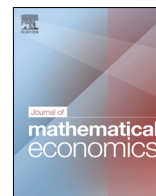




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journal homepage: www.elsevier.com/locate/jmatecoHybrid decision model and the ranking of experiments[☆]Edi Karni^{a,*}, Zvi Safra^b^a Johns Hopkins University, Department of Economics, United States of America^b Warwick Business School, University of Warwick, United Kingdom

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ABSTRACT

Departing from the reduction of compound lotteries axiom on multi-stage lotteries, this paper proposes a new hybrid model to analyze decision trees. Applied to multi-stage decision trees induced by experiments, Blackwell's (1953) definition of the relation "more informative" on the set of information structures is equivalent to experiments being more valuable to a class of non-expected utility preferences. This result extends Blackwell's theorem and provides new insights regarding the evaluation of information produced by experiments.

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1. Introduction

From a decision making point of view, experimentation is valuable because it provides information that helps decision makers choose courses of actions whose payoffs are higher in the states that are more likely to obtain. Blackwell (1953) formalized this perception as a binary relation: "more informative than" on the set of experiments. According to Blackwell, one experiment is more informative than another if, for every set of feasible actions, it yields a richer menu of experiment-wise expected payoffs (i.e., expected-loss vectors) each of which corresponds to an action taken contingent on the experimental observations. Blackwell characterized this relation by proving that one experiment is more informative than another if and only if the information content of the latter is obtained by garbling the information content of the former. Equivalently, an experiment is more informative if it allows choices that have higher expected utility. We refer to this equivalence as Blackwell's theorem.¹

Seen in this way, being better informed seems unambiguously beneficial. Thus, the equivalence between ranking experiments by their information content and their ranking by the expected utility criterion seems oddly restrictive. This equivalence is particularly disconcerting in view of experimental evidence suggesting that subjects systematically violate the tenets of expected

utility theory – the sure thing principle and the independence axiom – and the proliferation, over the last 40 years, of non-expected utility models of decision making under risk and under uncertainty.

To grasp the issue, consider a decision maker facing a choice among feasible actions whose consequences depend on the realization of some underlying states. Suppose that the likelihood of the various states materializing is quantified by a (prior) probability distribution function. Before choosing an action, the decision maker receives a signal (i.e., an observation), produced by an experiment, that informs him about the likely realization of the states. Upon receiving such signal, the decision maker invokes Bayes' rule to update the prior state probabilities and then proceeds to choose an action from the feasible set. This process may be thought of as two-stage compound lottery. In the first stage, the experiment produces a signal, according to some probability distribution on the set of signals, following which the decision maker chooses an action. In the second stage, a state is selected (according to the posterior distribution) and the decision maker is awarded the prize that corresponds to the image of the selected state under the chosen action.² The question is how decision makers perceive this two-stage lottery.

We argue that the critical aspect of the expected utility model that underlies Blackwell's theorem is the way these compound lotteries are handled. The standard way of handling compound

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* Corresponding author.

E-mail addresses: karni@jhu.edu (E. Karni), zvi.safra@wbs.ac.uk (Z. Safra).

¹ Cremer (1982), Leshno and Spector (1992), and de Oliveira (2018) provide simple proofs of Blackwell's theorem.

² If the prize itself is a lottery ticket then the procedure described above amounts to three-stage lottery in which, in the third and final stage, the lottery corresponding to the image of the selected state under the chosen action is played out to determine the prizes.

lotteries is to apply the reduction of compound lotteries axiom (henceforth, RCLA). This axiom asserts that a multi-stage lottery is reduced to a single-stage lottery by attributing to each ultimate payoff a probability equal to the product of the probabilities on the events that lead to it. In the context of the choice of experiments, the RCLA assigns the ultimate outcome the probability of the signal multiplied by the posterior probabilities of the states to which the chosen course of action assigns that outcome.

Analysis that treats the two-stage process as equivalent to its one-stage reduction runs the risk of ignoring subtleties that beset the extensive form decision process. Wakker (1988) and Safra and Sulganik (1995) showed that departing from the independence axiom and maintaining the RCLA implies a widespread and robust aversion towards information.

An alternative procedure of handling compound lotteries is to use the certainty-equivalent reduction. In this procedure, multi-stage lotteries are reduced to single-stage lotteries by folding back the lottery tree, replacing the lotteries along the branches by their certainty equivalents. Schlee (1990) and Safra and Sulganik (1995) showed that in non-expected utility theories, this procedure implies that information is not always valuable.

We propose a hybrid decision model that invokes the RCLA in the second stage of the decision-making process and a procedure analogous to certainty-equivalent reduction in the first stage. Application of the RCLA, which seems compelling when the transition between the stages is automatic, seems less so when the two stages are separated by an intermediate decision.

To formalize this idea, we propose a new model, dubbed the *hybrid model*, and show that it identifies a class of preferences that unambiguously value one experiment over another if and only if the information content of the latter is obtained by garbling that of the former. The expected utility model is a special case of this class. In fact, it is the only hybrid model that is consistent with the RCLA. Application of the hybrid model provides new insight into the manner in which information is evaluated. While admitting the possibility that some decision makers may not consider information to be unambiguously beneficial, it maintains that, in Blackwell's analytical framework, such behavior is unreasonable.

The surprising (difficult) aspect of Blackwell's theorem is that a more informative experiment (that is, one that affords better decisions by the expected utility criterion) implies clearer signals. In this paper, informativeness corresponds to an experiment being more valuable in the sense of affording better decisions for a broader set of preferences, including expected utility preferences. Consequently, this direction of the proof relies on Blackwell's theorem. The novelty of this paper is the observation that the full power of expected utility – in particular, the RCLA – is not needed for Blackwell's result. To the best of our knowledge, ours is the only nontrivial model in which a large class of non-expected utility preferences can satisfy this direction of Blackwell's theorem.³

The rest of the paper is organized as follows. The next section provides a brief review of Blackwell's (1953) theorem. Section 3 reviews the reduction procedures. Section 4 introduces and characterizes the hybrid decision model. Section 5 extends Blackwell's theorem. Section 6 discusses the value of information and reviews the related literature.

