Comparative Incompleteness: Measurement, Behavioral Manifestations and Elicitation

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Abstract

This paper introduces measures of overall incompleteness of preference relations under risk and uncertainty, as well as measures of incompleteness of beliefs and tastes. These measures are used to define "more incomplete than" relations among different preference relations. We show how greater incompleteness is manifested in the representations of decision makers' preferences and illustrate its behavioral implications. In addition, the paper introduces incentive compatible schemes of eliciting the degrees of overall incompleteness and those of beliefs and tastes.

Keywords: Incomplete Preferences; Knightian Uncertainty; Comparative Incompleteness; Elicitation Mechanisms.

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> "It is conceivable - and may even in a way be more realistic - to allow for cases where the individual is neither able to state which of two alternatives he prefers nor that they are equally desirable." von Neumann and Morgenstern

> "When you can measure what you are speaking about, and express it in numbers, you know something about it, when you cannot express it in numbers, your knowledge is of a meager and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely, in your thoughts advanced to the stage of science." Lord Kelvin

> "By measurement to knowledge." — Heike Kamerlingh Onnes

1 Introduction

There are situations in which the inability of decision makers to state a clear preference is undeniable. For example, having to decide between two treatments of a decease, one that is expected to expand your life span by 20 years at 70 percent quality of life and another that is expected to expand your life span by 15 years at 90 percent quality of life, a decision maker might have difficulty expressing a clear preference between the two treatments.[1] Incompleteness of preferences is a prevalent feature of actual choice behavior and to assume otherwise does not seem justified on either positive or normative grounds. "Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from a normative viewpoint." (Aumann [1962], p. 446).

During the last couple of decades, there has been growing appreciation of the significance of incomplete preferences and recognition of the potential behavioral implications thereof. As a result, there has been an increasing interest in the modeling, analysis and economic applications of incomplete preferences.[2] However, to the best of our knowledge, measures that would allow comparisons of the incompleteness of distinct preference relations have not yet been provided. In view of the role of mea-

[1] See Attema, Bleichrodt, l'Haridon, and Lipman (2020) for an experimental investigation.

[2] The study of the representation of incomplete preferences under risk and under uncertainty was pioneerd by Aumann (1962) and Bewley (2002). More recently, the issue has been addressed in the works of Dubra, Maccheroni and Ok (2004), Baucells and Shapley (2006), Nau (2006), Seidenfeld, Schervish and Kadane (1995), Galaabaatar and Karni (2013), Ortoleva, Ok and Riella (2013), Karni (2020a). For an analysis of the implications of incomplete beliefs for equilibrium in financial markets see Rigotti and Shanon (2005).

surement in scientific inquiry, the lack of measures of incompleteness is a significant lacuna in decision theory.

In this paper we propose measures of incompleteness of preferences under risk and under uncertainty. These include measures of incompleteness of beliefs, incompleteness of risk attitudes, and overall incompleteness of preference relations under uncertainty. When preferences have multi-prior subjective expected multi-utility representations, we show how these measures of incompleteness capture the sets of subjective probabilities and utilities that constitute the representations of decision makers' preferences. The local properties, or "incompleteness in the small," are investigated as well.

We proceed to introduce measures of comparative incompleteness. We define what it means for one preference relation to be more incomplete than another, both in terms of beliefs, risk attitudes, and overall. We also show how greater incompleteness manifests itself in the representation of preferences.

We illustrate the behavioral implications of greater incompleteness in the context of a simple portfolio choice model. The behavioral manifestations of incompleteness include the range of unpredictability of the decision maker's portfolio position and the level of inertia exhibited in response to changes in security prices. We show that greater incompleteness according to our measures corresponds to both greater inertia and greater unpredictability.

Finally, we introduce incentive compatible mechanisms – modified scoring rules – by which the proposed measures of incompleteness may be elicited.

A natural and intuitive idea is to regard one preference relation as displaying greater incompleteness than another if all alternatives that are non-comparable according to the latter are non-comparable according to the former, but not necessarily vice versa. Our measures are consistent with this direct ranking of incompleteness. However, this direct ranking does not fully capture the essence of comparative incompleteness. To illustrate, consider a situation where one preference relation is complete, while another relation is incomplete. Clearly, the complete relation will be able to compare any two alternatives, and we can comfortably state that the complete relation is less incomplete, even when the two decision makers are not necessarily comparing the same alternatives. The only alternatives among which the complete preferences may be indecisive (alternatives that are non-comparable) belong to the same indifference class. There is no reason why the same alternatives should be noncomparable according to the less complete preference relation.

The paper is structured as follows: Section 2 introduces our measures of incompleteness, connects them to properties of multi-prior subjective expected multi-utility representations, and investigates local behavior of the measures. Section 3 defines comparative incompleteness, shows how it manifests itself in the representation of preferences, and illustrates its behavioral implications in the context of a simple portfolio choice problem. Section 4 introduces incentive compatible mechanisms by which the measures of incompleteness may be elicited. Concluding remarks apper in Section 5 . The proofs are collected in the Appendix.

2 Measuring Incompleteness

2.1 Preliminaries

Let S be a finite set of *states* and denote by ΔX the set of simple probability distributions, dubbed *lotteries*, on a set of *outcomes*, X.[3] Although it is not always necessary, to simplify the exposition we assume that X is the set of reals representing monetary payoffs. Subsets of S are *events* and S is the *universal event*. Maps from S to ΔX are *acts*. Constant acts are identified with corresponding elements of ΔX. We denote by F the set of all acts. Denote by $\delta_x \in \Delta X$ the constant act whose payoff is the outcome x in every state. Henceforth, we identify $x \in X$ with the constant act δ_x . Hence, $X \subset \Delta X$. A *bet* on an event E is the act $x_E y \in F$ such that $(x_E y)(s) = x$ for all $s \in E$, and $(x_E y)(s) = y$ otherwise, where $x > y$. A lottery $\ell(r; x, y) \in \Delta X$, is a constant act that pays x with probability r and y with probability $(1-r)$.

A *strict preference relation* is an asymmetric, irreflexive and transitive binary relation $\succ$ on F. We assume throughout that the strict preference relation is neither negatively transitive nor is it empty. Define the induced *incomparability relation* $\asymp$ on F as follows: For all $f, g \in F$,

$$f \asymp g \text{ if } \neg(f \succ g) \text{ and } \neg(g \succ f). \tag{1}$$

Then, $\asymp$ is symmetric, reflexive and intransitive.[4]

The strict preference relation is continuous if the upper and lower contour sets, $\{f \in F \mid f \succ g\}$ and $\{f \in F \mid g \succ f\}$, are open (in the topology of $\mathbb{R}^n$) for all $g \in F$. Note that if $\succ$ is continuous then, for all $g \in F$, the non-comparable subsets, $\{f \in F \mid f \asymp g\}$ are closed. We assume throughout that the strict preference relation is continuous. We also assume that it is monotonic with respect to first-order stochastic dominance: For all $p, q \in \Delta(X)$, if p first-order stochastically dominates q, then $p \succ q$.[5]

[3]A simple probability distribution is a probability distribution with finite support.

[4]The intransitivity of $\asymp$ of F is an implication of $\succ$ not being negatively transitive.

[5]The lottery p first-order stochastically dominates the lottery q if $\sum_{\{z|z\succ x\}} p(z) \geq \sum_{\{z|z\succ x\}} q(z)$ for all $x \in X$ with strict inequality for some $x \in X$.

