Incomplete Preferences and Random Choice Behavior: Axiomatic Characterizations

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September 27, 2021

Abstract

This paper proposes axiomatic characterizations of random choice behavior that is due to incomplete preferences. It proposes a model of irresolute choice and examines its applications to decision making under certainty, uncertainty and risk.

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Keywords: Random choice, incomplete preferences, irresolute choice, Knightian uncertainty, multi-prior expected multi-utility representations, multi-utility representations.

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1 Introduction

Situations in which decision makers find the feasible alternatives difficult, if not impossible, to compare and choose from are common. As von Neumann and Morgenstern, (1947) admitted, "It is conceivable – and even in a way more realistic – to allow for cases where the individual is neither able to state which of two alternatives he prefers nor that they are equally desirable." Depending on the context, this difficulty may be due to the complexity of the alternatives or, for lack of experience, the inability to assess their, potentially long-run, consequences. A topical example is the decision whether or not to vaccinate against COVID-19 and, if the decision is to vaccinate, which vaccine to choose.

The appropriateness of the postulate that all alternatives are readily comparable (i.e., that the preference relation is complete) was broached by Leonard Savage in a letter to Karl Popper dated March 25, 1958, in which Savage discusses his work on the choice-based foundations of subjective probabilities. Savage wrote: "There is, though, a postulate that insists that economic situations can be ranked in a linear order by the subject, and I freely admit that this seems to me to be a source of much difficulty in my theory. This stringent postulate is in conflict with the common experience of vagueness and indecision, and if I knew a good way to make a mathematical model of those phenomena, I would adopt it, but I despair of finding one."[1]

Aumann (1962) questioned not only the descriptive validity of the completeness postulate but also its normative justification. "Of all the axioms of utility theory," he wrote, "the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from the normative viewpoint."

Danan and Ziegelmeyer (2006), Sautua (2017), and Cettolin and Riedl (2019) provide evidence of the prevalence of incomplete preferences in experimental settings is provided . Yet with few exceptions, the theories of individual decision making – under certainty, risk, or uncertainty – presume that the preference relations depicting individual choice behavior are complete.

When the preference relations are complete, all alternatives are comparable and, in general, decision makers exhibit resolute choice behavior. By

[1] This correspondence is reproduced by Carlo Zappia (2020).

contrast, when the preference relations are incomplete, there are alternatives that are noncomparable, and, facing a choice between such alternatives, decision makers display indecisiveness (e.g., procrastination, hesitation, and irresolute choice). Bewley (2002) suggests that if among the noncomparable alternatives there is one that may be regarded as the status quo, or default, alternative, it is chosen.[2] Danan (2010) analyzes the implications of choice behavior that invokes deliberate randomization.[3] Evren et al. (2019) model choice behavior based on secondary criterion of the top-cycle among all undominated alternatives in the feasible set relative to a complete and transitive binary relation. In the present work I address the same issue with a new axiomatic model, dubbed *irresolute choice model* (henceforth ICM). Taking preference relations on choice sets for a primitive ingredient and departing form the completeness postulate, the model describes random choice behavior between noncomparable alternatives by a collection of nested partial orders each depicting different choice probabilities.

The literature offers a variety of axiomatic models characterizing the representations of incomplete preferences under certainty (Ok [2002] Evren and Ok [2011]); under risk (Shapley and Baucells [1998] and Dubra et al. [2004]); and under uncertainty (Bewley [2002], Seidenfeld et al. [1995], Nau [2006], Ok et al. [2012], Galaabaatar and Karni [2013], and Riella [2015]). Unlike the case of complete preferences, in which the alternative that commends the highest representation value is chosen, in the case of incomplete preferences, the representations do not, in general, single out a preferred alternative. The main result of this paper connects the representations to choice behavior.

The underlying premise of this work is that when facing a choice among noncomparable alternatives, decisions are triggered by impulses, or signals, that are inherently random, or appear to be random to an observer who is not privy to the workings of the decision maker's mind. In either case, insofar as the observer is concerned, the decision maker's choices appear to be random. The model I propose has predictable *probabilistic choice behavior.* In other words, facing a choice between noncomparable alternatives, the model predicts the probabilities that each alternative is selected.

The main novelty of this work is conceptual rather than technical, it is the approach to modeling of random choice behavior. More specifically, the incompleteness of the preference relation is modeled as a continuum of

[2] See also Masatlioglu and Ok (2005).

[3] See further discussion of this work in the concluding section.

strict partial orders on the relevant choice sets depicting the binary relations "one alternative is strictly preferred over another with probability that is at most $\alpha \in [0,1]$." These strict partial orders are linked by a monotonicity requirement. The results are characterizations of probabilistic choice representations.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 applies the model to decision making under certainty. Section 4 applies the model to subjective expected utility theory. Section 5 provides sme concluding remarks and a brief review of the relevant literature.

2 A Model of Random Choice

2.1 Preliminaries

Let A denote a *choice set.* Elements of A are *alternatives*. Denote by $\succ$ irreflexive and transitive binary relation on A, dubbed *strict preference relation*. For any alternatives $a, a' \in A$, $a \succ a'$ is the proposition that, facing a choice between these two alternatives, a decision maker characterized by $\succ$ chooses the alternative a. This behavior has the usual interpretation that a is *strictly preferred* over a'. I assume throughout that $\succ$ on A is nonempty.

The strict preference relation, $\succ$, induces the following derived binary relations on A. For all $a, a' \in A$,

(a) The *weak preference relation,* $\succcurlyeq$, is defined by: $a \succcurlyeq a'$ if, for all $a'' \in A$, $a'' \succ a$ implies that $a'' \succ a'$.[4]

(b) The *indifference relation*, $\sim$, is defined by $a \sim a'$ if $a \succcurlyeq a'$ and $a' \succcurlyeq a$.

(c) The *noncomparability relation* $\bowtie$, is defined by: $a \bowtie a'$ if $\neg (a \succcurlyeq a')$ and $\neg (a' \succcurlyeq a)$.

(d) The *negation of* $\succ$, denoted $\succeq$, is defined by $a \succeq a'$ if $\neg (a' \succ a)$.[5]

It is natural to suppose that if presented with a choice between two alternatives, a and a', a decision maker would choose the former act if $a \succcurlyeq a'$ and

[4] Clearly, $a \succ a'$ implies that $a \succcurlyeq a'$.

[5] Note that $\succeq$ is reflexive but not necessarily transitive. The weak preference relation defined here was introduced in Galaabaatar and Karni (2013). Its significance and implications were investigated and discussed in Karni (2011), who showed that the relations $\succeq$ and $\succcurlyeq$ agree if and only if $\succ$ is negatively transitive and $\succcurlyeq$ is complete. Note that $\succ$ is not the asymmetric part of $\succcurlyeq$. The indifference relation defined here, introduced in Galaabaatar and Karni (2013), is equivalent to that of Eliaz and Ok (2006).

$\neg(a' \succcurlyeq a)$. However, if $a \bowtie a'$ or $a' \sim a$, then the preference relation does not indicate which of the two alternatives will be chosen. Moreover, since $\succsim \supseteq \bowtie \cup \sim$, $a \succsim a'$ does not imply that a will be chosen form the subset $\{a, a'\}$.