³ As pointed out by a referee, if the RCLA is used and either commitment is possible or decision makers are not allowed to change their choices when information is revealed, Blackwell's theorem holds regardless of lottery preferences. See Li and Zhou (2016) for an analysis with commitment in the context of ambiguity.

2. The analytical framework and Blackwell's theorem

2.1. The analytical framework

Let $S = \{s_1, \dots, s_n\}$ be a finite set of states and denote by $\Delta(S)$ the simplex in \mathbb{R}^n . Let X be a set of outcomes and let \mathcal{H} denote the set of all mappings from S to X , referred to as acts. Acts represent potential courses of action. For a given $f \in \mathcal{H}$ and $\tau \in \Delta(S)$, the act-probability pair (f, τ) induces a lottery $\sum_{s \in S} \tau(s) \delta_{f(s)} \in \Delta(X)$, where $\Delta(X)$ is the set of all simple probability distribution on X .⁴

Let \mathbb{Y} be a set (with cardinality $N < \infty$) whose generic element, y , is a signal. An experiment, also called an information structure, is identified with a conditional probability distribution μ on $\mathbb{Y} \times S$ such that, for all $s \in S$, $\mu(y | s)$ is the conditional probability of $y \in \mathbb{Y}$ given s . Let \mathbb{M} denote the set of all experiments. The support of $\mu \in \mathbb{M}$ is the set $Y = \{y \in \mathbb{Y} | \exists s \in S : \mu(y | s) > 0\}$ of all potentially realized signals under μ . The experiment μ can be identified with the $|S| \times |Y|$ right-stochastic matrix, $P(\mu)$, whose generic element is $\mu(y | s)$. For a given prior $\pi \in \Delta(S)$, denote $\mu(y) = \sum_{s \in S} \pi(s) \mu(y | s)$.

2.2. Blackwell's theorem

Consider an expected utility-maximizing decision maker characterized by a utility function u in \mathcal{U} , the set of all real-valued functions on X . Consider a fixed and strictly positive probability distribution $\pi \in \Delta(S)$ and a set of feasible acts $B \in \mathcal{B}$, where \mathcal{B} is the set of all non-empty, compact and convex subsets of \mathcal{H} .⁵ Assume that, before choosing an act from B , the decision maker observes the signal y , generated by an experiment $\mu \in \mathbb{M}$, and updates the prior distribution π according to Bayes' rule to obtain the posterior probability distribution $\pi(s | y) = \pi(s) \mu(y | s) / \mu(y)$, for all $s \in S$. The decision maker's ex post problem is:

$$\max_{f \in B} \sum_{s \in S} \pi(s | y) u(f(s)). \tag{1}$$

Denoting the maximal value by $U(u, B)$, the expected utility associated with the experiment μ is given by:

$$\hat{U}(\mu; u, B) := \sum_{y \in Y} \mu(y) U(u, B). \tag{2}$$

Definition 1. An experiment μ is more informative than another experiment μ' if, for all $(u, B) \in \mathcal{U} \times \mathcal{B}$,

$$\hat{U}(\mu; u, B) \geq \hat{U}(\mu'; u, B).$$

Let \mathcal{M} be the set of $|Y| \times |Y'|$ right-stochastic matrices dubbed *garbling matrices*. If there exists a garbling matrix $M \in \mathcal{M}$ such that $P(\mu)M = P'(\mu')$, then M introduces noise that blurs the information in $P(\mu)$. That is, for each entry $\mu'(y'_r | s)$ of $P'(\mu')$, $\mu'(y'_r | s) = \sum_{y_j \in Y} \mu(y_j | s) m_{jr}$ (where $\sum_{\{r | y_r \in Y'\}} m_{jr} = 1$ for all $\{j | y_j \in Y\}$).

Definition 2. An experiment μ is sufficient for μ' if the corresponding information structures satisfy $P(\mu)M = P'(\mu')$, for some $M \in \mathcal{M}$.

With these definitions in mind Blackwell's theorem is stated as follows:

⁴ Simple probability distributions are probability distribution functions with finite supports. $\Delta(X)$ is a mixture set: for all $p, q \in \Delta(X)$ and $\alpha \in [0, 1]$, $(\alpha p + (1 - \alpha) q)(x) = \alpha p(x) + (1 - \alpha) q(x)$, for all $x \in X$. $\delta_x \in \Delta(X)$ is a Dirac measure on X , identified with $x \in X$.

⁵ In this literature it is commonly assumed that feasible sets are convex hulls of non-empty finite sets.

Blackwell's Theorem: An experiment μ is more informative than another experiment μ' if and only if μ is sufficient for μ' .

That is, the relation “being sufficient” is equivalent to being ranked higher by all preference relations that admit expected utility representations.