An event E is *null* if $\neg(x_E y \succ y)$, for all $x, y \in X$ such that $x \succ y$. An event E is *nonnull* if it is not null. Thus, if there are $x, y \in X$ for which $x_E y \succ y$, then E is nonnull.

The analysis that follows is based on the axiomatic characterizations of multi-prior expected multi-utility (MPEMU) representations of incomplete preference relations on the set of Anscombe-Aumann acts of Galaabaatar and Karni (2013). The incomplete preference relation $\succ$ on F has a MPEMU product representation if the following holds: For all $f, g \in F$,

$$f \succ g \Leftrightarrow \Sigma_{s \in S} U(f(s))\pi(s) > \Sigma_{s \in S} U(g(s))\pi(s), \forall (\pi, U) \in \Pi \times \mathcal{U}, \tag{2}$$

where Π a unique closed convex set of subjective probability measures on S and $\mathcal{U}$ is a set of real-valued, affine, functions on ΔX. This representation in (2) includes two special cases: (a) Bewley's (2020) *Knightian uncertainty* in which $\succ$ on the subset of constant acts (that is, on ΔX) is negatively transitive and, consequently, $\mathcal{U}$ is a singleton set, and (b) the case of complete beliefs in which Π is a singleton set.

Incomplete preferences under uncertainty stem from two sources: incomplete beliefs and incomplete tastes. The former source expresses the decision makers' ambiguous beliefs concerning the likelihoods of events. The latter source expresses the decision makers' indecisiveness regarding the appropriate criterion for the evaluation of risky prospects. When both sources are present, they generally interact. Correspondingly, we develop measures of the incompleteness of beliefs and of tastes as well as measures of the overall degree of incompleteness.

2.2 Measure of belief incompleteness

Borel (1924), Ramsey (1931) and de Finetti (1937) were the first to propose the idea that subjective probabilities may be inferred from the odds a decision maker is just willing to offer when betting on events. To the extent that the subjective probabilities reflect the decision makers' beliefs about the likelihood of the events, the corresponding betting odds measure these beliefs. In the case of incomplete beliefs a decision maker may entertain a set of possible beliefs about the likelihood of an event. Building on the aforementioned idea, we define a measure of incompleteness of a decision maker's beliefs of an event by the range of the odds she considers possible when betting on the said event.

For each event, $E \in 2^S$ such that neither E nor its complement $E^c = S \backslash E$ are null, and for any $x, y \in X$, define

$$R^{\succ}(x_E y) = \{r \in [0,1] \mid x_E y \asymp \ell(r; x, y)\}. \tag{3}$$

The elements of $R^{\succ}(x_E y)$ are the winning probabilities of lotteries that, according to $\succ$, are not comparable to a bet on the event E with the same stakes.

Since $\succ$ is monotone with respect to first-order stochastic dominance and continuous,

$$R^{\succ}(x_E y) = [\underline{r}^{\succ}(x_E y), \bar{r}^{\succ}(x_E y)],$$

where $\underline{r}^{\succ}(x_E y) = \sup\{r \mid x_E y \succ \ell(r; x, y)\}$ and $\bar{r}^{\succ}(x_E y) = \inf\{r \mid \ell(r; x, y) \succ x_E y\}$. That $\underline{r}^{\succ}(x_E y)$ and $\bar{r}^{\succ}(x_E y)$ exist is an implication of the boundedness (that is, $r \in [0, 1]$) and the fact that the sets are non-empty, (that is, $0 \in \{r \mid x_E y \succ \ell(r; x, y)\}$ and $1 \in \{r \mid \ell(r; x, y) \succ x_E y\}$). Hence, $R^{\succ}(x_E y)$ is a compact interval.

Since $\succ$ is irreflexive, we have that for every null E, $R^{\succ}(x_E y) = \{0\}$ and for every E, for which $S \backslash E$ is either null or empty, we have that $R^{\succ}(x_E y) = \{1\}$, for all $x, y \in X$. For null events E, we thus define $\underline{r}^{\succ}(x_E y) = \bar{r}^{\succ}(x_E y) = 0$, and while for events E for which $S \backslash E$ is null or empty, we define $\underline{r}^{\succ}(x_E y) = \bar{r}^{\succ}(x_E y) = 1$, for all $x, y \in X$. With this in mind we make the following definition.

Definition 1 *For every $E \in 2^S$, and $x, y \in X$, the measure of* ***belief incompleteness*** *of $\succ$* ***at*** $\mathbf{x_E y}$ *is $m_b(x_E y; \succ) = \bar{r}^{\succ}(x_E y) - \underline{r}^{\succ}(x_E y)$.*

Definition 1 captures the preference relation's incompleteness that arises from the decision maker being unsure of how a subjective bet on event E compares to objective lotteries. The payoffs of the bet, x and y, constitute a "measuring rod" of the incompleteness of beliefs. If E is null or the empty set then $m_b(x_E y; \succ) = 0$. If $\succ$ is negatively transitive then $m_b(x_E y; \succ) = 0$ for all E. Clearly, $m_b(x_E y; \succ) = m_b(x_{E^C} y; \succ)$, for all $E \in 2^S$ and $x, y \in X$. Definition 1 does not rule out that the measure $m_b(x_E y; \succ)$ depends on the measuring rod being used. However, as we show in Theorem 1 below, if the decision maker's preferences admit MPEMU representation then $m_b(x_E y; \succ)$ is independent of the choice of x and y, or the "measuring rod" being used, and of the decision maker's risk attitudes.

Given $\succ$ on F, let Π be the corresponding set of subjective priors that figure in the representation (2). For each $E \in 2^S$, let $\bar{\pi}(E) := \max_{\pi \in \Pi} \pi(E)$ and $\underline{\pi}(E) := \min_{\pi \in \Pi} \pi(E)$. Then $\bar{\pi}(E) - \underline{\pi}(E)$ represents the range of beliefs that, according to $\succ$, the true state is in E.[6] We show next that, for MPEMU preferences, the probability measure of belief-incompleteness in Definition 1 is equal to the length of the interval of subjective probabilities of E.

[6]That $\bar{\pi}(E)$ and $\underline{\pi}(E)$ exist is an implication of the compactness of $\Pi(E)$ and the linearity of the preference functional.

Theorem 1 *If an incomplete preference relation $\succ$ on F has MPEMU representation, then the measure of belief incompleteness at E, $m_b(x_E y; \succ)$, is independent of the outcomes x and y and of the set of utility functions $\mathcal{U}$ in the representation. Furthermore, $m_b(x_E y; \succ) := m_b(E; \succ) = \bar{\pi}(E) - \underline{\pi}(E)$.*

The proof is in the Appendix. It worth underscoring that this result also holds if instead of MPEMU preferences the decision maker's preference relation displays probabilistic sophistication a la Machina and Schmeidler (1995).

2.3 Measure of taste incompleteness

Consider next the measurement of incompleteness of preference relations under risk,[7] by restricting $\succ$ to ΔX. For every $p \in \Delta X$, define

$$C^{\succ}(p) = \{c \in \mathbb{R} \mid p \asymp \delta_c\}. \tag{4}$$

The elements of $C^{\succ}(p)$ are certain amounts that, according to $\succ$, are not comparable to the lottery p. Then

$$C^{\succ}(p) = \left[\underline{c}^{\succ}(p), \bar{c}^{\succ}(p)\right], \tag{5}$$

where $\bar{c}^{\succ}(p) = \inf\{c \in \mathbb{R} \mid \delta_c \succ p\}$ and $\underline{c}^{\succ}(p) = \sup\{c \in \mathbb{R} \mid p \succ \delta_c\}$. That $\bar{c}^{\succ}(p)$ and $\underline{c}^{\succ}(p)$ exist is an implication of $C^{\succ}(p)$ being closed (it is the complement of an open set), the support of p being finite and, hence, bounded, and the fact that $\succ$ satisfies first-order stochastic dominance. We use these notations to define a measure of taste incompleteness (i.e. of the incompleteness of the decision maker's risk attitudes).