2.2 Irresolute choice model

The basic premise of this work is that, facing a choice between noncomparable or indifferent alternatives. the decision maker behaves *as if* he is awaiting a signal that would determine his choice and, thereby, resolve his indecision. The signal is presumed to be generated by a stochastic process whose nature is not specified extraneously. The behavioral manifestation of this presumption is that having to choose between noncomparable or indifferent alternatives, the decision maker may procrastinate while waiting for the signal and then choose in a manner that reflects the underlying randomness of the signal-generating process.[6] Consequently, to the outside observer, the decision maker displays stochastic choice behavior.

To formalize this idea, I model *irresolute choice behavior* as a set $\{\succ^\alpha \mid \alpha \in [0,1]\}$ of binary relations on A, dubbed *probabilistic choice relations.* For each $\alpha \in [0,1]$, the derived relations $\succcurlyeq^\alpha, \sim^\alpha, \bowtie^\alpha$ and $\succsim^\alpha$ are defined follows: $a \succcurlyeq^\alpha a'$ if, for all $a'' \in A$, $a'' \succ^\alpha a$ implies that $a'' \succ^\alpha a'$; $a \sim^\alpha a'$ if $a \succcurlyeq^\alpha a'$ and $a' \succcurlyeq^\alpha a$; $a \bowtie^\alpha a'$ if and only if $\neg(a \succcurlyeq^\alpha a')$ and $\neg(a' \succcurlyeq^\alpha a)$; $a \succsim^\alpha a'$ if $\neg(a' \succ^\alpha a)$.

Given any $a, a' \in A$, the interpretation of $a \succ^\alpha a'$ is as follows: Facing a choice between the alternatives a and a', alternative a is is strictly preferred and, hence, chosen, over a' with probability that is smaller or equal to α. In other words, for all $\alpha' < \alpha$, $a \succ^\alpha a'$ implies that $a \succ^{\alpha'} a'$ (that is, $\succ^\alpha \subseteq \succ^{\alpha'}$) and for no $\alpha'' > \alpha$ it holds that $a \succ^{\alpha''} a'$. To grasp how this interpretation is related to choice behavior, observe that if $a \succcurlyeq^\alpha a'$ then $a \succ^{\alpha'} a'$ for all $\alpha' < \alpha$. Thus, except in the case in which $a \sim a'$, which implies that $a \sim^\alpha a'$ for all $\alpha \in [0,1]$, $a \succcurlyeq^\alpha a'$ implies that $\alpha = \sup\{\alpha' \in [0,1] \mid a \succ^{\alpha'} a'\}$.[7] Consequently, if $\neg(a \sim a')$, then α is the exact probability that a is chosen from the set $\{a, a'\}$. Clearly, $a \succ a'$ implies that $a \succcurlyeq^1 a'$.[8] Hence, $a \succcurlyeq^1 a'$

[6] For example, the underlying process may have the structure of the drift-diffusion model, in which the procrastination is measured to the response time. See, for example, Ian Krajbich et al. (2014) and Baldassi et al. (2020).

[7] That the supremum exists follows from the fact that the set is bounded and that $\neg(a' \sim a)$ implies that there is $\alpha' \in [0,1]$ such that $a \succ^{\alpha'} a'$. Hence, the set is nonempty.

[8] That is, $a \succcurlyeq^{\bar{\alpha}} a'$ where $\bar{\alpha} = 1$. However, to maintain coherence, henceforth I use the

implies that a is chosen from the set $\{a, a'\}$ with probability one. If $a \sim a'$ then, insofar as the probability of a chosen over a' is concerned, the model is silent.

The IMC may also be stated as a set of binary relations $\{\succcurlyeq^{\alpha} \mid \alpha \in [0,1]\}$ on A. The interpretation of $a \succcurlyeq^{\alpha} a'$ is as follows: Facing the choice between the alternative a and a', the alternative a is chosen with probability no greater than $(1-\alpha)$. Hence, for all $\alpha' \geq \alpha$, $a \succcurlyeq^{\alpha} a'$ implies that $a \succcurlyeq^{\alpha'} a'$ (that is, $\succcurlyeq^{\alpha} \subseteq \succcurlyeq^{\alpha'}$) and for no $\alpha'' < \alpha$ it holds that $a \succcurlyeq^{\alpha''} a'$.

To analyze the behavioral implications of the ICM I invoke the formulation that is consistent with the underlying decision model under consideration.

3 Irresolute Choice Behavior

3.1 An axiomatic characterization

Let the choice set A be a nonempty topological space, and denote by $\succcurlyeq$ a preorder on A. For any $a \in A$, the upper and lower $\succcurlyeq$-contour sets of a are defined as $\mathbb{U}_{\succcurlyeq}(a) = \{a' \in A \mid a' \succcurlyeq a\}$ and $\mathbb{L}_{\succcurlyeq}(a) = \{a' \in A \mid a \succcurlyeq a'\}$. The preorder $\succcurlyeq$ is *continuous* if $\mathbb{U}_{\succcurlyeq}(a)$ and $\mathbb{L}_{\succcurlyeq}(a)$ are closed, for all $a \in A$. A nonempty set $\mathcal{U}$ of real-valued functions on A is said to *represent* $\succcurlyeq$ if, for all $a, a' \in A$, $a \succcurlyeq a'$ if and only if $u(a) \geq u(a')$ for all $u \in \mathcal{U}$.

Let $\{\succ^{\alpha} \mid \alpha \in [0,1]\}$ be a set of probabilistic choice relations on A, and $\{\succcurlyeq^{\alpha} \mid \alpha \in [0,1]\}$ the corresponding model expressed in terms of the negations of $\succ^{\alpha}$. For each $\alpha \in [0,1]$ the structure of $\succcurlyeq^{\alpha}$ is depicted axiomatically as follows:

(P1) (**Partial Order**) For each $\alpha \in [0,1]$ $\succcurlyeq^{\alpha}$ is transitive and reflexive.

(P2) (**Continuity**) For each $a \in A$ and $\alpha \in [0,1]$, $\mathbb{U}_{\succcurlyeq^{\alpha}}(a)$ and $\mathbb{L}_{\succcurlyeq^{\alpha}}(a)$ are closed, for all $a \in A$.

The representation of irresolute choice behavior requires that the random choice relations in the set $\{\succcurlyeq^{\alpha} \mid \alpha \in [0,1]\}$ be linked. The next axiom provides this link.

(P3) (**Monotonicity**) For all $\alpha, \alpha' \in [0,1]$, $\succcurlyeq^{\alpha} \subseteq \succcurlyeq^{\alpha'}$ if and only if $\alpha' \geq \alpha$.

symbol $\succ^{1}$ instead of $\succ$ to denote the strict preference relation.