3. Informative signals and reduction procedures

3.1. Signals

According to Blackwell's theorem more, informative experiments produce clearer signals in the following sense: When comparing two experiments, every posterior distribution produced by the less informative experiment is a weighted average of the posterior distributions produced by the more informative experiment (i.e., the sufficient experiment). Formally, let $\mu, \mu' \in \mathbb{M}$ with supports Y and Y' , respectively. Suppose that μ is sufficient for μ' , hence there exists $M \in \mathcal{M}$ such that $P(\mu)M = P'(\mu')$. Observe that the equalities

$$\mu'(y'_r | s) = \sum_{\{j|y_j \in Y\}} \mu(y_j | s) m_{jr}, \quad (3)$$

for all $y'_r \in Y'$ and $s \in S$, imply

$$\mu'(y'_r) = \sum_{\{j|y_j \in Y\}} \mu(y_j) m_{jr}. \quad (4)$$

Therefore, by Bayes' rule,

$$\begin{aligned} \pi(s | y'_r) &= \frac{\pi(s) \mu'(y'_r | s)}{\sum_{s' \in S} \pi(s') \mu'(y'_r | s')} \\ &= \frac{\pi(s) \sum_{\{j|y_j \in Y\}} \mu(y_j | s) m_{jr}}{\mu'(y'_r)} \\ &= \frac{\sum_{\{j|y_j \in Y\}} \pi(s) \mu(y_j | s) m_{jr}}{\mu'(y'_r)} \\ &= \frac{\sum_{\{j|y_j \in Y\}} \frac{\mu(y_j)}{\mu'(y'_r)} \pi(s) \mu(y_j | s) m_{jr}}{\mu'(y'_r)} \\ &= \frac{\sum_{\{j|y_j \in Y\}} \mu(y_j) \pi(s | y_j) m_{jr}}{\mu'(y'_r)} \\ &= \sum_{\{j|y_j \in Y\}} \frac{\mu(y_j) m_{jr}}{\mu'(y'_r)} \pi(s | y_j). \end{aligned} \quad (5)$$

and the posterior distributions satisfy,

$$\pi(\cdot | y'_r) = \sum_{\{j|y_j \in Y\}} \frac{\mu(y_j) m_{jr}}{\mu'(y'_r)} \pi(\cdot | y_j).$$

Consequently, for each act-posterior probability pair $(f, \pi(\cdot | y'_r)) \in \mathcal{H} \times \Delta(S)$ that is feasible under the less informative experiment, corresponds a set $\{(f, \pi(\cdot | y_j)) | y_j \in Y\} \subset \mathcal{H} \times \Delta(S)$ of act-posterior probability pairs of the more informative experiment. Thus, from the ex ante viewpoint, the more informative experiment offers a richer set of opportunities to match feasible acts to the perceived likelihood of the states depicted by their posterior probabilities.

3.2. Reduction procedures

Consider the set $\Delta(\Delta(X))$ of simple two-stage compound lotteries on X with generic element $\tilde{q} = \sum_p \tilde{q}(p) \delta_p$ ($p \in \Delta(X)$) (\tilde{q} has a finite support). The most common way to reduce such a two-stage lottery to a one-stage lottery in $\Delta(X)$ is the *reduction of compound lotteries axiom* (RCLA). This reduction procedure identifies \tilde{q} with the one-stage mixture lottery $\sum_p \tilde{q}(p) p \in \Delta(X)$. The

axiomatic structure underlying expected utility theory implicitly utilizes RCLA.

Another way to reduce two-stage lotteries is the *certainty-equivalent reduction*.⁶ This reduction procedure assumes that the decision maker possesses a preference relation over $\Delta(X)$ and that, for every lottery $p \in \Delta(X)$, there exists an element $c(p) \in X$, the *certainty equivalent* of p , such that the decision maker is indifferent between p and $\delta_{c(p)}$.⁷ The certainty-equivalent reduction identifies \tilde{q} with the one-stage mixture lottery $\sum_p \tilde{q}(p) \delta_{c(p)} \in \Delta(X)$. Like RCLA, this reduction is implicit in expected utility theory.

In general, the resulting one-stage lotteries $\sum_p \tilde{q}(p) p$ and $\sum_p \tilde{q}(p) \delta_{c(p)}$ are not equivalent. However, under expected utility they are closely related in the sense that the decision maker is always indifferent between them.

To see how these reductions affect the evaluation of experiments, let a strictly positive probability distribution $\pi \in \Delta(S)$ and a set $B \in \mathcal{B}$ of feasible acts be given, and consider a decision maker facing an experiment $\mu \in \mathbb{M}$. From the decision maker's viewpoint, this decision problem requires choosing acts in B contingent on the realization of signals produced by μ . This problem can be described as a two-stage compound lottery. In the first stage, a signal $y \in Y$ is drawn according to the distribution μ . Contingent on the signal and the corresponding posterior distribution $\pi(\cdot | y)$, any act $f \in B$ can be chosen. Hence, in the second stage the decision maker is faced with the lottery $\sum_{s \in S} \pi(s | y) \delta_{f(s)} \in \Delta(X)$ induced by the pair $(f, \pi(\cdot | y))$, in which a state s is selected according to the posterior distribution $\pi(\cdot | y)$ and an outcome $f(s) \in X$ is awarded.

It is worth underscoring that the sequence of events described above is depicted, as it should be, from the point of view of the decision maker who receives the signal and must act upon the information it provides. The problem may be reformulated so that the state is drawn first according to the prior π , followed by a signal drawn from Y according to the conditional distribution $\mu(\cdot | s)$. Because the state is not observable, however, the decision maker may not be able to distinguish between nodes resulting from different states that potentially yield the same signal (while he must act solely on the basis of the information produced by the signal). Consequently, from the decision maker's point of view, reformulating the problem so that the state is selected in the first stage followed by the draw of a signal is substantively equivalent to the formulation above but would complicate the notations and clutter the exposition.