Definition 2 *For every lottery $p \in \Delta X$, the measure of* ***taste incompleteness*** *of $\succ$* ***at*** $\mathbf{p}$ *is $m_t(p; \succ) = \bar{c}^{\succ}(p) - \underline{c}^{\succ}(p)$.*

The measure in Definition 2 captures the degree to which a decision maker is unsure of how a lottery compares to certain amounts. In other words, it captures the degree to which a decision maker is unable to evaluate the riskiness of p.

Let $\mu(p)$ denote the expected value, or mean, of p. Define

$$\bar{\xi}^{\succ}(p) := \mu(p) - \underline{c}^{\succ}(p) \tag{6}$$

and

$$\underline{\xi}^{\succ}(p) := \mu(p) - \bar{c}^{\succ}(p), \tag{7}$$

which are, respectively, the highest and lowest risk premiums of the lottery p according to $\succ$. Then,

$$m_t(p; \succ) = \bar{\xi}^{\succ}(p) - \underline{\xi}^{\succ}(p). \tag{8}$$

[7] See Dubra, Maccheroni and Ok (2004) and Baucells and Shapley (2006) for an axiomatic characterization of expected utility representations with incomplete preferences under risk.

2.4 Measure of overall incompleteness

The overall degree of incompleteness of a preference relation at E amalgamates the incompleteness of beliefs and of tastes. A decision maker may be unsure of how a subjective bet on E compares to objective lotteries, and also of how to assess the risk represented by these lotteries. That is, for a subjective bet on E, there is a set of non-comparable lotteries, and for each of these non-comparable lotteries, there is a range of sure payoffs that are non-comparable to the lottery. Because a bet on E corresponds to a set of non-comparable lotteries the question arises how to incorporate the values of the certain payoffs into the measure of the overall incompleteness of the preference relation at E.

To measure a decision maker's degree of overall incompleteness at E, we invoke the range of monetary outcomes that are non-comparable to a bet on E. For each event, $E \in 2^S$ and $x, y \in X$, define

$$O^{\succ}(x_E y) = \{c \in \mathbb{R} \mid x_E y \asymp \delta_c\}. \tag{9}$$

The elements of $O^{\succ}(x_E y)$ are certain payoffs that, according to $\succ$, are not comparable to the bet $x_E y$. Then,

$$O^{\succ}(x_E y) = [\underline{c}(x_E y; \succ), \bar{c}(x_E y; \succ)],$$

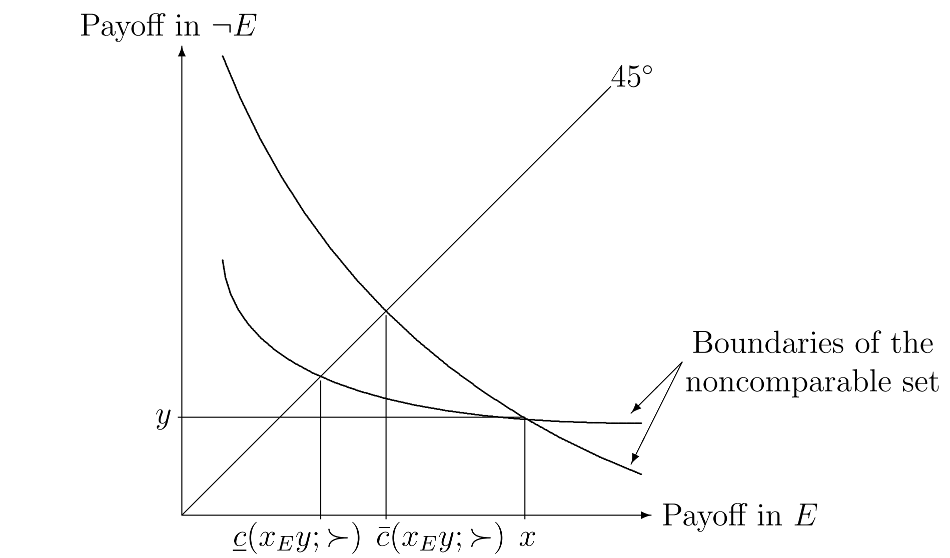
where $\bar{c}(x_E y; \succ) = \inf\{c \in \mathbb{R} \mid \delta_c \succ x_E y\}$ and $\underline{c}^{\succ}(x_E y; \succ) = \sup\{c \in \mathbb{R} \mid x_E y \succ \delta_c\}$. That $\bar{c}(x_E y; \succ)$ and $\underline{c}(x_E y; \succ)$ exist when neither E nor its complement is null follows from the fact that, by first-order stochastic dominance $\{c \in \mathbb{R} \mid \delta_c \succ x_E y\}$ is non-empty and bounded below by y and, similarly, $\{c \in \mathbb{R} \mid x_E y \succ \delta_c\}$ is non-empty and bounded above by x. If E is the universal event define $\bar{c}(x_E y; \succ) = \underline{c}(x_E y; \succ) = x$ and if E is null define $\bar{c}(x_E y; \succ) = \underline{c}(x_E y; \succ) = y$. Using these notations we make the following definition:

Definition 3 *For every $E \in 2^S$, the measure of **overall incompleteness** of $\succ$ **at** $\mathbf{x_E y}$ is $M(x_E y; \succ) = \bar{c}(x_E y; \succ) - \underline{c}(x_E y; \succ)$.*

The measure of overall incompleteness at $x_E y$ is illustrated in Figure 1.

If E is either a null event or the universal event then $M(x_E y; \succ) = 0$ for all x, y. If $\succ$ is negatively transitive then, $M(x_E y; \succ) = 0$ for all E and for all x, y. Unlike the measure of incomplete beliefs, the measure of the overall incompleteness of a preference relation at $x_E y$ depends on the "measuring rod", that is, the payoffs x and y that are used to construct it. This dependence is a consequence of the fact that the magnitudes of the payoffs determine the riskiness of the bet. Because the measure of overall incompleteness incorporates the decision maker's risk attitudes, it must be sensitive to the risk of the bet.

Figure 1: Illustration of the Measure of Overall Incompleteness at $x_E y$



2.5 Measures of incompleteness in the small

Consider next the local version of our measures of incompleteness. This analysis allows us to express the measures of incompleteness in terms of the properties of the subjective probabilities and the utility functions that figure in the MPEMU representation. Fix a probability r and consider a lottery $\ell(r; x, y)$. Denote its mean by $\mu_r(x, y)$ and its variance by $\sigma_r^2(x, y)$. Let u denote the Bernoulli utility function corresponding to U, so that $U(p) = \sum_{x \in \text{supp}(p)} p(x)u(x)$ for all $p \in \Delta(X)$.

We first consider our measure of taste incompleteness as $x - y \to 0$ while keeping the mean of the lottery constant. Theorem 2 shows that, locally around $\mu_r(x, y)$, the measure of taste incompleteness is proportional to the largest difference in the Arrow-Pratt measure of absolute risk-aversion evaluated at $\mu_r(x, y)$ displayed by the utility functions that figure in the representation.

