena must have clear testable predictions and implications. To render the proposed ICM meaningful, I describe briefly some of its behavioral implications in the context of a simple portfolio problem. I also describe experiments designed to test qualitative and quantitative predictions of the ICM, pointing out the kind of observations that would contradict it.

4.1 | A simple portfolio problem

Let there be two financial assets: a risk-free asset, whose rate of return is zero, and a risky asset whose rates of return are r_1 or r_2, in the states s_1 and s_2, respectively, where $r_1 > 0 > r_2$. Consider a risk-averse decision maker displaying Knightian uncertainty, and let the set of his subjective probabilities of state s_1 be $[\underline{\pi}, \bar{\pi}]$. Suppose that the decision maker's initial wealth is w_0, which he must allocate between the two assets. Denote by B the investment in the risky asset, which may be positive or negative depending on whether the decision maker buys or sells the risky asset. The decision maker's problem is to choose B.

[23] $\langle \mathcal{U}^\alpha \rangle$ denotes the closure, in $\mathbb{R}^{|X|}$, of the cone generated by $\mathcal{U}^\alpha$ and the constant real-valued functions on X.

According to the ICM, the decision is triggered by a signal $\pi \in [\underline{\pi}, \bar{\pi}]$ that induces a choice of B that maximizes $\pi u(w_0 + Br_1) + (1 - \pi)u(w_0 + Br_2)$. If the decision maker is sufficiently risk averse, there is a unique internal solution, denoted $B^*(\pi; r_1, r_2)$, given by the necessary and sufficient condition:

$$\pi u'(w_0 + B^*(\pi; r_1, r_2)r_1)r_1 + (1 - \pi)u'(w_0 + B^*(\pi; r_1, r_2)r_2)r_2 = 0.$$

Clearly, $B^*(\cdot; r_1, r_2)$ is a monotonic increasing function of π.

The prediction of the ICM is that the choice of B is random and is depicted by a cumulative distribution function $G(B^*(\pi; r_1, r_2)) = F(\pi)$, for all r_1, r_2. Moreover, a change in the rates of returns may induce a random change in the portfolio position triggered by a new signal $\pi' \in [\underline{\pi}, \bar{\pi}]$. Specifically, consider a decrease in the positive return from r_1 to r_1'. Define $\hat{\pi}$ by the equation

$$\hat{\pi}\, u'\Big(w_0 + B^*\Big(\pi'; r_1', r_2\Big)r_1'\Big)r_1' + (1 - \hat{\pi})u'\Big(w_0 + B^*\Big(\pi'; r_1', r_2\Big)r_2\Big)r_2 = 0.$$

Then decision maker chooses *to increase or decrease* the investment in the risky asset depending on whether π' is larger or smaller than $\hat{\pi}$. Specifically, $B^*\Big(\pi'; r_1', r_2\Big)$ $>(\leq)B^*(\pi; r_1, r_2))$ if and only if $\pi'(>) \leq \hat{\pi}$.

The random choice behavior described above is different from Bewley's dictum "if in doubt do nothing." Applied to the initial portfolio choice, Bewley's dictum predicts that the decision maker will choose to stay put, not buy, or sell the risky asset unless $\underline{\pi}\, r_1 + (1 - \underline{\pi})r_2 > 0$ or $\bar{\pi}\, r_1 + (1 - \bar{\pi})r_2 < 0$, respectively. Suppose that the decision maker invested in the risky asset (i.e., $B^*(\pi; r_1, r_2) > 0$). Then, unlike the prediction of the ICM, Bewley's dictum predicts that the decision maker displays inertia by not adjusting his portfolio position if the variations in the rate of return r_1' are in the range $\underline{r}_1 < r_1' < \bar{r}_1$, where $\bar{r}_1$ and $\underline{r}_1$ are defined, by $\underline{\pi}\, \bar{r}_1 + (1 - \underline{\pi})r_2 = 0$ and $\bar{\pi}\, \underline{r}_1 + (1 - \bar{\pi})r_2 = 0$, respectively.

4.2 | Experiments

Generally speaking, testing the proposed ICM requires that alternatives the decision maker considers to be noncomparable be identified and the agreement between the observed choices among such alternatives and the probabilistic choices predicted by the model evaluated. In the context of decision making under uncertainty, Karni and Vierø (2023) introduced incentive compatible schemes by which the incompleteness displayed by a preference relation may be elicited.