Consider a non-expected utility decision maker whose preferences over $\Delta(X)$ are represented by some utility functional v and, for each signal y , let $f_y^* \in B$ be a maximizer of v over the set of one-stage lotteries $\{\sum_{s \in S} \pi(s | y) \delta_{f(s)} | f \in B\}$ (that is, $f_y^* \in \arg \max_{f \in B} v(\sum_{s \in S} \pi(s | y) \delta_{f(s)})$). Applying RCLA, the two-stage compound lottery induced by the experiment $\mu \in \mathbb{M}$ is reduced to the one-stage lottery

$$\sum_{y \in Y} \mu(y) \sum_{s \in S} \pi(s | y) \delta_{f_y^*(s)} = \sum_{s \in S} \left(\sum_{y \in Y} \mu(y) \pi(s | y) \right) \delta_{f_y^*(s)}. \quad (6)$$

The application of the RCLA procedure is commonly used, in conjunction with consequentialism, to analyze behavior of non-expected utility decision makers. The procedure implies a widespread and robust aversion to information. The reason for this aversion is that the RCLA, in conjunction with consequentialism, imply that preference relations display dynamic consistency (that

⁶ For more detailed discussion and application of certainty equivalent reduction, see Segal (1987).

⁷ The existence of certainty equivalents depends on the richness of the set of outcomes X .

is, the optimal continuation of the sequential choice process as of any decision node agrees with the optimal contingent plan, made at the outset of the process, for that node) if and only if it satisfies the independence axiom of expected utility theory.

To grasp this assertion, consider the following simple example. A decision maker faces a choice between a lottery A and an equal chance of receiving another lottery, B , or a choice between two lotteries, C and D (all lotteries are in $\Delta(X)$). At the initial decision node, the decision maker must choose between A and two contingent plans, one calling for the choice of C and the other the choice of D in case he finds himself at the second decision node. Under the RCLA, the former contingent plan induces the lottery $0.5B + 0.5C$, and the latter the lottery $0.5B + 0.5D$. If the preference relation does not satisfy the independence axiom, there are always lotteries C and D such that $C \succ D$ and $0.5B + 0.5D \succ 0.5B + 0.5C$. Suppose further that, $0.5B + 0.5D \succ A \succ 0.5B + 0.5C$. Then the decision maker rejects lottery A in favor of pursuing the contingent plan that calls for the choice of D in the second decision node. However, if he happens to find himself at that node and is satisfying consequentialism, he will choose C , thereby displaying dynamic inconsistency. Furthermore, realizing this dynamic inconsistency, the decision maker would prefer A over the contingent plan that yields $0.5B + 0.5C$.⁸ Consequently, in our case, evaluating the lotteries induced by the pairs $(f_y^*, \pi(\cdot | y))$ of optimal acts and posterior probabilities from an *ex ante* point of view, by looking at the one-stage lottery (6), the decision maker who does not abide by the independence axiom may find out that he prefers to replace some of these *ex post* optimal acts by others. However, when he finds himself at the second stage, optimality dictates that he choose the acts f_y^* .

In the sequel, for simplicity and with a slight abuse of notation, we will also use f_y^* to denote the lottery $\sum_{s \in S} \pi(s | y) \delta_{f_y^*(s)}$ (the exact meaning will be clear from the context). Applying the certainty-equivalent reduction, the two-stage compound lottery induced by the experiment is reduced to the one-stage lottery

$$\sum_{y \in Y} \mu(y) \delta_{c(f_y^*)} \quad (7)$$

where $c(f_y^*)$ is the certainty equivalent of f_y^* for the utility functional v . Under this procedure, information is not always valuable for non-expected utility decision makers. Unlike the RCLA, however, the reason here is not dynamic inconsistency (it is easy to verify that, viewing the one-stage lottery (7) *ex ante*, the decision maker is always in agreement with his planned *ex post* optimal acts f_y^*). It is that the mixture and certainty-equivalent operators are not commutative for non-expected utility decision makers. Thus, having larger sets of options under a sufficient experiment does not necessarily translate to being in a preferable situation.

An alternative procedure, which we propose and study in this paper, applies the RCLA to the second stage and a reduction analogous to the certainty equivalent to the first stage.⁹ Under this *hybrid procedure*, which is characterized in the next section, the value of the experiment μ is given by

$$\sum_{y \in Y} \mu(y) v(f_y^*).$$

The justification for applying distinct procedures to the different stages is the nature of the uncertainties involved. In the second stage, given the act and the (updated) state probabilities, the outcome is selected “algorithmically” without interference by

⁸ For a detailed analysis see Karni and Schmeidler (1991), and Karni and Safra (1989).

⁹ The analogous reduction does not require the existence of certainty equivalents.

the decision maker. By contrast, after the first stage, corresponding to each signal there is an interim step at which the decision maker interferes by updating the state probabilities and choosing an act. This aspect of the dynamic process suggests that decision makers may regard the first stage as qualitatively distinct from the later stages and, consequently, treat them differently. Specifically, according to the hybrid procedure, assessing the value of experiments, decision makers envision the acts they would choose contingent on the signals, assign these acts utility values, and take the mean utility values as the value of the experiment.

4. Characterization of the hybrid representation

Assume that X is a compact topological space and consider a probability distribution $\pi \in \Delta(S)$, a non-empty compact set $B \in \mathcal{B}$, and an experiment $\mu \in \mathbb{M}$. As above, the experiment can be identified with a two-stage decision tree in which a signal $y \in Y$ is realized in the first stage and, in the second stage, at the decision node associated with y , the decision maker chooses an act from B , inducing the lottery f_y^* . We also assume that in the second stage the RCLA is applied. From an *ex ante* viewpoint, the *ex post* choices are seen as the $|Y| + 1$ tuple $(\mu, (f_y^*)_{y \in Y}) \in \Delta(Y) \times \Delta(X)^{|Y|}$, and the whole experiment can be identified with a subset of all $N + 1$ tuples $(\mu, (p(y))_{y \in Y}) \in \Delta(Y) \times \Delta(X)^N$ (note that the cardinality of Y , the set of all signals, is equal to $N < \infty$). A decision maker is characterized by two complete and transitive, preference relations: (i) an *ex ante* preference relation \succsim on $\Delta(Y) \times \Delta(X)^N$, which ranks decision trees, and (ii) an *ex post* preference relation \succsim^* on $\Delta(X)$, that is used to choose optimal acts at the realized decision nodes. Note that for the preference relation \succsim to have the domain indicated above, it is implicitly assumed that the decision maker is able to express preferences over all conceivable information structures, so that every conceivable distribution on Y is attainable.