Equivalently, the monotonicity postulates asserts that, for each $a \in A$, the upper contour set of a according to $\succcurlyeq^{\alpha}$ is contained in that of $\succcurlyeq^{\alpha'}$ if and only if $\alpha' \leq \alpha$. Formally,

Lemma 1. The irresolute choice model $\{\succcurlyeq^{\alpha} | \ \alpha \in [0,1]\}$ satisfies monotonicity if and only if for every $a \in A$, $\mathbb{U}_{\succcurlyeq^{\alpha'}}(a) \subseteq \mathbb{U}_{\succcurlyeq^{\alpha}}(a)$ if and only if $\alpha' \geq \alpha$.

Proof. Monotonicity is equivalent to the proposition, for all $a, a' \in A$, $a' \succcurlyeq^{\alpha} a$ implies that $a' \succcurlyeq^{\alpha'} a$, if and only if $\alpha' \geq \alpha$. Equivalently, for all $a, a' \in A$, $\neg \left(a' \succcurlyeq^{\alpha'} a\right)$ implies $\neg \left(a' \succcurlyeq^{\alpha} a\right)$ if and only if $\alpha' \geq \alpha$. But the last statement is equivalent to the proposition $a \in \mathbb{U}_{\succcurlyeq^{\alpha'}}(a')$ implies that $a \in \mathbb{U}_{\succcurlyeq^{\alpha}}(a')$ if and only if $\alpha' \geq \alpha$, for all $a' \in A$. Thus, monotonicity holds if and only if, for all $a' \in A$, $\mathbb{U}_{\succcurlyeq^{\alpha'}}(a') \subseteq \mathbb{U}_{\succcurlyeq^{\alpha}}(a')$ if and only if $\alpha' \geq \alpha$. ▲

The following theorem extends Evren and Ok (2011) Corollary 1, to include irresolute choice behavior.[9]

Theorem 1: *Let A be a locally compact separable metric space and $\{\succcurlyeq^{\alpha} | \ \alpha \in [0,1]\}$ be binary relations on A. Then, the following conditions are equivalent:*

(i) *For every $\alpha \in [0,1]$, $\succcurlyeq^{\alpha}$ satisfies (P1) and (P2) and jointly $\succcurlyeq^{\alpha}$, $\alpha \in [0,1]$, satisfy (P3).*

(ii) *For every $\alpha \in [0,1]$ there exists a set $\mathcal{U}^{\alpha}$ of real-valued, continuous and strictly $\succcurlyeq^{\alpha}$ $-$increasing, functions such that $\mathcal{U}^{\alpha}$ represents $\succcurlyeq^{\alpha}$ and $\alpha \geq \alpha'$ if and only if $\mathcal{U}^{\alpha} \supseteq \mathcal{U}^{\alpha'}$.*

Proof. $(i) \Rightarrow (ii)$. Suppose that A is a locally compact separable metric space and $\{\succcurlyeq^{\alpha} | \ \alpha \in [0,1]\}$ be binary relations on A satisfying (P1) and (P2) then, by Evren and Ok (2011) Corollary 1, for each $\alpha \in [0,1]$, there exists a set $\mathcal{U}^{\alpha}$ of real-valued continuous functions such that $\mathcal{U}^{\alpha}$ represents $\succcurlyeq^{\alpha}$, every $u \in \mathcal{U}^{\alpha}$ is strictly $\succcurlyeq^{\alpha}$ $-$increasing . By (P3) and Lemma 1, $\alpha' \leq \alpha$ if and only if $\mathbb{U}_{\succcurlyeq^{\alpha'}}(a) \supseteq \mathbb{U}_{\succcurlyeq^{\alpha}}(a)$. But, by the representation, $\mathbb{U}_{\succcurlyeq^{\alpha'}}(a) \supseteq \mathbb{U}_{\succcurlyeq^{\alpha}}(a)$ if and only if $\mathcal{U}^{\alpha} \subseteq \mathcal{U}^{\alpha'}$. Hence, $\alpha' \leq \alpha$ if and only if $\mathcal{U}^{\alpha} \supseteq \mathcal{U}^{\alpha'}$.

$(ii) \Rightarrow (i)$. Suppose that (ii) holds. Then, by Evren and Ok (2011) Corollary 1 each $\succcurlyeq^{\alpha}$, $\alpha \in [0,1]$, satisfies (P1) and (P2*)*. Suppose that $\alpha' \leq \alpha$ if and only if $\mathcal{U}^{\alpha} \supseteq \mathcal{U}^{\alpha'}$. By the representation , $\mathcal{U}^{\alpha} \supseteq \mathcal{U}^{\alpha'}$ if and only if $\mathbb{U}_{\succcurlyeq^{\alpha'}}(a) \supseteq \mathbb{U}_{\succcurlyeq^{\alpha}}(a)$. Hence, $\alpha' \leq \alpha$ if and only if $\mathbb{U}_{\succcurlyeq^{\alpha'}}(a) \supseteq \mathbb{U}_{\succcurlyeq^{\alpha}}(a)$, for all $a \in A$, which, by Lemma 1 is equivalent to (P3). ■

The uniqueness of the representation is as follows: Given any nonempty subset $\mathcal{U}^{\alpha}$ of $\mathbb{R}^{A}$, define the map $\Upsilon_{\mathcal{U}^{\alpha}} : A \rightarrow \mathbb{R}^{\mathcal{U}^{\alpha}}$ by $\Upsilon_{\mathcal{U}^{\alpha}}(a)(u) := u(a)$.

[9] Other results of Evren and Ok (2011), including their Theorem 1 and Corollaries 2 and 3, may be extended in the same way.

Two nonempty subsets $\mathcal{U}^\alpha$ and $\mathcal{V}^\alpha$ of continuous real-valued functions on A represent the same preorder if, and only if, there exists an $f : \Upsilon_{\mathcal{U}^\alpha}(A) \to \Upsilon_{\mathcal{V}^\alpha}$ such that (i) $\Upsilon_{\mathcal{V}^\alpha} = f(\Upsilon_{\mathcal{U}^\alpha})$; and (ii) for every $b, c \in \Upsilon_{\mathcal{U}^\alpha}(A)$, $b > c$ if and only if $f(b) > f(c)$.[10]

Remark: Let $\mathcal{V}^\alpha$ and $\mathcal{V}^{\alpha'}$ be arbitrary multi-utility representations of two preorders, $\succcurlyeq^\alpha$ and $\succcurlyeq^{\alpha'}$, respectively, and suppose that $\succcurlyeq^\alpha \subset \succcurlyeq^{\alpha'}$ (i.e., $\alpha' > \alpha$). In general, this does not imply that $\mathcal{V}^\alpha \subset \mathcal{V}^{\alpha'}$. However, one can always find representations $\mathcal{U}^\alpha$ and $\mathcal{U}^{\alpha'}$ such that $\mathcal{U}^\alpha \subset \mathcal{U}^{\alpha'}$, as is required in Theorem 1. To see this, let $\mathcal{U}^\alpha$ be the set of all (continuous) real functions u such that $a \succcurlyeq^\alpha a'$ implies $u(a) \geq u(a')$ and $\mathcal{U}^{\alpha'}$ be the set of all continuous real functions u such that $a \succcurlyeq^{\alpha'} a'$ implies $u(a) \geq u(a')$. Then $\succcurlyeq^\alpha \subset \succcurlyeq^{\alpha'}$ implies that if $u \in \mathcal{U}^{\alpha'}$ then $u \in \mathcal{U}^\alpha$. Thus, $\mathcal{U}^\alpha \subset \mathcal{U}^{\alpha'}$.[11] Henceforth, the property $\alpha' > \alpha$ if and only if $\mathcal{U}^{\alpha'} \supset \mathcal{U}^\alpha$ is dubbed *nestedness*.