Under uncertainty monotonicity of the preference relations with respect to first-order stochastic dominance transcends individual idiosyncratic risk attitudes. Consequently, the multiprior expected multiutility model with incomplete preferences displays probabilistic choice monotonicity with respect to first-order stochastic dominance. Formally, if an act h first-order stochastically dominates an act g and f is noncomparable to either h or g, then the probability, $p(f, \{f, g\})$, that f is selected from the pair $\{f, g\}$ is greater than the probability, $p(f, \{f, h\})$, that it is selected from the pair $\{f, h\}$. Moreover, for all acts j and $\lambda \in (0, 1)$,

$$\begin{aligned}&p(\lambda f+(1-\lambda)j,\{\lambda f+(1-\lambda)j,\lambda g+(1-\lambda)j\})\\&-p(\lambda f+(1-\lambda)j,\{\lambda f+(1-\lambda)j,\lambda h+(1-\lambda)j\})\\&\quad=p(f,\{f,g\})-p(f,\{f,h\}).\end{aligned}$$

Similar reasoning applies in the case of decision making under certainty in which the alternatives are multiattribute goods and the incompleteness is due to the inability of the decision maker to compare alternatives that have different attributes. Formally, if an alternative a dominates an alternative a' in the sense that it has more of the positive attributes and/or less of the negative ones, and a'' is an alternative that is noncomparable to either a or a', then the probability that a'' is selected from the pair $\{a'', a'\}$ is greater than the probability that it is selected from the pair $\{a'', a\}$.

An experimental test of the probabilistic choice monotonicity hypothesis consists of two parts: In the first part, a set $J = \{1, ..., n\}$ of subjects is recruited and the ranges of incompleteness of bets (i.e., xEy, where $x > y$) are elicited using the scheme of Karni and Vierø (2023). In the second part, the subjects are asked to choose, repeatedly, between a bet on xEy and sure payoffs that are noncomparable to the bet. The prediction of the ICM is that the relative frequency of choosing the bet decreases monotonically with the values of the sure payoffs.[24]

The experiments described above are designed to test a qualitative property of the ICM, namely, probabilistic choice monotonicity that transcends the idiosyncratic variations of individual stochastic signal-generating processes. They are not designed to quantify the change in the probabilistic choice behavior in response to variations in the sets of alternatives. To grasp the nature of quantitative constraints imposed by the ICM model on subjects' choice behavior, consider the following experiment. Let $b = xEy$, $b' = x'Ey'$, and $b'' = x''Ey''$ be three bets on E, where $y'' < y' < y < x < x' < x''$, and suppose that no two of these bets are comparable.[25] The subjects are asked to choose, repeatedly, from the binary set $\{b, b'\}$, and $\{b, b''\}$. Let $p(b', \{b, b'\})$ and $p(b'', \{b, b''\})$ denote the relative frequency of choosing the b' from the set $\{b, b'\}$ and b'' from the set $\{b, b''\}$. Then the ICM model predicts that: (a) If $p(b'', \{b, b''\}) \geq p(b', \{b, b'\})$, then facing a choice among the three bets, the subject chooses b'' with probability $p(b'', \{b, b', b''\}) = p(b'', \{b, b''\})$, b with probability $p(b, \{b, b', b''\}) = (1 - p(b'', \{b, b''\}))$ and $p(b', \{b, b', b''\}) = 0$ (i.e., b' is a dominated bet in the set $\{b, b', b''\}$). (b) If $p(b'', \{b, b''\}) < p(b', \{b, b'\})$, then facing the choice among the three bets, the subject chooses b'' with probability, $p(b'', \{b, b''\})$, b with probability $(1 - p(b', \{b, b'\}))$, and b' with probability $p(b', \{b, b'\}) - p(b'', \{b, b''\})$.

5 | CONCLUDING REMARKS

This paper proposes a novel approach to modeling decision making under certainty, risk, and uncertainty in situations in which the preference relations are incomplete. The indecisiveness, due to the noncomparability of the alternatives under consideration, is captured by a set of

[24]This method is discussed in Loomes and Sugden (1998) and was implemented in a study by Loomes et al. (2002). To provide the subjects with an incentive to consider the choice seriously, one of each subject's choices is randomly selected, and the subject is rewarded according to the outcome of the selected alternative.

[25]The bets are chosen after the range of incompleteness at E is elicited, using the scheme described in Karni and Vierø (2023).

partial strict orders on the corresponding choice sets and a Borel probability measure on it. The implied stochastic choice behavior is characterized.[26]

The preference relations of different decision makers may not agree on the sets of alternatives that are noncomparable. For example, one decision maker may strictly prefer an alternative a over a', displaying resolute choice, while another decision maker may find the same alternatives noncomparable and display irresolute choice behavior. Even if the decision makers are indecisive with regard to two alternatives, they may still exhibit distinct random choice patterns, due to distinct underlying signal-generating processes. To grasp this, let the ICM of the one decision maker be $K = \{\succ^{\alpha} | \alpha \in [0, 1]\}$ and that of another be $\hat{K} = \{\hat{\succ}^{\alpha} | \alpha \in [0, 1]\}$. Suppose that the probability measures on K and $\hat{K}$ are η and $\hat{\eta}$, respectively. Even if both ICMs agree that a and a' are noncomparable, it may still be that $a \succcurlyeq^{\bar{\alpha}(a,a')} a'$ and $a \hat{\succcurlyeq}^{\bar{\alpha}'(a,a')} a'$, for $\bar{\alpha}(a, a') \neq \bar{\alpha}'(a, a')$. According to the ICM, the former decision maker chooses a with probability $p(a, \{a, a'\}) = \eta([0, \bar{\alpha}(a, a')))$, and the latter with probability $p'(a, \{a, a'\}) = \hat{\eta}([0, \bar{\alpha}'(a, a')])$.