Theorem 2 *For small $x - y$, the measure of taste incompleteness of $\succ$ at $\ell(r; x, y)$ satisfies*

$$\begin{aligned} m_t(\ell(r; x, y); \succ) &= \left[\max_{U \in \mathcal{U}} \left(-\frac{u''(\mu_r(x, y))}{u'(\mu_r(x, y))} \right) - \min_{U \in \mathcal{U}} \left(-\frac{u''(\mu_r(x, y))}{u'(\mu_r(x, y))} \right) \right] \frac{\sigma_r^2(x, y)}{2} \\ &+ o(\sigma_r^2(x, y)). \end{aligned}$$

Recall that the measure of taste incompleteness captures the degree to which a decision maker is unable to evaluate the riskiness of lotteries. Theorem 2 shows that this inability is reflected in the range of risk attitudes the decision maker may have at the mean of the lottery.

When preferences exhibit both taste and belief incompleteness, the mean of a bet $x_E y$ is not uniquely defined, nor is the Arrow-Pratt coefficient of risk aversion. We therefore consider what happens as $y \to x$. Theorem 3 describes the local behavior of the measure of overall incompleteness.

Theorem 3 *For small $x - y$, the measure of overall incompleteness of $\succ$ at $x_E y$ satisfies*

$$\begin{aligned}
& M(x_E y; \succ) = (\bar{\pi}(E) - \underline{\pi}(E))(x-y) \\
+ \quad & \frac{1}{2}\left[\max_{U \in \mathcal{U}}\left(-\frac{u''(\mu_{\underline{\pi}(E)}(x,y))}{u'(\mu_{\underline{\pi}(E)}(x,y))}\right)\sigma^2_{\underline{\pi}(E)}(x,y) - \min_{U \in \mathcal{U}}\left(-\frac{u''(\mu_{\bar{\pi}(E)}(x,y))}{u'(\mu_{\bar{\pi}(E)}(x,y))}\right)\sigma^2_{\bar{\pi}(E)}(x,y)\right] \\
+ \quad & o((x-y)^2).
\end{aligned}$$

The first term in the square brackets is the variance of the bet according to the DM's belief assigning lowest probability to E times the largest Arrow-Pratt coefficient of absolute risk aversion at the mean of the bet according to that belief. The second term in the square brackets is the variance of the bet according to the belief assigning highest probability to E times the smallest Arrow-Pratt coefficient of absolute risk aversion at the mean of the bet according to that belief. Theorem 3 states that, locally the measure of overall incompleteness can be decomposed into the difference between these terms and the measure of belief incompleteness weighted by the stakes of the bet. Formally, by Theorem 1,[8] $\bar{\pi}(E) - \underline{\pi}(E) = \bar{r}^{\succ}(E) - \underline{r}^{\succ}(E)$ and, by Pratt (1964),

$$\begin{aligned}
& \bar{\xi}^{\succ}(\ell(\underline{r}^{\succ}(E); x, y)) - \underline{\xi}^{\succ}(\ell(\bar{r}^{\succ}(E); x, y)) \\
= \quad & \max_{U \in \mathcal{U}}\left(-\frac{u''(\mu_{\underline{\pi}(E)}(x,y))}{u'(\mu_{\underline{\pi}(E)}(x,y))}\right)\frac{\sigma^2_{\underline{\pi}(E)}(x,y)}{2} - \min_{U \in \mathcal{U}}\left(-\frac{u''(\mu_{\bar{\pi}(E)}(x,y))}{u'(\mu_{\bar{\pi}(E)}(x,y))}\right)\frac{\sigma^2_{\bar{\pi}(E)}(x,y)}{2}.
\end{aligned}$$

Hence, we have that

$$M(x_E y; \succ) = (\bar{r}^{\succ}(E) - \underline{r}^{\succ}(E))(x-y) + (\bar{\xi}^{\succ}(\ell(\underline{r}^{\succ}(E); x, y)) - \underline{\xi}^{\succ}(\ell(\bar{r}^{\succ}(E); x, y))).$$

In the case of complete beliefs, $\underline{\pi}^{\succ}(E) = \bar{\pi}^{\succ}(E)$ and the term in the square brackets in Theorem 3 equals the local measure of taste incompleteness in Theorem

[8]We have shown that when preferences hace MPEMU representation, $m_b(x_E y, \succ)$ is independent of x, y, therefore we can write $\bar{r}^{\succ}(x_E y) = \bar{r}^{\succ}(E)$ and $\underline{r}^{\succ}(x_E y) = \underline{r}^{\succ}(E)$.

2 (times 2). When beliefs are incomplete and $\underline{\pi}^{\succ}(E) \neq \bar{\pi}^{\succ}(E)$, the term in the square brackets can be positive or negative depending on the local curvature of the utility functions at the highest and lowest mean of the bet. Thus, while $M(x_E y; \succ)$ measures the combined effect of incomplete beliefs and tastes, even locally and with the measure of belief incompleteness weighted by the stakes of the bet, it is not additive in the measures of belief and taste incompleteness. This is because the belief and taste incompleteness interact. Taste incompleteness is defined at a particular lottery, and belief incompleteness means that two different lotteries are evaluated. The exact nature of the interaction is described in Theorem 3.

Theorem 4 shows that the derivative of the measure of overall incompleteness of $\succ$ at $x_E y$, evaluated at $y = x$, equals the measure of belief incompleteness of $\succ$ at E. This is intuitive, since when the stakes of the bet are zero, the preference relation displays risk neutrality and, consequently, the decision maker's risk attitudes are unambiguous. Consequently, in the limit the tastes are complete, and the only source of incompleteness is the belief. Thus, in the limit the model reduces to Knightian uncertainty.

Theorem 4 *The derivative of the measure of overall incompleteness of $\succ$ at $x_E y$, evaluated at $y = x$, equals the measure of belief incompleteness of $\succ$ at E. That is,*

$$\lim_{y \to x} \frac{M(x_E y; \succ)}{x - y} = \bar{r}^{\succ}(E) - \underline{r}^{\succ}(E) = m_b(E; \succ).$$

3 Comparative Incompleteness: Measurement and Behavioral Manifestations

Corresponding to the measures of incompleteness of the preceding section, we define binary relations "more incomplete than" on the set of preference relations. There is a similarity between measuring risk aversion and measuring incompleteness. The Arrow-Pratt measures of absolute and relative risk aversion are local (at every level of wealth). Consequently, interpersonal comparisons of risk attitudes are defined locally and if the local relationship "more risk averse" holds at every level of wealth, then the comparison is global. Our measures of incompleteness are also defined locally. In the case of incomplete beliefs the measure is defined locally at events and in the case of incomplete tastes it is defined locally at lotteries. Interpersonal comparisons of the degree of incompleteness are defined locally and if the local relationship "more incomplete" holds at each event (for beliefs) or lottery (for tastes) then the comparison is global.

3.1 Comparative incompleteness: General measures

In this subsection, we introduce measures that can be applied generally in the sense that the set of incomparable objects for one of the relations need not be a subset of that for the other relation. Specifically, in Definition 4, we introduce a notion of comparative belief incompleteness that applies to any two preference relations. The "greater belief incompleteness at E" relation is itself a complete binary relation on the set of preference relations on F.