3.2 The indifference relation

The case in which the alternatives under consideration belong to the same indifference class requires special attention. By definition, $a \sim^1 a'$ if and only if $a \succcurlyeq^1 a'$ and $a' \succcurlyeq^1 a$.

Lemma 2: For all $a, a' \in A$,

$$a \succcurlyeq^1 a' \;\Leftrightarrow u(a) \geq u(a'),\text{ for all } u \in \mathcal{U}^1.$$

Proof. By definition $a \succcurlyeq^1 a'$ if $\hat{a} \succ^1 a$ then $\hat{a} \succ^1 a'$, for all $\hat{a} \in A$. Hence, by definition, $a \succcurlyeq^1 a''$ implies $a' \succcurlyeq^1 a''$. By Theorem 1, this is equivalent to $u(a) \geq u(a'')$ implying that $u(a') \geq u(a'')$ for all $u \in \mathcal{U}^1$.

Consider a sequence $(a''_n) \subset A$ such that $a \succcurlyeq^1 a''_n$ for $n = 1, 2, ...$ and $a = \lim_{n\to\infty} a''_n$. Then $a' \succcurlyeq^1 a''_n$ for $n = 1, 2, ...$ By continuity, $a \succcurlyeq^1 a'$. Hence, by Theorem 1, $a \succcurlyeq^1 a' \;\Leftrightarrow u(a) \geq u(a')$, for all $u \in \mathcal{U}^1$. ▲

By definition of $\sim^1$, and Lemma 2,

$$a \sim^1 a' \;\Leftrightarrow u(a) = u(a'),\text{ for all } u \in \mathcal{U}^1.$$

By Theorem 1, $\mathcal{U}^\alpha \subseteq \mathcal{U}^1$ for all $\alpha \in [0,1]$. Thus, for all $a, a' \in A$, $a \sim^1 a'$ implies that $a \sim^\alpha a'$, $a \sim^{\alpha'} a'$, for all $\alpha, \alpha' \in [0,1]$. Consequently, the irresolute choice model is silent with regard to the probability of selection of any alternatives belonging to the same indifference class.

[10] See Evren and Ok (2011).

[11] I thank Özgür Evren for this remark.

3.3 Canonical signal space

The premise underlying the stochastic choice behavior depicted by the ICM is that choices between noncomparable or indifferent alternatives are governed by unspecified process of randomly generating signals. Consider the choice between two alternatives, a and a' such that $\neg(a \sim a')$, then the probability of a signal that would resolve the indecision in favor of a is $\bar{\alpha} = \sup\{\alpha' \in [0,1] \mid a \succ^{\alpha'} a'\}$. By the representation of the ICM this is the case if and only if $u(a) \geq u(a')$, for all $u \in \mathcal{U}^{\bar{\alpha}}$. In other words, the decision maker behaves *as if* a function u is selected from $\mathcal{U}^1$ and a is chosen if $u \in \mathcal{U}^{\bar{\alpha}}$ and a' is chosen if $u \in \mathcal{U}^1 \backslash \mathcal{U}^{\bar{\alpha}}$. Under this interpretation, $\bar{\alpha}$ is the probability of the set $\mathcal{U}^{\bar{\alpha}}$. Therefore, the set $\mathcal{U}^1$ may be taken to be the *canonical signal space*.

3.4 Probabilistic choice

Many decision problems require the decision maker to choose from finite sets of feasible alternatives that include more than two elements. To apply the ICM to choice from such sets, let $M = \{a_1, ..., a_m\} \subset A$ be a feasible set of alternatives and, to simplify the exposition, suppose that no two alternatives in M belong to the same indifference class. An alternative $a \in M$ is said to be *dominated* if for no $\alpha \in [0,1]$ it holds that $a \succcurlyeq^{\alpha} a'$, $\forall a' \in M \backslash \{a\}$. Let $D(M)$ denote the subset of dominated alternatives in M and let $U(M) = M \backslash D(M)$ denote the subset of *undominated* alternatives in M. Note that $U(M)$ is nonempty.

For each $a_i \in U(M)$ define $\Lambda_i(M) = \{\alpha \in [0,1] \mid a_i \succcurlyeq^{\alpha} a_j, \forall a_j \in M \backslash \{a_i\}\}$. In words, $\Lambda_i(M)$ is the set of indices of the random choice relations that rank the alternative a_i higher than any other alternative in the menu M. For each $a_i \in U(M)$ let $\alpha^*(a_i) := \sup \Lambda_i(M)$.[12] Then, by Theorem 1, $a_i \succcurlyeq^{\alpha^*(a_i)} a_j$, for all $a_j \in M \backslash \{a_i\}$ if and only if $u(a_i) \geq u(a_j)$, for all $u \in \mathcal{U}^{\alpha^*(a_i)}$.

Without loss of generality assume that the elements of $U(M)$ are permuted so they are rearranged in a descending order of α^*. (i.e., $\alpha^*(a_1) > \alpha^*(a_2) > ... > \alpha^*(a_m) > 0$, $a_i \in U(M)$, $i = 1, ...m$). Note that $\alpha^*(a_1) = 1$ and let $\alpha^*(a_{m+1}) := 0$. By the nestedness and Theorem 1, $a_i \succcurlyeq^{\alpha^*(a_i)} a_j$ if and only if $u \in \mathcal{P}_i(M) := \mathcal{U}^{\alpha^*(a_i)} \backslash \mathcal{U}^{\alpha^*(a_{i+1})}$, $i = 1, ..., m-1$, and $\mathcal{P}_m(M) :=$

[12]That the supremum exist since the set $\Lambda_i(M)$ is bounded and, because a_i is undominated, $\Lambda_i(M)$ nonempty.

$\mathcal{U}^{\alpha^*(a_m)}$. Then $u \in \mathcal{P}_i(M)$ if $u(a_{i-1}) > u(a_i) \geq u(a_{i+1})$. Thus, $\mathcal{P}(M) := \{\mathcal{P}_1(M), ..., \mathcal{P}_m(M)\}$ is a partition of $\mathcal{U}^1$.[13]

Since $\mathcal{U}^1$ is the canonical signal space, the probability of receiving a signal $u \in \mathcal{P}_i(M)$ is $p(a_i) := \alpha^*(a_i) - \alpha^*(a_{i+1})$, $i = 1, ..., m$. Thus, when facing a choice form M, the decision maker behaves *as if* a utility function $u \in \mathcal{U}^1$ is selected according to the distribution $p(a_i)$, $i = 1, ..., m$, and the *undominated alternative*, a_i is chosen if $u \in \mathcal{P}_i(M)$, $i = 1, ..., m$. Hence, $a_i \in U(M)$ is chosen with probability $p(a_i)$ and dominated alternatives are never chosen. Stated differently, $\alpha^*(a_i)$ is the probability that an alternative in the set $A_i := \{a_i, a_{i+1}, ..., a_m\}$.