5.1 | Related literature

The recognition that, in many settings, choices are observed to display stochastic patterns led, in recent years, to a revival of interest in modeling and testing stochastic choice behavior.[27] Much of this work—including recent contributions by Echenique and Saito (2019), Ahumada and Ulku (2018), and Horan (2021)—builds on, and extends, the seminal model proposed by Luce (1959). A common feature of these models is a primitive stochastic choice function that is assumed to be the observable object being studied and characterized. The stochastic choice relations are the analog concept in this paper. Like the stochastic choice function, these relations are a primitive concept. In every other respect, however (i.e., the axiomatic structure and the representations), the ICM is different from Luce's original model and its extensions. In particular, Luce's (1959) model requires that the choice probabilities must always be strictly positive. In many instances the empirical choice probability is zero. Consequently, this requirement is regarded as a weakness of the model. Indeed, the aforementioned extensions of Luce's model are intended to address and overcome this weakness by admitting "editing" processes that qualify the supports of the images of the choice functions by eliminating dominated alternatives. By contrast, the ICM naturally admits dominated alternatives that are assigned zero probability.

Another characteristic of Luce's model is the constancy of probability ratios. Formally, the ratio of the choice probabilities of any two alternatives a_i and a_j is independent of the menu to which they belong. This condition which is neither natural nor intuitively compelling is not required by the stochastic choice function induced by the ICM. In other words, according to the stochastic choice function induced by the ICM, richer menu may decrease the probabilities of choosing existing alternatives. The decreases, however, are not necessarily equiproportional.

[26]The ICM can be applied, using the same approach, to nonexpected utility theories with incomplete preferences (e.g., the dual theory, Maccheroni, 2004; probabilistically sophisticated choice, Karni, 2020; weighted utility theory, Karni & Zhou, 2021).

[27]See Gul et al. (2014), Fudenberg et al. (2015), and Frick et al. (2019).

Consequently, the ICM is consistent with a richer set of stochastic choice behaviors than Luce's model.

Danan (2010) modeled a two-stage decision-making process according to which, in the first stage, any two alternatives are either ranked in the strict sense or judged as being equally valuable. If no judgment is rendered comparing their values, the two alternatives are determined to be noncomparable. In the second stage, the alternative that is ranked higher, if such an alternative exists, is selected. Otherwise, one of the alternatives is chosen either by deliberate randomization or selectively. In the case of deliberate randomization, choice behavior is based on a signal produced by a randomization device. In terms of the ICM the signal space of the randomizing device is mapped onto the canonical signal space by ascribing to the sets of utility functions that rank one alternative over the other the probability that the first alternative is selected by the randomization device.

Recent experimental studies documented evidence of deliberate randomization (Agranov et al., 2022; Agranov & Ortoleva, 2017; Dwenger et al., 2018; Feldman & Rehbeck, 2022; Halevy et al., 2023). One interpretation of this evidence is that preferences are incomplete and decision makers relegate the decisions to the randomizing devices. The empirical counterpart of randomization is probability distributions on the sets of relevant sets of feasible alternatives. Deliberate randomization is different from the stochastic impulses that govern choice behavior in the ICM. Yet, if the outside observer is not aware of the randomization, the probability of selecting an alternative may be interpreted as the probability of the set of utility functions that rank the selected alternative above all others in the feasible set.

Ok and Tserenjigmid (2020) model random choice behavior as random choice functions, which they define and characterize for stochastic choices induced by indifference, indecisiveness, and experimentation. The first two are closely related to the phenomena modeled in this paper. Ok and Tserenjigmid merely assert that the maximal elements of the menu will be chosen with positive probability.[28]

Karni and Safra (2016) study stochastic choice under risk and under uncertainty based on the notion that decision makers' actual choices are governed by randomly selected *states of mind*. They provide axiomatic characterization of the representation of decision makers' perceptions of the stochastic process underlying the selection of their state of mind. In the context of decision making under uncertainty with incomplete preferences, the states of mind are probability-utility pairs in the set Φ.[29] The stochastic choice process corresponds to a subjective probability measure, λ, of the sets Φ^{λ}. Thus, the work of Karni and Safra (2016) may be regarded as providing axiomatic foundations of a subjective version of the ICM.

Finally, although not involving random choice behavior, the idea of nested family of preorders, was explored by Hill (2016). In the context of Knightian uncertainty, Hill proposed that the larger the stake involved, the more confidence the decision maker must have in his judgment before making a decision. Formally, this takes the form of nested preorders with the set of prior corresponding to a higher stake decision being contained in that of the lower stake one (Baucells & Shapley, 2008).

[28]Ok and Tserenjigmid (2021) propose making rationality comparisons between stochastic choice rules by means of a partial ordering method. According to their method, the stochastic choice model of this paper is maximally rational.
[29]In the special cases of Knightian uncertainty and complete beliefs, the sets of states of mind are Π and $\mathcal{U}^1$, respectively.

CONFLICT OF INTEREST STATEMENT

The authors declare no conflict of interest.

DATA AVAILABILITY STATEMENT

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