Continuity: (of \succsim and \succsim^* with respect to the corresponding Euclidean topology) The sets $\{(\mu', (p'(y))_{y \in Y}) \in \Delta(Y) \times \Delta(X)^N \mid (\mu, (p(y))_{y \in Y}) \succsim (\mu', (p'(y))_{y \in Y})\}$ and $\{(\mu', (p'(y))_{y \in Y}) \in \Delta(Y) \times \Delta(X)^N \mid (\mu', (p'(y))_{y \in Y}) \succsim (\mu, (p(y))_{y \in Y})\}$ are closed for all $(\mu, (p(y))_{y \in Y}) \in \Delta(Y) \times \Delta(X)^N$, and the sets $\{p \in \Delta(X) \mid q \succsim^* p\}$, $\{p \in \Delta(X) \mid p \succsim^* q\}$ are closed for all $q \in \Delta(X)$.

Henceforth we assume that both \succsim and \succsim^* are continuous binary relations. To capture consequentialism, we assume that the *ex post* preference relation \succsim^* is independent of the realized signal y and of the probability distribution μ .

The next axiom asserts that if a signal y is certain to obtain, then the *ex ante* ranking of any two lotteries agrees with their *ex post* ranking. Formally, define $\mathbf{p} = (p(y))_{y \in Y} \in \Delta(X)^N$, then

Consistency: For all $\mathbf{p}, \mathbf{q} \in \Delta(X)^N$ and $y \in Y$, $(\delta_y, \mathbf{p}) \succsim (\delta_y, \mathbf{q})$ if and only if $p(y) \succsim^* q(y)$.

The next axiom asserts that independence applies to the first stage of the decision-making process. To formalize this idea we introduce the partial mixture operation on $\Delta(Y) \times \Delta(X)^N$: For all $(\mu, \mathbf{p}), (v, \mathbf{p}) \in \Delta(Y) \times \Delta(X)^N$ and $\lambda \in [0, 1]$, define $\lambda(\mu, \mathbf{p}) + (1 - \lambda)(v, \mathbf{p}) = (\lambda\mu + (1 - \lambda)v, \mathbf{p})$. The mixture $\lambda\mu + (1 - \lambda)v \in \Delta(Y)$ depicts a potential distribution on the signals space induced by an experiment. Alternatively, it can be interpreted as a choice between two experiments that is decided by a coin flip.

Consider a decision maker who is indifferent between the alternatives (μ, \mathbf{p}) and (μ', \mathbf{q}) and prefers the alternative (v, \mathbf{p})

over (v', \mathbf{q}) . Suppose that, facing a choice between the decision trees $A = (\lambda\mu + (1-\lambda)v, \mathbf{p})$ and $B = (\lambda\mu' + (1-\lambda)v', \mathbf{q})$ he reasons that if the event whose probability is λ obtains and he has chosen A , he is faced with the alternative (μ, \mathbf{p}) ; if he has chosen B , he faces the alternative (μ', \mathbf{q}) . Conditional on the realization of this event, he is indifferent between A and B . By the same logic, he would prefer A over B conditional on the realization of the complementary event whose probability is $1 - \lambda$. Consequently, he prefers A over B unconditionally.¹⁰ Formally,

First-Stage Independence: For all $(\mu, \mathbf{p}), (v, \mathbf{p}), (\mu', \mathbf{q}), (v', \mathbf{q})$ in $\Delta(\mathbb{Y}) \times \Delta(X)^N$ and $\lambda \in [0, 1]$, if $(\mu, \mathbf{p}) \sim (\mu', \mathbf{q})$ then $(v, \mathbf{p}) \succsim (v', \mathbf{q})$ if and only if $(\lambda\mu + (1-\lambda)v, \mathbf{p}) \succsim (\lambda\mu' + (1-\lambda)v', \mathbf{q})$.

The next theorem provides the hybrid representation. It shows that an ex ante preference relation satisfying consistency (with respect to an ex post preference relation) and first-stage independence is representable as a weighted sum of the ex post utilities.

Theorem 1. Let \succsim and \succsim^* be binary relations on $\Delta(\mathbb{Y}) \times \Delta(X)^N$ and $\Delta(X)$, respectively, such that $\succ \neq \emptyset$. Then the following conditions are equivalent:

- (a) \succsim and \succsim^* are complete, transitive, and continuous, jointly they satisfy consistency and \succsim satisfies first-stage independence.
- (b) There exist continuous non-constant functions $V : \Delta(\mathbb{Y}) \times \Delta(X)^N \rightarrow \mathbb{R}$ and $v : \Delta(X) \rightarrow \mathbb{R}$, such that V represents \succsim , v represents \succsim^* and, for all $(\mu, \mathbf{p}) \in \Delta(\mathbb{Y}) \times \Delta(X)^N$,

$$V(\mu, \mathbf{p}) = \sum_{y \in \mathbb{Y}} \mu(y) v(p(y)). \tag{8}$$

Moreover, if there are functions $\bar{V} : \Delta(\mathbb{Y}) \times \Delta(X)^N \rightarrow \mathbb{R}$ and $\bar{v} : \Delta(X) \rightarrow \mathbb{R}$ such that \bar{V} represents \succsim , \bar{v} represents \succsim^* and $\bar{V}(\mu, \mathbf{p}) = \sum_{y \in \mathbb{Y}} \mu(y) \bar{v}(p(y))$ then $\bar{v} = bv + a$, $b > 0$.