4 Irresolute Choice Behavior under Uncertainty

4.1 The analytical framework

For over half a century subjective expected utility theory has been the dominant model of decision making under uncertainty. Because of its prominent role and rich analytical framework, I explore the application of the ICM to subjective expected utility theory, invoking the model of Galaabaatar and Karni (2013). This model admits incomplete beliefs and tastes, and includes Bewley's Knigthian uncertainty model (i.e., complete tastes and incomplete beliefs) and the subjective expected multi-utility model (i.e., complete beliefs and incomplete tastes) as special cases.

The analytical framework is that of Anscombe and Aumann (1963). Let S be a finite set of *states.* Subsets of S are *events.* Let X be a finite set of *outcomes* and denote by $\Delta(X)$ the set of all probability distributions on X. For each $q, q' \in \Delta(X)$ and $\gamma \in [0,1]$ define $\gamma q + (1-\gamma) q' \in \Delta(X)$ by $(\gamma q + (1-\gamma) q')(x) = \gamma q(x) + (1-\gamma) q'(x)$, for all $x \in X$.

The choice set is $H := \Delta(X)^S$ (i.e., the set of mapping from S to $\Delta(X)$). The elements of H are *acts.* For all $h, h' \in H$ and $\gamma \in [0,1]$, define $\gamma h + (1-\gamma) h' \in H$ by $(\gamma h + (1-\gamma) h')(s) = \gamma h(s) + (1-\gamma) h'(s)$. Under this definition H is a convex subset of the linear space $\mathbb{R}_+^{|X| \times |S|}$. A constant act, $h \in H$, is an act such that $h(s) = q$ for all $s \in S$, where $q \in \Delta(X)$. Henceforth,

[13] Since indifference is not allowed, there is no ambiguity with regard to which element of the partition each utlity function belongs to.

I identify the subset of constant acts with $\Delta(X)$. Hence, $\Delta(X) \subset H$.

Let $\{\succ^{\alpha} \mid \alpha \in [0,1]\}$ be random choice relations on H depicting irresolute choice behavior. The choice set H is said to be *bounded* if there exist $\bar{h}$ and $\underline{h}$ in H such that $\bar{h} \succ^1 h \succ^1 \underline{h}$, for all $h \in H - \{\bar{h}, \underline{h}\}$.

For each $\alpha \in [0,1]$, let $\mathcal{U}^{\alpha}$ be a nonempty closed set off real-valued functions on X and, for every $u \in \mathcal{U}^{\alpha}$, let $\Pi^{\alpha}(u)$ be a nonempty closed set of probability measures on S. Define $\Phi^{\alpha} = \{(\pi, u) \mid u \in \mathcal{U}^{\alpha},\ \pi \in \Pi^{\alpha}(u)\}$. Then $\{\Phi^{\alpha} \mid \alpha \in [0,1]\}$ is said to *represent* the ICM $\{\succ^{\alpha} \mid \alpha \in [0,1]\}$ if the following conditions hold:

(a) For all $h \in H$ and $(\pi, u) \in \Phi^1$,

$$\sum_{s\in S}\pi(s)\sum_{x\in X}\bar{h}(x,s)u(x) > \sum_{s\in S}\pi(s)\sum_{x\in X}h(x,s)u(x) > \sum_{s\in S}\pi(s)\sum_{x\in X}\underline{h}(x,s)u(x), \tag{1}$$

(b) For all $h, h' \in H$,

$$h \succ^{\alpha} h' \Leftrightarrow \sum_{s\in S}\pi(s)\sum_{x\in X}h(x,s)u(x) > \sum_{s\in S}\pi(s)\sum_{x\in X}h'(x,s)u(x),\ \forall (\pi,u) \in \Phi^{\alpha}. \tag{2}$$

4.2 Axiomatic characterization

Following Galaabaatar and Karni (2013) I assume that the random choice relations $\succ^{\alpha}$, $\alpha \in [0,1]$ have a structure depicted by the following axioms. The first three axioms are well-known and require no elaboration.

(A.1) (**Strict partial order**) For every $\alpha \in [0,1]$, the $\succ^{\alpha}$ is transitive and irreflexive.

(A.2) (**Archimedean**) For all $f, g, h \in H$, if $f \succ^{\alpha} g$ and $g \succ^{\alpha} h$ then there exist $\beta, \gamma \in (0,1)$ such that $\beta f + (1-\beta) h \succ^{\alpha} g$ and $g \succ^{\alpha} \gamma f + (1-\gamma) h$.

(A.3) (**Independence**) For all $f, g, h \in H$ and $\alpha \in (0,1]$, $f \succ^{\alpha} g$ if and only if $\alpha f + (1-\alpha) h \succ^{\alpha} \alpha g + (1-\alpha) h$.

For each $h \in H$ and every $s \in S$, denote by h^s the constant act whose payoff is $h(s)$ in every state. The next axiom asserts that if every possible consequence of h, taken as a constant act, is an element of the lower contour set of g according to irresolute choice relation $\succ^{\alpha}$ then the convexity of the lower contour sets implies that any convex combination of the consequences of h is dominated by g. Think of h as representing a subset of the simplex in $\mathbb{R}^{|S|}$ whose elements correspond to subjective probabilities on S that the decision maker may entertain. Since any such combination is $\succ^{\alpha}$ $-$dominated by g, so is h. Formally,

(A.4) (**Dominance**) For all $h, g \in H$, and $\alpha \in [0,1]$, if $g \succ^{\alpha} h^s$ for every $s \in S$, then $g \succ^{\alpha} h$.

The next axiom restates (P3) in terms of the present model.

(A.5) (**Monotonicity**) For all $\alpha, \alpha' \in [0,1]$, $\succ^{\alpha} \subseteq \succ^{\alpha'}$ if and only if $\alpha' \leq \alpha$.

The following theorem characterizes irresolute choice behavior:

Theorem 2: *Let* $\{\succ^{\alpha} \mid \alpha \in [0,1]\}$ *be a set of binary relations on* H*. Then the following conditions are equivalent:*

(i) *H is $\succ^{\alpha}$-bounded and for each $\alpha \in [0,1]$, $\succ^{\alpha}$satisfies (A.1)–(A.4) and jointly $\succ^{\alpha}$, $\alpha \in [0,1]$, satisfy (A.5).*

(ii) *For each $\alpha \in [0,1]$ $\succ^{\alpha}$ is represented by (1) and (2) and $\alpha \geq \alpha'$ if and only if $\Phi^{\alpha} \supseteq \Phi^{\alpha'}$.*

The proof that $\succ^{\alpha}$ satisfies (A.1)–(A.4) if and only if $\succ^{\alpha}$ is represented by (1) and (2) is an immediate implications of Theorem 1 of Galaabaatar and Karni (2013). The proof that (A.5) holds if and only if $\alpha \geq \alpha'$ if and only if $\Phi^{\alpha} \supseteq \Phi^{\alpha'}$ is by the same argument as in Theorem 1 above. The uniqueness of the representation is described in Galaabaatar and Karni (2013)and is not replicated here.