Definition 4 *Preference relation $\succ_1$ displays **greater** (strictly greater) **belief incompleteness at** **E** than preference relation $\succ_2$ if $m_b(E;\succ_1) \geq (>)\ m_b(E;\succ_2)$. It displays greater (strictly greater) belief incompleteness on F if it displays greater (strictly greater) belief incompleteness at E for all nonnull $E \in 2^S \setminus S$.*

To motivate Definition 4, note that the property $m_b(E;\succ_1) \geq m_b(E;\succ_2)$ in the definition is equivalent to the property that for any ϵ, if $\ell(r+\epsilon;x,y) \succ_1 x_E y$, for all $r \in R^{\succ_1}(E)$, then $\ell(r+\epsilon;x,y) \succ_2 x_E y$, for all $r \in R^{\succ_2}(E)$. The intuition behind the property is as follows: For a preference relation $\succ_i$, $i = 1, 2$, a bet on event E is non-comparable to lotteries with odds in $R^{\succ_i}(E)$. However, if the odds in the lottery are improved sufficiently, the lottery would become so attractive that a strict preference would emerge in favour of the lottery and the decision maker would no longer find the bet and the lottery incomparable. Now, consider any of the lotteries that are incomparable to the bet according to the preference relation $\succ_1$, and consider an increase ϵ in the odds of winning, which is large enough to always break incomparability for all $r_1 \in R^{\succ_1}(E)$. If the same increase in odds will also always break incomparability for a preference relation $\succ_2$, we conclude that $\succ_2$ is less incomplete than $\succ_1$. In other words, it takes a smaller increase in odds for the preference relation $\succ_2$ to be able to compare the lottery and the bet and state a strict preference between the two objects than it does for the preference relation $\succ_1$.

Definition 4 applies to situations where the set of incomparable objects for one decision maker is not necessarily a subset of that for the other decision maker. The case where one preference relation $\succ_i$ is, in fact, complete, and consequently, the $R^{\succ_i}(E)$ is a singleton set, is a special case. Also, all preference relations exhibit complete beliefs at S, the universal event, and at every event E that is null.

Corollary 1 below states that $\succ_1$ displays greater belief incompleteness at E than $\succ_2$ if and only if the length of the interval of priors for $\succ_1$ is greater than that for $\succ_2$.

Corollary 1 *If preference relations $\succ_1$ and $\succ_2$ on F both admit MPEMU representations, then $\succ_1$ displays greater belief incompleteness at E than $\succ_2$ if and only if*

$\bar{\pi}_1(E) - \underline{\pi}_1(E) \geq \bar{\pi}_2(E) - \underline{\pi}_2(E)$*. It displays strictly greater belief incompleteness at* E *if and only if the inequality is strict.*

The result is an immediate consequence of Definition 4 and Theorem 1.

Consider next the binary relation "greater taste incompleteness at p" on the set of preference relations on ΔX.

Definition 5 *On* ΔX *preference relation* $\succ_1$ *displays* ***greater*** *(strictly greater)* ***taste incompleteness at* p** *than preference relation* $\succ_2$ *if* $m_t(p; \succ_1) \geq (>)\ m_t(p; \succ_2)$*. It displays greater (strictly greater) taste incompleteness on* ΔX *if it displays greater (strictly greater) taste incompleteness at all nondegenerate* $p \in \Delta(X)$*.*

Definition 5 can be motivated as follows: If a decision maker is unable to compare a lottery to a certain monetary payoff, there will in general be increases or decreases in the certain payoff large enough that the decision maker is able to state a clear preference between the two. The intuition behind "more incomplete than" in Definition 5 is as follows: if any change in the certain monetary payoff that is large enough to break incomparability for preference relation $\succ_1$ always breaks incomparability for preference relation $\succ_2$, then $\succ_1$ is more incomplete than $\succ_2$.

Theorem 5 below states that for low-variance lotteries p, $\succ_1$ displays greater taste incompleteness at p than $\succ_2$ if and only if, when evaluated at the mean of p, the largest difference in the Arrow-Pratt coefficient of risk aversion among the utility functions representing $\succ_1$ is greater than among the utility functions representing $\succ_2$.

Theorem 5 *Suppose preference relations* $\succ_1$ *and* $\succ_2$ *on* ΔX *both admit expected multi-utility representations. For low-variance lotteries* p*,* $\succ_1$ *displays greater taste incompleteness at* p *than* $\succ_2$ *if and only if*

$$\max_{U \in \mathcal{U}_1} \left\{ -\frac{u''(\mu(p))}{u'(\mu(p))} \right\} - \min_{U \in \mathcal{U}_1} \left\{ -\frac{u''(\mu(p))}{u'(\mu(p))} \right\} \geq \max_{U \in \mathcal{U}_2} \left\{ -\frac{u''(\mu(p))}{u'(\mu(p))} \right\} - \min_{U \in \mathcal{U}_2} \left\{ -\frac{u''(\mu(p))}{u'(\mu(p))} \right\}.$$

It displays strictly greater taste incompleteness at p *if and only if the inequality is strict.*

Theorem 5 highlights the intuition that greater taste incompleteness is reflected in a larger range of risk attitudes.

Definition 6 below characterizes the binary relation "greater overall incompleteness than" on the set of all preference relation on F.

Definition 6 *Preference relation $\succ_1$ displays* ***greater*** *(strictly greater)* ***overall incompleteness at*** $\mathbf{x_E y}$ *than preference relation $\succ_2$ if $M(x_E y;\succ_1) \geq (>) M(x_E y;\succ_2)$. It displays greater (strictly greater) overall incompleteness at E if it displays greater (strictly greater) overall incompleteness at $x_E y$ for all $x, y \in X$ such that $x > y$. It displays greater (strictly greater) overall incompleteness on F if it displays greater (strictly greater) incompleteness at E for all nonnull $E \in 2^S \setminus S$.*

The motivation of Definition 6 is similar to that of Definition 5: If any change in the certain monetary payoff that is large enough to break incomparability for preference relation $\succ_1$ always breaks incomparability for the preference relation $\succ_2$, then $\succ_1$ is more incomplete than $\succ_2$.

3.2 Comparative Incompleteness in the Strong Sense

We now turn our attention to comparison of preference relations for which every lottery that is incomparable to a bet for one of the relations is also incomparable to the bet for the other relation. This set-inclusion concept of "more incomplete than" is a partial binary relation on the set of preference relations on F. Therefore, in contrast to Definitions 4 through 6 it will not be able to rank all preference relations, even locally. However, if two relations are comparable in this strong sense, it has strong behavioral implications, as will be apparent in Section 3.3.

Definition 7 below states that one preference relation is "more belief incomplete in the strong sense" than another at E if every lottery that is non-comparable to a bet on E according to the latter is non-comparable to the same bet on E according to the former. Unlike definitions 4-6, according to which greater incompleteness is defined by the measures, in the definitions below are they are defined directly in terms of the preference relations.

Definition 7 *A preference relation $\succ_1$ displays* ***greater belief incompleteness in the strong sense at*** $\mathbf{E}$ *than preference relation $\succ_2$ if $x_E y \asymp_2 \ell(r;x,y)$ implies $x_E y \asymp_1 \ell(r;x,y)$, for all bets on E. It displays strictly greater belief-incompleteness in the strong sense at E if it displays greater belief-incompleteness in the strong sense and, in addition, for some $\ell(r;x,y)$, $x_E y \asymp_1 \ell(r;x,y)$ and $\neg(x_E y \asymp_2 \ell(r;x,y))$.*

Applying the same idea to the preference relations on $\Delta(X)$, Definition 8 states that one preference relation is "more taste incomplete in the strong sense" than another at p if every certain amount that is non-comparable to p according to the latter is non-comparable to p according to the former.