Proof. The proof is omitted, as it is identical to that of Theorem 1 of Karni and Safra (2000), with the additional assumption that the ex post preference relation \succsim^* is signal-independent.

Remark. The behavior of a decision maker with such a pair of ex ante and ex post preference relations displays *dynamic consistency*. Formally, if \mathbf{p} and \mathbf{q} differ only when signal y occurs and $\mu(y) > 0$, $p(y)$ is preferred to $q(y)$ ex post if and only if \mathbf{p} is preferred to \mathbf{q} ex ante.

The case in which the ex post preferences satisfy the independence axiom is of special interest. Let $U(p) = \sum_{x \in X} u(x) p(x)$ be an expected utility representation of \succsim^* . Then, by Theorem 1, the ex ante preferences \succsim are represented by

$$V(\mu, \mathbf{p}) = \sum_{y \in \mathbb{Y}} \mu(y) w(U(p(y))), \tag{9}$$

where w is strictly increasing. This representation is more general than expected utility (for \succsim) and is reduced to it when w is affine.¹¹ It resembles the class of non-expected utility models that have representations known as smooth ambiguity attitudes.¹² Note, however, that ambiguity aversion requires that the function w be concave. In the present context, a concave w implies aversion to the spread of signal-contingent payoffs that more informative experiments afford. See further discussion of this issue at the end of the next section.

¹⁰ This axiom is analogous to the constrained independence axiom defined in Karni and Safra (2000).

¹¹ In the social choice framework, Grant et al. (2010) provide an axiomatization of such preferences.

¹² See Klibanoff et al. (2005) and Seo (2009).

5. Blackwell's theorem extended

Consider all pairs (\succ, \succ^*) of ex ante and ex post preferences satisfying the requirements of Theorem 1(a). Let \mathcal{V} be the set of all pairs (V, v) satisfying (8), and let $\mathcal{V}_{cx} \subset \mathcal{V}$ consist of all pairs in which v is convex (the need for convexity will be clarified in the next theorem). As above, for a given strictly positive and fixed $\pi \in \Delta(S)$, $B \in \mathcal{B}$ and $\mu \in \mathbb{M}$, let f_y^* be the one-stage lottery that maximizes the ex post utility v at the decision node associated with y . As the ex ante preferences \succ admit the hybrid representation and are dynamically consistent, the ex ante value of μ is given by $V(\mu, (f_y^*)_{y \in \mathbb{Y}}) = \sum_{y \in \mathbb{Y}} \mu(y) v(f_y^*)$. For every $((V, v), B) \in \mathcal{V}_{cx} \times \mathcal{B}$, the value of an experiment $\mu \in \mathbb{M}$ with support Y is defined by:

$$\hat{V}(\mu; (V, v), B) := \sum_{y \in Y} \mu(y) v(f_y^*). \tag{10}$$

Definition 3. An experiment μ is *more hybrid-valuable* than another experiment μ' if, for all $((V, v), B) \in \mathcal{V}_{cx} \times \mathcal{B}$,

$$\hat{V}(\mu; (V, v), B) \geq \hat{V}(\mu'; (V, v), B).$$

The next theorem extends Blackwell's (1953) theorem.

Theorem 2. An experiment μ is more hybrid-valuable than another experiment μ' if and only if μ is sufficient for μ' .

The proof below rests on two properties of Blackwell's result: (i) that the expected utility under the sufficient experiment is larger and (ii) that the set of expected utility payoffs generated by the sufficient experiment constitutes a larger "spread" of expected utility values. With v monotonic increasing and convex, these properties make the expected value of v larger for the sufficient experiment.

Proof. (Sufficiency) Suppose that μ is sufficient for μ' . Let Y and Y' denote the supports of μ and μ' , respectively. Fix $((V, v), B) \in \mathcal{V}_{cx} \times \mathcal{B}$. Then,

$$\begin{aligned} \hat{V}(\mu'; (V, v), B) &= \sum_{y' \in Y'} \mu'(y') v(f_{y'}^*) \\ &= \sum_{\{r|y'_r \in Y'\}} \mu'(y'_r) v\left(\sum_{s \in S} \pi(s | y'_r) \delta_{f_{y'_r}^*(s)}\right) \\ &= \sum_{\{r|y'_r \in Y'\}} \mu'(y'_r) v\left(\sum_{s \in S} \left[\sum_{\{j|y_j \in Y\}} \frac{\mu(y_j) m_{jr}}{\mu'(y'_r)} \pi(s | y_j)\right] \delta_{f_{y'_r}^*(s)}\right) \\ &= \sum_{\{r|y'_r \in Y'\}} \mu'(y'_r) v\left(\sum_{\{j|y_j \in Y\}} \frac{\mu(y_j) m_{jr}}{\mu'(y'_r)} \left[\sum_{s \in S} \pi(s | y_j) \delta_{f_{y'_r}^*(s)}\right]\right) \\ &\leq \sum_{\{r|y'_r \in Y'\}} \mu'(y'_r) \sum_{\{j|y_j \in Y\}} \frac{\mu(y_j) m_{jr}}{\mu'(y'_r)} v\left(\sum_{s \in S} \pi(s | y_j) \delta_{f_{y'_r}^*(s)}\right) \\ &= \sum_{\{r|y'_r \in Y'\}} \sum_{\{j|y_j \in Y\}} \mu(y_j) m_{jr} v\left(\sum_{s \in S} \pi(s | y_j) \delta_{f_{y'_r}^*(s)}\right) \\ &\leq \sum_{\{r|y'_r \in Y'\}} \sum_{\{j|y_j \in Y\}} \mu(y_j) m_{jr} v\left(\sum_{s \in S} \pi(s | y_j) \delta_{f_{y_j}^*(s)}\right) \\ &= \sum_{\{j|y_j \in Y\}} \mu(y_j) v\left(\sum_{s \in S} \delta_{f_{y_j}^*(s)} \pi(s | y_j)\right) \sum_{\{r|y'_r \in Y'\}} m_{jr} \\ &= \sum_{y \in Y} \mu(y) v(f_y^*) = \hat{V}(\mu; (V, v), B) \end{aligned} \tag{11}$$

where the second line equality follows from (5); the third line equality follows from RCLA; the first inequality holds as v is convex; the second inequality holds since, by definition, $v(\sum_{s \in S} \pi$