4.3 Special cases

The theory of subjective expected utility with incomplete preferences includes two special cases: the case in which the incompleteness is due solely to incomplete beliefs and the case in which it is due solely to incomplete tastes.

The case of incomplete beliefs was axiomatized by Bewley (2002), who dubbed it Knightian uncertainty. Tastes completeness, or unambiguous risk attitudes, requires that the restriction of the preference relation to constant acts exhibits negative transitivity. Let $p \in \Delta(X)$ denotes the constant act that pays off p in every state. Then tastes completeness is captured by the following:

(A.6) (**Unambiguous risk attitudes**) For all constant acts $p, q, r \in \Delta(X)$, $\neg(p \succ^1 q)$ and $\neg(q \succ^1 r)$ imply $\neg(p \succ^1 r)$.

The corollary below is implied by Theorem 2.

Corollary 1: *Let* $\{\succ^\alpha \mid \alpha \in [0,1]\}$ *be a set of binary relations on* H. *Then* H *is bounded and, for each* $\alpha \in [0,1]$, $\succ^\alpha$ *satisfies (A.1)–(A.4), jointly* $\succ^\alpha$, $\alpha \in [0,1]$, *satisfy (A.5), and* $\succ^1$ *satisfies (A.6) if and only if* $\succ^\alpha$ *is represented by (1) and (2) with* $\Theta^\alpha = \{u\} \times \Pi^\alpha$ *and* $\alpha \geq \alpha'$ *if and only if* $\Pi^\alpha \supseteq \Pi^{\alpha'}$. *Moreover,* u *is unique up to positive affine transformation, the closed convex hull of* Π^α *is unique and, for each* $\pi \in \Pi^\alpha$, $\pi(s) > 0$ *for all* $s \in S$.

Consider next the case of complete beliefs and ambiguous risk attitudes. For each event E, denote by rEq the act whose payoff is r for all $s \in E$ and q for all $s \in S - E$. Denote by $r\gamma q \in \Delta(X)$ the constant act whose payoff in every state is $\gamma r + (1-\gamma) q$. A *bet* on an event E is the act rEq, whose payoffs satisfy $r \succ^1 q$, where $r, q \in \Delta(X)$.[14]

Suppose that the decision maker considers the constant act $r\gamma q$ preferable to the bet rEq. Because the payoffs are the same, this preference indicates that he believes that γ exceeds the likelihood of E. This belief is said to be *coherent* if it holds that $r'\gamma q'$ is preferable to the bet $r'Eq'$ for all constant acts r' and q' such that $r' \succ^1 q'$. By the same logic a preference of a bet rEq over the constant act $r\gamma q$ means that the decision maker believes the probability of E to exceed γ. A binary relation $\succ^1$ on H is said to exhibit *coherent beliefs* if, for all events E and $r, q, r', q' \in \Delta(X)$ such that $r \succ^1 q$ and $r' \succ^1 q'$, $r\gamma q \succ^1 rEq$ if and only if $r'\alpha q' \succ^1 r'Eq'$, and $rEq \succ^1 r\gamma q$ if and only if $r'Eq' \succ^1 r'\gamma q'$. Note that the structure of a binary relation $\succ^1$ depicted by

(A.1)–(A.4) implies that the decision maker's beliefs are coherent.

The idea of complete beliefs is captured by the following axiom, which is due to Galaabaatar and Karni (2013).

(A.7) (**Complete beliefs**) For all events E and $\gamma \in [0,1]$, and constant acts r and q such that $r \succ^1 q$, either $r\gamma q \succ^1 rEq$ or $rEq \succ^1 r\gamma' q$, for every $\gamma > \gamma'$.

If the decision maker's beliefs are complete, then the incompleteness of the random choice relations $\succ^\alpha$, $\alpha \in [0,1]$, on H is due entirely to the

[14] By monotonicity, $r \succ^1 q$ implies that $r \succ^\alpha q$, for all $\alpha \in [0,1]$.

incompleteness of his tastes. To state the next result I introduce the following additional notations. Let $\langle \mathcal{U}^\alpha \rangle := cl\{con\,(\mathcal{U}^\alpha) + \{\theta \mathbf{1}_X\}_{\theta \in \mathbb{R}}$ (i.e., $\langle \mathcal{U}^\alpha \rangle$ denotes the closure, with respect to the sup-norm topology, of the cone generated by $\mathcal{U}^\alpha$ and the constant real-valued functions on X). The next Corollary is an implication of Theorem 2.

Corollary 2: *Let* $\{\succ^\alpha \mid \alpha \in [0,1]\}$ *be binary relations on* H. *Then* H *is bounded and, for each* $\alpha \in [0,1]$, $\succ^\alpha$ *satisfies (A.1)–(A.4), jointly* $\succ^\alpha$, $\alpha \in [0,1]$, *satisfy (A.5) and* $\succ^1$ *satisfies (A.7), if and only if* $\succ^\alpha$ *is represented by (1) and (2) with* $\Theta^\alpha = \mathcal{U}^\alpha \times \{\pi\}$ *and* $\alpha \geq \alpha'$ *if and only if* $\mathcal{U}^\alpha \supseteq \mathcal{U}^{\alpha'}$. *Moreover, the probability measure,* π, *is unique and* $\pi(s) > 0$, *for all* $s \in S$, *and if* $\mathcal{V}^\alpha$ *is another set of real-valued functions on* X *that represent* $\succ^\alpha$ *in the sense of (2) then* $\langle \mathcal{V}^\alpha \rangle = \langle \mathcal{U}^\alpha \rangle$.

5 Concluding Remarks

This paper proposes a novel approach to modeling decision making under certainty, risk and uncertainty in situations in which the preference relations are incomplete. The indecisiveness that is due to the noncomparability of the alternatives under consideration, is captured by a set of partial strict orders on the corresponding choice sets. The implied probabilistic choice behavior was characterized.

5.1 Interpersonal comparisons

Different decision makers may exhibit distinct random choice behaviors because of different attributes of the ICMs that depict their decision-making processes. Specifically, the preference relations may not agree on the sets of alternatives that are noncomparable. For example, one decision maker may strictly prefer an alternative a over a', displaying resolute choice, while another decision maker may find the same alternatives noncomparable and display irresolute choice behavior. Even if the decision makers are indecisive with regard to the two alternatives, they may still exhibit distinct random choice patterns due to distinct underlying signal-generating processes. To grasp this let the ICM model of the one decision maker be $\{\succ^\alpha \mid \alpha \in [0,1]\}$ and that of another be $\{\hat{\succ}^\alpha \mid \alpha \in [0,1]\}$. Suppose both models agree that a and a' are noncomparable. It may still be that $a \not\succ^\alpha a'$ and $a \hat{\succ}^{\alpha'} a'$, for

$\alpha \neq \alpha'$. According to the ICM, the former decision maker chooses a with probability α and the latter with probability α'.