Definition 8 *On ΔX, a preference relation $\succ_1$ displays* ***greater taste incompleteness in the strong sense at* p** *than preference relation $\succ_2$ if $p \asymp_2 \delta_c$ implies $p \asymp_1 \delta_c$. It displays strictly greater taste incompleteness in the strong sense at p if it displays greater taste incompleteness in the strong sense and, in addition, for some δ_c, $p \asymp_1 \delta_c$ and $\neg(p \asymp_2 \delta_c)$.*

Similarly, one preference relation is "more incomplete overall in the strong sense" than another at E if every certain amount that is non-comparable to a bet on E according to the latter is non-comparable to the same bet on E according to the former.

Definition 9 *A preference relation $\succ_1$ displays* ***greater overall incompleteness in the strong sense at* E** *than preference relation $\succ_2$ if $x_E y \asymp_2 \delta_c$ implies $x_E y \asymp_1 \delta_c$ for all x, y, such that $x > y$. It displays strictly greater overall incompleteness in the strong sense at E if it displays greater overall incompleteness in the strong sense at E and, in addition, for some δ_c, x, y such that $x > y$, $x_E y \asymp_1 \delta_c$ and $\neg(x_E y \asymp_2 \delta_c)$.*

The following are immediate implications of Definitions 7, 8, and 9, respectively:

1. The preference relation $\succ_1$ on F displays greater belief-incompleteness in the strong sense at E than $\succ_2$ if and only if $R^{\succ_2}(E) \subseteq R^{\succ_1}(E)$.

2. The preference relation $\succ_1$ displays greater taste incompleteness in the strong sense at p than $\succ_2$ if and only if $C^{\succ_2}(p) \subseteq C^{\succ_1}(p)$.

3. The preference relation $\succ_1$ displays greater overall incompleteness in the strong sense than $\succ_2$ at E if and only if $O^{\succ_2}(x_E y) \subseteq O^{\succ_1}(x_E y)$ for all x, y, such that $x > y$.

As the names suggest, displaying greater incompleteness in the strong sense is a special case of displaying greater incompleteness: If $\succ_1$ displays greater belief incompleteness in the strong sense at E than $\succ_2$, as in Definition 7, then $\succ_1$ displays greater belief incompleteness at E than $\succ_2$ in the general sense from Definition 4. If $\succ_1$ displays greater taste incompleteness in the strong sense at p than $\succ_2$, as in Definition 8, then $\succ_1$ displays greater taste incompleteness at p than $\succ_2$ in the general sense from Definition 5. If $\succ_1$ displays greater overall incompleteness in the strong sense at E than $\succ_2$, as in Definition 9, then $\succ_1$ displays greater overall incompleteness at E than $\succ_2$ in the general sense from Definition 6. Furthermore, we have the following result linking the measure of greater belief incompleteness in the strong sense and the beliefs in the MPEMU representations:

Corollary 2 *Suppose preference relations $\succ_1$ and $\succ_2$ on F both admit MPEMU representations. The preference relation $\succ_1$ on F displays greater (strictly greater) belief incompleteness in the strong sense at E than $\succ_2$ if and only if $[\underline{\pi}_2(E), \bar{\pi}_2(E)] \subseteq [\underline{\pi}_1(E), \bar{\pi}_1(E)]$.*

According to Corollary 2, $\succ_1$ displays greater belief incompleteness in the strong sense than $\succ_2$ if and only if the set of prior beliefs for $\succ_2$ is a subset of the set of prior beliefs for $\succ_1$.

In general, one decision maker may display greater belief incompleteness but smaller taste incompleteness than another or vice versa. This makes the comparison of the overall incompleteness depend on the relative magnitudes of the incompleteness of beliefs and tastes (or risk attitudes) of the decision makers being compared. If one decision maker displays greater incompleteness of both beliefs and tastes in the strong sense then, not surprisingly, she displays greater overall incompleteness in the strong sense. Formally, we have the following result:

Theorem 6 *If a preference relation $\succ_1$ displays greater belief and taste incompleteness in the strong sense than preference relation $\succ_2$ then it displays greater overall incompleteness in the strong sense.*

Incomplete beliefs and tastes have distinct effects on the overall measure of incompleteness. This can be easily grasped by observing that even if the tastes are complete and the beliefs of two decision makers are incomplete to the same degree, unless their *Bernoulli utility functions belong to the same equivalence class for risk attitudes*, the overall measure of incompleteness may be different due to possible distinct risk attitudes. For example, fix a bet $x_E y$ on E, and consider preference relations $\succ_i$, $i = 1, 2$ exhibiting Knightian uncertainty. Assume that $\Pi_1 = \Pi_2$ and suppose that $\succ_1$ displays greater absolute risk aversion at $\mu(\underline{\pi}; x, y)$ and smaller absolute risk aversion at $\mu(\bar{\pi}; x, y)$ than $\succ_2$. Then, $\xi(l(\bar{\pi}; x, y); \succ_1) < \xi(l(\bar{\pi}; x, y); \succ_2)$ and $\xi(l(\underline{\pi}; x, y); \succ_1) > \xi(l(\underline{\pi}; x, y); \succ_2)$. Thus, $M(x_E y; \succ_1) > M(x_E y; \succ_2)$.

3.3 Portfolio choice

The main behavioral manifestations of incomplete preferences are inertia and unpredictability. Loosely speaking, inertia means that to take an action, a decision maker must be persuaded that the action dominates not taking it (i.e. sticking to the status quo) according to all the possible values he may attribute to the outcomes of the action and the beliefs he entertains about the likelihoods of these outcomes.

Unpredictability means that when a decision maker decides that a change is called for, it is impossible to predict which of a set of feasible actions he will take.

Invoking the definitions of comparative incompleteness in the strong sense, we study the levels of inertia and unpredictability in the context of a simple portfolio selection model. More specifically, we show that the strong measures of comparative incompleteness characterize the level of unpredictability of a decision maker's portfolio choice behavior (that is, the size of the set of portfolio positions she may choose) and the level of inertia she displays.

Let $S = \{1, 2\}$, then an act is depicted by the point in $\mathbb{R}^2_+$ whose coordinates are the payoffs in the two states. Consider a decision maker whose preference relation $\succ$ on $\mathbb{R}^2_+$ is incomplete and has a multi-prior expected multi-utility representation. With slight abuse of notation, let the decision maker's set of priors be $\Pi = \{(\pi, 1-\pi) \mid \pi \in [\underline{\pi}, \bar{\pi}]\}$, where $[\underline{\pi}, \bar{\pi}]$ denotes the range of subjective probabilities of state 1, and denote by $\mathcal{U}$ the set of Bernoulli utility functions corresponding to $\succ$. We assume that the decision makers display risk aversion. Formally, assume that the elements of $\mathcal{U}$ are monotonic increasing, concave, real-valued functions on $\mathbb{R}_+$.

Let there be two Arrow securities, a_1 and a_2, with a_s paying one dollar contingent on the realization of state $s \in \{1, 2\}$. Denote by q the relative price of a_1 in terms of a_2, (i.e., a_2 is the numeraire Arrow security). Suppose that the decision maker's initial endowment consists of an equal number, w_0, of the two Arrow securities and denote the corresponding budget set $\{(w_{1,}w_2) \in \mathbb{R}_2 \mid qw_1 + w_2 \leq qw_0 + w_0\}$ by $B(w_{0,}q)$.