$(s | y_j) f_{y_j}^*(s) \leq v \left(\frac{f_{y_j}^*}{y_j} \right) = v(\sum_{s \in S} \pi(s | y_j) f_{y_j}^*(s))$, and last equality holds as $\sum_{\{r | y_r \in Y'\}} m_{jr} = 1$.

(Necessity) When v is affine, the hybrid representation with $(V, v) \in \mathcal{V}_{cx}$ reduces to an expected utility representation, which implies that the set of expected utility preferences is a subset of the set of preference relations that have hybrid representations. Hence, necessity is implied by the necessity part of Blackwell's theorem. ♣

The ingredients of the representation of the hybrid model for affine v in (9) are similar to those of the class of non-expected utility models known as smooth ambiguity models. Smooth ambiguity aversion requires that the analogue of the function w be concave. By contrast, our extension of Blackwell's theorem requires that decision makers exhibit *information proclivity*, an attitude captured by the convexity of the function w . This difference between the two models may be explained by the fact that more informative experiments offer larger spreads of expected utility payoffs to choose from. Consequently, in addition to the unambiguous advantage of allowing decision makers to choose acts whose payoffs are higher in the states that are more likely to obtain, information affects the decision-making process through the decision makers' attitudes towards wider spread of expected payoffs. We showed that the value of information increases if the decision maker displays information proclivity (that is, prefers a wider spread of expected payoffs). Information may be less valuable if the decision maker displays payoff-spread aversion, which is captured by concave w . The case of expected utility-maximizing behavior (i.e., the function w is linear) exhibits neutrality towards the payoffs spread.

Ambiguity aversion in smooth ambiguity models seems justifiable because decision makers are confronted with a spread of the priors that introduce payoff variations that they cannot exploit. By the same token, information proclivity is justifiable because decision makers value the opportunity to choose from, and thereby exploit, the wider spread of expected payoffs afforded by the more informative experiments.¹³

To grasp this point, let $S = \{s_1, s_2\}$, $Y = \{y_1, y_2\}$, $\pi = (\frac{1}{2}, \frac{1}{2})$, and assume that v is strictly convex. Choose $p, q \in \Delta(X)$ such that $v(\frac{1}{2}p + \frac{1}{2}q) < \frac{1}{2}v(p) + \frac{1}{2}v(q)$, and consider $B = \{f\}$ where $f(s_1) = p$ and $f(s_2) = q$. Let μ be the fully informative experiment (that is, $\mu(y_j | s_j) = 1, j = 1, 2$) and μ' be the totally uninformative experiment ($\pi(\cdot | y_j) = \pi(\cdot), j = 1, 2$). Clearly, μ is sufficient for μ' . Then,

$$\hat{V}(\mu; (V, v), B) = \frac{1}{2}v(p) + \frac{1}{2}v(q) > v\left(\frac{1}{2}p + \frac{1}{2}q\right) = \hat{V}(\mu'; (V, v), B)$$

In addition to affording better choice of acts, information thus has intrinsic value in the sense that, ex ante, the decision maker prefers a wider spread of expected payoffs, even if their expectations is the same.

This example can be used to demonstrate the necessity of the convexity of v . Assume that v is not convex, and choose $p, q \in \Delta(X)$ such that $v(\frac{1}{2}p + \frac{1}{2}q) > \frac{1}{2}v(p) + \frac{1}{2}v(q)$. Then

$$\hat{V}(\mu; (V, v), B) = \frac{1}{2}v(p) + \frac{1}{2}v(q) < v\left(\frac{1}{2}p + \frac{1}{2}q\right) = \hat{V}(\mu'; (V, v), B)$$

¹³ Gensbittel et al. (2015) analyzed the ambiguity case and provided conditions under which information is desirable within the maxmin model. Our main result does the same for the smooth ambiguity model. In both their paper and ours, the decision maker cannot commit ex ante to executing ex post choices.

and μ , the fully informative experiment, becomes less hybrid-valuable than the totally uninformative experiment μ' .

Finally, the following example demonstrates the necessity of the first-stage independence axiom.

Example. Let $S = \{s_1, s_2\}$, $Y = \{y_1, y_2\}$ and $\pi = (\frac{1}{2}, \frac{1}{2})$. Assume that v is a rank dependent utility functional defined by $v(p) = \int_{z \in X} z dg(F_p(z))$, where F_p denotes the cumulative distribution functions of p and $g : [0, 1] \rightarrow [0, 1]$ is increasing and onto. For a two-outcome lottery $p = t\delta_x + (1-t)\delta_y$, ($x < y$), $v(p) = g(t)x + (1-g(t))y$. It is known that v is convex if and only if g is concave.¹⁴ Choose g to be the concave piecewise linear function

$$g(t) = \begin{cases} (2-\tau)t & t < 0.5 \\ 1-\tau+\tau t & t \geq 0.5 \end{cases}$$

($0 < \tau < 1$) and consider $B = \{f\}$ where $f(s_1) = \delta_1$ and $f(s_2) = \delta_2$. Let μ be the fully informative experiment and μ' be the less informative experiment satisfying $\mu'(y_1 | s_1) = 1$, $\mu'(y_2 | s_1) = 0$, $\mu'(y_1 | s_2) = \varepsilon$, $\mu'(y_2 | s_2) = 1 - \varepsilon$, where $\varepsilon \in (0, 1)$. Note that $\mu'(y_1) = \frac{1+\varepsilon}{2}$ and $\mu'(y_2) = \frac{1-\varepsilon}{2}$. Clearly, μ is sufficient for μ' .