5.2 Behavioral implications

Any meaningful theory that purports to describe natural or social phenomena must be accompanied by clear testable implications. To render the proposed ICM meaningful I describe briefly below experiments designed to test qualitative and quantitative properties of the model. Generally speaking, testing the proposed ICM requires that alternatives the decision maker considers noncomparable be identified and the agreement between the observed choices among such alternatives and the probabilistic choices predicted by the model evaluated.

In the cases of decision making under risk and under uncertainty, monotonicity of the preference relations with respect to first-order stochastic dominance is a property that transcends individual idiosyncratic attitudes towards risk or uncertainty. Consequently, the multi-prior expected multi-utility model with incomplete preferences displays *probabilistic choice monotonicity with respect to first-order stochastic dominance*. Formally, if an act h first-order stochastically dominates an act g, and f in noncomparable to either h or g, then the probability that f is selected from the pair (f, g) is greater than the probability that it is selected from the pair (f, h).

The degree of incompleteness of a decision maker's preference relation is a personal characteristic. Therefore, to obtain testable implications of the ICM, one needs formal measures of the degree of incompleteness and an elicitation scheme by which it is possible to determine the individual degree of incompleteness. Karni and Vierø (2021) introduced such measures and as well as incentive compatible mechanisms by which the incompleteness displayed by a preference relation may be elicited.

The experimental test of the probabilistic choice monotonicity hypothesis consists of two parts. In the first part, a set $J = \{1, ..., n\}$ of subjects is recruited and the their ranges of incompleteness of bets on an event, E, are elicited using the scheme of Karni and Vierø (2021). In the second part, the subjects are asked to choose, repeatedly, between a bet on E and a sure payoffs that are noncomparable to the bet. The prediction of the ICM is that the relative frequency of choosing the bet decreases monotonically with the values of the sure payoff.[15]

The experiments described above is designed to test a qualitative property of the ICM, namely, probabilistic choice monotonicity that transcends the idiosyncratic variations of individual stochastic signal-generating processes. They are not designed to quantify the change in the probabilistic choice behavior in response to variations in the sets alternatives.

To understand the kind of qualitative constrained imposed by the ICM model on subjects' choice behavior, consider the following experiment. A *bet on an event* E is an act that pays off x dollars if E obtains and y dollars otherwise, where $x > y$. Let $x_E y$, $x'_E y'$, and $x''_E y''$ be three bets on E, where $y'' < y' < y < x < x' < x''$, and suppose that no two of these bets are comparable.[16] The subjects are asked to choose, repeatedly, from the binary set $\{x_E y\, , \, x'_E y'\}$, $\{x'_E y'\, , \, x''_E y''\}$ and $\{x_E y\, , \, x''_E y''\}$. Let α, α' and α'' denote the relative frequency of choosing the $x_E y$ from the first set, $x'_E y'$ from the second, and $x''_E y''$ from the third. Then the ICM model predicts that:

a. If $\alpha \leq \alpha'$, then facing the choice among the three bets, the subject chooses $x_E y$ with probability α , and $x''_E y''$ with probability $(1 - \alpha)$. The probability that the subject chooses $x'_E y'$ is zero (i.e., $x'_E y'$ is a dominated bet in the set $\{x_E y\, , \, x'_E y', x''_E y''\}$).

b. If $\alpha > \alpha'$, then facing the choice among the three bets, the subject chooses $x_E y$ with probability α , $x''_E y''$ with probability $(1 - \alpha')$ and $x'_E y'$ with probability $\alpha' - \alpha$.

c. Suppose that payoffs of the bets are such that $y' < y'' < y < x < x'' < x'$ and no two bets are comparable then, facing the choice among the bets in $\{x_E y\, , \, x'_E y', x''_E y''\}$, the subject chooses $x_E y$ with probability $1 - \alpha''$, $x''_E y''$ with probability $\alpha'' - \alpha'$, and $x'_E y'$ with probability α'.

5.3 Related literature

Building on existing representation results, specific applications of the ICM to decision making under certainty and uncertainty were axiomatically characterized. In all instances, the ICM predicts random choice behavior and

[15] This method is discussed in Loomes and Sugden (1998) and was implemented in a study by Loomes, Moffatt, and Sugden (2002). To provide the subjects with incentive to consider the choice seriously, one of each subject's choices is randomly selected, and the subject is rewarded according to the outcome of the selecte bet.

[16] The bets are chosen afer the range of incompleteness at E is elicited, using the scheme described in Karni and Vierø (2021).

assigns probabilities to the different feasible alternatives depicting the likelihoods of their being selected. The ICM can be applied to nonexpected utility theories with incomplete preferences such as the dual theory (Maccheroni [2004]), the probabilstically sophisticated choice (Karni [2020]) and weighted utility theory (Karni and Zhou [2021]) using the same approach.

The recognition that, in many settings, observed choices display stochastic behavior lead, in recent years, to increased interest in modeling and testing stochastic choice behavior.[17] In general, these studies do not attribute stochastic choice behavior specifically to preference incompleteness. Exceptions include Danan (2010) and Ok and Tserenjigmid (2020).

Danan (2010) modeled a two-stage decision-making process according to which in the first stage any two alternatives that are evaluated are either ranked in the strict sense, or being judged as being equally valuable. If no judgment is rendered comparing their values, the two alternatives are determined to be noncomparable. In the second stage, the alternative that is ranked higher, if such an alternative exists, is selected. Otherwise, one of the alternatives is chosen either by deliberate randomization or selectively. Danan's analysis addresses the vulnerability of the decision process to being manipulated to produce sure losses through a process known as a money pump. In the case of deliberate randomization, choice behavior is based on a signal produced be a randomization device. In terms of the irresolute choice model of this paper, the signal space of the device is mapped onto the canonical signal space by ascribing to the sets of utility functions that rank one alternative over the other the probability that the first alternative is selected by the randomization device.

Ok and Tserenjigmid (2020) model random choice behavior as random choice functions, which they define and characterize for stochastic choices induced by indifference, indecisiveness, and experimentation. The first two are closely related to the phenomena modeled in this paper. Ok and Tserenjigmid merely assert that the maximal elements of the menu will be chosen with positive probability. By contrast, the ICM characterizes the random choice behavior.

Karni and Safra (2016) study stochastic choice under risk and under uncertainty based on the notion that decision makers' actual choices are governed by randomly selected *states of mind*. They provide axiomatic characterization of the representation of decision makers' perceptions of the stochas-

[17]See Luce (1959), Gul et al. (2014), Fudenberg et al. (2015), Frick et al. (2019).

tic process underlying the selection of their state of mind. In the context of decision making under uncertainty with incomplete preferences, the states of mind are probability-utility pairs in the set Φ.[18] The stochastic choice process corresponds to a subjective probability measure, λ, of the sets Φ^λ. To the extent that the decision maker's introspective perception of the random choice process agrees with the actual random choice process, (that is, $\lambda = \alpha$). The work of Karni and Safra (2016) may be regarded as an alternative axiomatic foundations of the ICM.

Ok and Tserenjigmid (2021) propose to make rationality comparisons between stochastic choice rules by means of a partial ordering method. According to their method, the stochastic choice model of this paper is maximally rational.[19]

[18]In the special cases of Knightian uncertainty and complete beliefs, the sets of states of mind are Π and $\mathcal{U}^1$, respectively.