The decision maker's problem is to choose a portfolio $(w_1^*, w_2^*) \in B(w_{0,}q)$ of Arrow securities such that, for no other $(w_{1,}w_2) \in B(w_{0,}q)$,

$$\pi u(w_1) + (1-\pi)u(w_2) > \pi u(w_1^*) + (1-\pi)u(w_2^*), \forall(\pi, u) \in [\underline{\pi}, \bar{\pi}] \times \mathcal{U}. \qquad (10)$$

That is, there is no feasible portfolio that is strictly preferred to (w_1^*, w_2^*).

To find the set of portfolios that solve the decision maker's problem, consider the following: Given the budget set $B(w_{0,}q)$, there corresponds to each $(\pi, u) \in \Pi \times \mathcal{U}$ an optimal portfolio position given by the solution to

$$\left(w_1^{(\pi,u)}(w_{0,}q), w_2^{(\pi,u)}(w_{0,}q)\right) := \arg \max_{(w_1, w_2) \in B(w_{0,}q)} \left[\pi u(w_1) + (1-\pi)u(w_2)\right].$$

Denote the set of solutions by

$$W(w_{0,}q) = \left\{\left(w_1^{(\pi,u)}(w_{0,}q), w_2^{(\pi,u)}(w_{0,}q\right)) \mid (\pi, u) \in [\underline{\pi}, \bar{\pi}] \times \mathcal{U}\right\}.$$

The set $W(w_{0,}q)$ captures the unpredictability corresponding to a decision maker characterized by $[\underline{\pi}, \bar{\pi}] \times \mathcal{U}$.

The necessary and sufficient condition for $(w_1, w_2) \in W(w_{0,}q)$ is:

$$\frac{\pi u'(w_1)}{(1-\pi)u'(w_2)} = q$$

for some $(\pi, u) \in [\underline{\pi}, \bar{\pi}] \times \mathcal{U}$. Let $(\bar{w}_1(w_{0,}q),\ \underline{w}_2(w_{0,}q))$ and $(\underline{w}_1(w_{0,}q), \bar{w}_2(w_{0,}q))$ be implicitly defined by the equations

$$\frac{\bar{\pi}}{1-\bar{\pi}} \sup_{u \in \mathcal{U}} \frac{u'(\bar{w}_1(w_{0,}q))}{u'(\underline{w}_2(w_{0,}q))} = q$$

and

$$\frac{\underline{\pi}}{1-\underline{\pi}} \inf_{u \in \mathcal{U}} \frac{u'(\underline{w}_1(w_{0,}q))}{u'(\bar{w}_2(w_{0,}q))} = q.$$

Thus, $(\bar{w}_1(w_{0,}q), \underline{w}_2(w_{0,}q))$ is the point on the budget line at which the decision maker's largest marginal rate of substitution equals the slope of the budget line. Likewise, $(\underline{w}_1(w_{0,}q), \bar{w}_2(w_{0,}q))$ is the point on the budget line at which the decision maker's smallest marginal rate of substitution equals the slope of the budget line. Therefore, given $B(w_{0,}q)$, $(\bar{w}_1(w_{0,}q), \underline{w}_2(w_{0,}q))$ and $(\underline{w}_1(w_{0,}q), \bar{w}_2(w_{0,}q))$ are the extreme points of the set of portfolio positions in the set $W(w_0, q)$ that may be chosen by a preference relation $\succ$ with MPEMU representation $[\underline{\pi}, \bar{\pi}] \times \mathcal{U}$.

If $\bar{\pi}/(1-\bar{\pi}) < q$ then $w_1 < w_0 < w_2$, for all $(w_1, w_2) \in W(w_0, q)$ (that is, $W(w_0, q)$ is contained in the cone above the certainty line). If $\bar{\pi}/(1-\bar{\pi}) > q > \underline{\pi}/(1-\underline{\pi})$ then $(w_0, w_0) \in W(w_0, q)$. If $\underline{\pi}/(1-\underline{\pi}) > q$ then $w_1 > w_0 > w_2$, for all $(w_1, w_2) \in W(w_0, q)$ (that is, $W(w_0, q)$ is contained in the cone below the certainty line). Figure 2 illustrates the unpredictability set for the case in which $\bar{\pi}/(1-\bar{\pi}) > q > \underline{\pi}/(1-\underline{\pi})$ so that $(w_0, w_0) \in W(w_0, q)$.

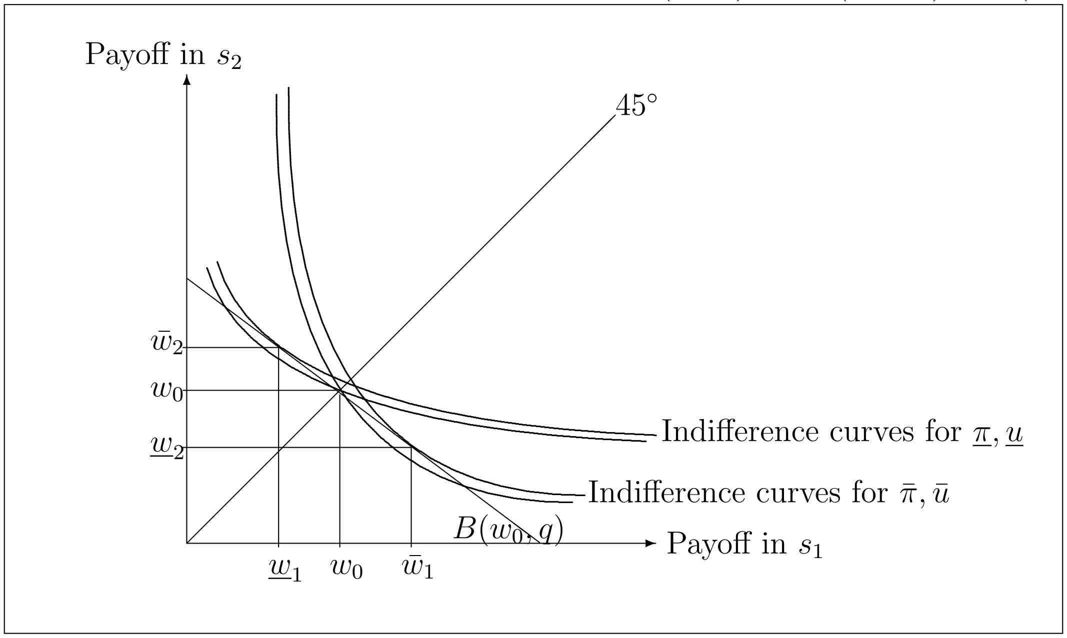
Proposition 1 shows that the level of unpredictability is higher the more incomplete a preference relation is in the strong sense.