Assume that instead of using representation (8) of Theorem 1, we apply the rank-dependent utility v to the first-stage, too. Doing so is equivalent to using the certainty-equivalent reduction twice. Denoting the resulting value of experiments by \check{V} and noting that $v(\delta_1) = 1$ and $v(\delta_2) = 2$, yields

$$\check{V}(\mu; (V, v), B) = g\left(\frac{1}{2}\right)v(\delta_1) + \left(1-g\left(\frac{1}{2}\right)\right)v(\delta_2) = 2 - g\left(\frac{1}{2}\right) = 1 + \frac{\tau}{2}.$$

Similarly,

$$\begin{aligned} \check{V}(\mu'; (V, v), B) &= g\left(\frac{1+\varepsilon}{2}\right)v\left(\frac{1}{1+\varepsilon}\delta_1 + \frac{\varepsilon}{1+\varepsilon}\delta_2\right) \\ &\quad + \left(1-g\left(\frac{1+\varepsilon}{2}\right)\right)v(\delta_2) \\ &= g\left(\frac{1+\varepsilon}{2}\right)\left[g\left(\frac{1}{1+\varepsilon}\right) + \left(1-g\left(\frac{1}{1+\varepsilon}\right)\right)2\right] \\ &\quad + \left(1-g\left(\frac{1+\varepsilon}{2}\right)\right)2 \\ &= 2 - g\left(\frac{1+\varepsilon}{2}\right)g\left(\frac{1}{1+\varepsilon}\right) \\ &= 2 - \left(1-\tau+\tau\frac{1+\varepsilon}{2}\right)\left(1-\tau+\tau\frac{1}{1+\varepsilon}\right). \end{aligned}$$

¹⁴ Note that, by integration by parts, $v(p) = \int_{z \in X} z dg(F_p(z)) = x_{\max} - \int_{z \in X} g(F_p(z)) dz$. Hence, for a $\frac{1}{2} : \frac{1}{2}$ mixture and concave g ,

$$\begin{aligned} v\left(\frac{1}{2}p + \frac{1}{2}q\right) &= x_{\max} - \int_z g\left(F_{\frac{1}{2}p + \frac{1}{2}q}(z)\right) dz = x_{\max} \\ &\quad - \int_z g\left(\frac{1}{2}F_p(z) + \frac{1}{2}F_q(z)\right) dz \\ &\leq x_{\max} - \int_z \left[\frac{1}{2}g(F_p(z)) + \frac{1}{2}g(F_q(z))\right] dz \\ &= \frac{1}{2}\left[x_{\max} - \int_z g(F_p(z)) dz\right] \\ &\quad + \frac{1}{2}\left[x_{\max} - \int_z g(F_q(z)) dz\right] \\ &= \frac{1}{2}v(p) + \frac{1}{2}v(q) \end{aligned}$$

(see Ma and Safra, 2019). The converse is similarly proved.

For $\tau = 0.2$ and $\varepsilon = 0.1$, we get

$$\check{V}(\mu; (V, v), B) = 1.1 < 1.106 = \check{V}(\mu'; (V, v), B).$$

Hence, information may have negative value when first-stage independence is not assumed.

This example also demonstrates that employing the certainty-equivalent reduction twice does not eliminate aversion to information even if v is convex.

6. Concluding remarks

6.1. The value of information

Lurking in the background of Blackwell's theorem are two tacit properties of expected utility theory: *consequentialism* and *reduction of compound lotteries*. The former maintains that, facing sequential decisions involving risky choices, decision makers are "forward looking" in the sense that, at every decision node, their preferences are unaffected by outcomes that did not materialize, ("roads not taken") along the decision-making path. The latter asserts that decision makers evaluate acts solely based on the ultimate probability distributions they induce on outcomes, regardless of whether the outcome is drawn, in a single step, from a known distribution by more convoluted trajectory that includes chance and decision nodes.

Wakker (1988) showed that departing from the independence axiom while maintaining consequentialism and reduction of compound lotteries (at all stages) and assuming quasi-convexity necessarily results in situations in which decision makers refuse free information related to finer partitions. Safra and Sulganik (1995) demonstrated that, maintaining consequentialism and reduction of compound lotteries throughout, violations of the relation of being more informative experiment à la Blackwell are quite robust. More specifically, for almost all pairs of experiments μ and μ' satisfying that μ is sufficient for μ' , there exists a non-expected utility preference that strictly prefers the latter.

Replacing the RCLA by the certainty-equivalent reduction at all stages of the decision problem is not sufficient to derive an analogue of Blackwell's theorem (see Schlee (1990) and Safra and Sulganik (1995)). This and the previous observations clarify the need for hybrid models.

6.2. Related literature

Segal (1990) was the first to propose a model of decision making under risk that replaces the reduction of compound lotteries with certainty equivalent reduction. Seo (2009) obtained smooth ambiguity averse representations of choice under uncertainty that departs from the reduction of compound lotteries axiom. Halevy (2007) presented experimental evidence suggesting that

subjects whose behavior violates the reduction of compound lotteries under risk are more likely to exhibit ambiguity aversion when facing decision making under uncertainty.

This paper argues that departing from the independence axiom of expected utility theory while maintaining dynamic consistency and the intuitive presumption that being better informed is unambiguously desirable justifies the departure from the RCLA in sequential decision situations in which, at the interim stages, information may be exploited by choosing acts that better match the underlying data.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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