[19]The stochastic choice function on binary menus that selects an alternative a from the menu $\{a, a'\}$ with probability smaller or equal to α if and only if $a \succ^\alpha a'$ is, by (P1), what Ok and Tserenjigmid (2021) call moderately stochastically transitive. Because it is restricted to binary menus, by their Lemma 3.3, it is maximally rational.

References

[1] Aumann, Robert J. (1962) "Utility Theory without the Completeness Axiom," *Econometrica* 30, 445-462.

[2] Anscombe, Francis J. and Aumann, Robert J. (1963) "A Definition of Subjective Probability," *Annals of Mathematical Statistics* 43, 199–205.

[3] Bewley, Truman F. (2002) "Knightian Decision Theory: Part I," *Decision in Economics and Finance,* 25, 79-110.

[4] Baldassi, Carlo, Cerreia-Vioglio, Simone, Maccheroni, Fabio,Marinacci Massimo, and Pirazzinia, Marco (2020) "A Behavioral Characterization of the Drift Diffusion Model and Its Multialternative Extension for Choice Under Time Pressure." *Management Science* 66, 5075–5093

[5] Cettolin, Elena and Riedl, Arno (2019) "Revealed Preferences under Uncertainty: Incomplete Preferences and Preferences for Randomization," *Journal of Economic Theory,* 181, 547-585.

[6] Danan, Eric and Ziegelmeyer, Anthony (2006) "Are Preferences Complete?: An Experimental Measurement of Indecisiveness under Risk," mimeo, Max Planck Inst. of Economics.

[7] Danan, Eric (2010) "Randomization vs. Selection: How to Choose in the Absence of Preference?" *Management Science* 56, 503–518

[8] Dubra, Juan, Maccheroni, Fabio and Ok, Efe A. (2004) "Expected Utility Theory without the Completeness Axiom," *Journal of Economic Theory,* 115, 118-133.

[9] Eliaz, Kfir and Ok, Efe, A. (2006) "Indifference or indecisiveness? Choice-theoretic Foundations of Incomplete Preferences," *Games and Economic Behavior*, 56, 61–86.

[10] Evren, Özgür and Ok, Efe, A. (2011) "On the Multi-Utility Representation of Preference Relations," *Journal of Mathematical Economics*, 47, 554-563.

[11] Evren, Özgür, Nishimura, Hiroki, and Ok, Efe A. (2019) "Top-Cycles and Revealed Preference Structures," unpublished manuscript.

[12] Frick, Mira, Iijima, Ryota, and Strzalecki, Tomasz (2019) "Dynamic Random Utility," *Econometrica*, 87, 1941–2002

[13] Fudenberg, Drew, Iijima, Ryota, and Strzalecki, Tomasz (2015) "Stochastic Choice and Revealed Perturbed Utility," *Econometrica*, 83, 2371–2409.

[14] Galaabaatar, Tsogbadral and Karni Edi (2012) "Expected Multi-Utility Representations," *Mathematical Social Sciences* 64: 242-246.

[15] Gul, Faruk, Natenzon, Paulo and Pesendorfer, Wolfgang (2014) "Random Choice as Behavioral Optimization," *Econometrica*, 82, 1873–1912.

[16] Karni, Edi (2011) "Continuity, Completeness and the Definition of Weak Preferences," *Mathematical Social Sciences* 62, 123-125.

[17] Karni, Edi (2020) "Probabilisitic Sophistication without Completeness," *Journal of Mathematical Economics*, 89, 8-13.

[18] Karni, Edi and Safra, Zvi (2016) "A Theory of Stochastic Choice under Uncertainty." *Journal of Mathematical Economics* 63, 164-173.

[19] Karni, Edi and Zhou, Nan (2021) "Weighted Utility Theory with Incomplete Preferences," *Mathematical Social Sciences*, 113, 116-135.

[20] Karni, Edi and Vierø, Marie-Louise (2021) "Comparative Incompleteness: Measurement, Behavioral Manifestations and Elicitation," unpublished manuscript

[21] Krajbich, Ian, Oud, Bastiaan and Fehr, Ernst (2014) "Benefits of Neuroeconomic Modeling: New Policy Interventions and Predictors of Preference," *American Economic Review* 104, 501-506.

[22] Loomes, Graham and Robert Sugden (1998). "Testing Alternative Stochastic Specifications for Risky Choice," *Economica* 65, 581–598.

[23] Loomes, Graham, Moffatt, Peter G. and Robert Sugden (2002) "A Microeconometric Test of Alternative Stochastic Theories of Risky Choice," *Journal of Risk and Uncertainty*, 24 103–130.

[24] Luce, Duncan, R. (1959) *Individual Choice Behavior: A Theoretical Analysis*. New York: Wiley.

[25] Maccheroni, Fabio (2004) "Yaari's Dual Theory without the Completeness Axiom," *Economic Theory* 23: 701-714.

[26] Masatlioglu, Yusufcan and Efe A. Ok (2005) "Rational Choice with Status Quo Bias," *Journal of Economic Theory*, 121, 1–29.

[27] Nau, Robert (2006) "The Shape of Incomplete Preferences," *Annals of Statistics,* 34, 2430-2448.

[28] Ok, Efe A. (2002) "Utility Representation of an Incomplete Preference Relation," *Journal of Economic Theory,* 104, 429-449.

[29] Ok, Efe A. Ortoleva, Pietro and Riella, Gil (2012) "Incomplete Preferences Under Uncertainty: Indecisiveness in Beliefs versus Tastes," *Econometrica*, 80, 1791-1808

[30] Ok, Efe A. and Tserenjigmid, Gerelt (2020) "Indifference, Indecisiveness, Experimentation and Stochastic Choice," *Theoretical Economics* (Forthcoming).

[31] Ok, Efe A. and Tserenjigmid, Gerelt (2021) "Comparative Rationality of Random Choice Behaviors," unpublished manuscript.

[32] Riella, Gil (2015) "On the Representation of Incomplete Preferences under Uncertainty with Indecisiveness in Tastes and Beliefs," *Economic Theory* 58, 571-600.

[33] Shapley, Lloyd S. and Baucells, Manel J. (1998) "Multiperson Utility," working paper

[34] Sautua, Santiago I. (2017) "Does Uncertainty Cause Inertia in Decision Making? An Experimental Study of the Role of Regret Aversion and Indecisiveness," *Journal of Economic Behavior and Organization,* 136, 1-14.

[35] Seidenfeld, Teddy, Schervish, Mark, J. and Kadane, Joseph, B. (1995) "A Representation of Partially Ordered Preferences," *Annals of Statistics,* 23, 2168-2217.

[36] von Neumann, John and Morgenstern, Oscar (1947) *Theory of games and Economic Behavior* Princeton University Press, Princeton.

[37] Zappia, Carlo (2020) "Paradox? What paradox? On a brief correspondence between Leonard Savage and Karl Popper, *Research in the History of Economic Thought and Methodology*, 38C, 161-177.