Proposition 1 *If preference relation $\succ_1$ on F displays greater belief and taste incompleteness in the strong sense than preference relation $\succ_2$ then $W_2(w_{0,}q) \subseteq W_1(w_{0,}q)$, for all $(w_{0,}q) \in \mathbb{R}^2_{++}$.*

Now consider the effect of a change in relative prices to $\hat{q}$. Starting from (w_1^*, w_2^*), the decision maker will change his portfolio position to some other $(\hat{w}_1^*, \hat{w}_2^*) \in B(w_1^*, w_2^*, \hat{q})$ if and only if

$$\pi u(\hat{w}_1^*) + (1-\pi)u(\hat{w}_2^*) > \pi u(w_1^*) + (1-\pi)u(w_2^*), \forall\, (\pi, u) \in [\underline{\pi}, \bar{\pi}] \times \mathcal{U}.$$

Figure 2: Illustration of the Unpredictability Set $W(w_0, q)$ when $(w_0, w_0) \in W(w_0, q)$



Let

$$\frac{\bar{u}'(w_1^*)}{\bar{u}'(w_2^*)} := \sup_{u \in \mathcal{U}} \left\{ \frac{u'(w_1^*)}{u'(w_2^*)} \right\} \text{ and } \frac{\underline{u}'(w_1^*)}{\underline{u}'(w_2^*)} := \inf_{u \in \mathcal{U}} \left\{ \frac{u'(w_1^*)}{u'(w_2^*)} \right\}.$$

It is easy to verify that if

$$\hat{q} \in \left[\frac{\underline{\pi}}{1 - \underline{\pi}} \frac{\underline{u}'(w_1^*)}{\underline{u}'(w_2^*)}, \frac{\bar{\pi}}{1 - \bar{\pi}} \frac{\bar{u}'(w_1^*)}{\bar{u}'(w_2^*)} \right] \tag{11}$$

then the decision maker will hold on to her position (w_1^*, w_2^*). To see this, note that the left endpoint of the interval in (11) is the slope of the flattest of the decision maker's indifference curves through (w_1^*, w_2^*), while the right endpoint of the interval is the slope of the steepest of the decision maker's indifference curves through (w_1^*, w_2^*). The decision maker will hold on to his portfolio (w_1^*, w_2^*) as long as the slope of the budget line, given by $\hat{q}$, falls within this range.

Define the *measure of inertia* for $\succ$ at (w_1^*, w_2^*) by the interval of prices at which the portfolio position is maintianed. Formally,

$$I_{\succ}(w_1^*, w_2^*) = \left[\frac{\underline{\pi}}{1 - \underline{\pi}} \frac{\underline{u}'(w_1^*)}{\underline{u}'(w_2^*)}, \frac{\bar{\pi}}{1 - \bar{\pi}} \frac{\bar{u}'(w_1^*)}{\bar{u}'(w_2^*)} \right].$$

In particular, if the initial endowment (w_0, w_0) is the status quo, or default, portfolio then the measure of inertia at (w_0, w_0) is:

$$I_{\succ}(w_0, w_0) = \left[\frac{\underline{\pi}}{1-\underline{\pi}}, \frac{\bar{\pi}}{1-\bar{\pi}} \right]. \tag{12}$$

With only two states, if $E = \{s_1\}$, then $E^C = \{s_2\}$. Thus, if $\succ_1$ displays greater incompleteness than $\succ_2$ at $\{s_1\}$ then it displays greater incompleteness. Therefore, in this two-state economy the measure of inertia need not be indexed by the conditioning event.

We now investigate the comparative statics properties of the measure of inertia $I_{\succ}(w_1^*, w_2^*)$.

Proposition 2 *Let $\succ_1$ and $\succ_2$ be preference relations on $\mathbb{R}_+^2$. If $\succ_1$ displays greater belief and taste incompleteness in the strong sense than $\succ_2$ then $I_{\succ_1}(w_1^*, w_2^*) \supseteq I_{\succ_2}(w_1^*, w_2^*)$, for all (w_1^*, w_2^*). Moreover, if $I_{\succ_1}(w_1^*, w_2^*) \supseteq I_{\succ_2}(w_1^*, w_2^*)$ for all (w_1^*, w_2^*), then $\succ_1$ displays greater belief incompleteness in the strong sense than $\succ_2$.*

Proposition 2 shows that a preference relation displaying greater belief and taste incompleteness exhibits a higher level of inertia. Thus, the portfolio position of a decision maker displaying greater belief and taste incomplete preferences is less sensitive to price fluctuations. If a preference relation $\succ_1$ displays either greater belief incompleteness or greater taste incompleteness than $\succ_2$, but not both, then it is possible that $\succ_2$ displays greater overall incompleteness than $\succ_1$, and thus it is possible for $\succ_2$ to display greater inertia than $\succ_1$. However, if $\succ_1$ exhibits a higher level of inertia than $\succ_2$ at (w_0, w_0), then it must be the case that it displays greater belief incompleteness. An immediate implication of Proposition 2 is that if the preference relations $\succ_1$ and $\succ_2$ display the same level of belief-incompleteness, then $\succ_1$ exhibits greater taste-incompleteness than $\succ_2$ if and only if $I_{\succ_1}(w_1^*, w_2^*) \supseteq I_{\succ_2}(w_1^*, w_2^*)$, for all (w_1^*, w_2^*).

4 Elicitation

The elicitation of the measures of incomplete beliefs $m_b(E; \succ)$, incomplete tastes $m_t(p; \succ)$, and overall incomplete preferences $M(x_E y; \succ)$ requires a formal model depicting the process of choosing among non-comparable alternatives. The elicitation mechanisms to be described and analyzed below presume that choice among non-comparable alternatives is random. More specifically, imagine a decision maker facing a choice among non-comparable alternatives and suppose that, before choosing, the decision maker receives a signal – a subconscious impulse or some exogenous information – drawn at random from a distribution function whose support is $[\underline{r}, \bar{r}]$ in the case of incomplete beliefs and $[\underline{c}, \bar{c}]$ in the case of incomplete tastes or overall incomplete preferences. In either case the merits of the alternatives are reassessed according to the value of the signal and the choice is made accordingly.[9] In what follows, we denote the signal's cumulative distribution function by η. We begin with a discussion of an elicitation mechanism of $m_b(E; \succ)$ invoking a scheme due to Karni (2020b). We then extend this scheme to construct mechanisms for the elicitation of $M(x_E y; \succ)$ and $m_t(p; \succ)$.

4.1 Elicitation under Knightian uncertainty

There is a substantial body of literature dealing with incentive compatible mechanisms designed to elicit experts' subjective probabilities of uncertain events. Beginning with the work of Brier (1950) and Good (1952) it was followed by Savage (1971), Kadane and Winkler (1988), Grether (1981), Karni (2009) and others.[10] Underlying all these mechanisms is the presumption that the experts' beliefs are depicted by a unique probability measure. Recently, however, incentive compatible mechanisms designed to elicit sets of priors or posterior probabilities have been proposed. Karni (2020b) proposed a modified proper scoring rule for the elicitation of the range, $R(E)$, of the probabilities of an event E. This mechanism allows a direct elicitation of the range of the beliefs of any preference relation that admits MPEMU representation.

To see how this mechanism works, fix an event E and let $[\underline{\pi}(E), \bar{\pi}(E)]$ denote the range of the subjective probabilities representing a subject's beliefs about the likelihood of E. At time $t = 0$ the subject is asked to report two numbers, $\underline{r}, \bar{r} \in [0, 1]$ with $\underline{r} < \bar{r}$. Then a random number, r, is drawn from a uniform distribution on $[0, 1]$. In the interim period, $t = 1$, the subject is awarded the bet $x_E y$ if $r \leq \underline{r}$ and the lottery $\ell(r; x, y)$ if $r \geq \bar{r}$, where $x > y$. If $r \in (\underline{r}, \bar{r})$, then the subject is allowed to choose between the bet $(x - \theta)_E(y - \theta)$ and the lottery $\ell(r; x - \theta, y - \theta)$, where $\theta > 0$. In the last period, $t = 2$, after it is verified whether or not the event E obtained and the outcome of the lottery is revealed, all payments are made. Denote this mechanism $\mathcal{M}_b$.

Karni (2020b) proved the following result:

[9] This idea was formalized and the existence of such random selection process was proved in Karni and Safra (2016).

[10] For a recent review, see Chambers and Lambert (2017).

Theorem Karni (2020b): *Given the mechanism $\mathcal{M}_b$, in the limit, as $\theta \to 0$, the subject's unique dominant strategy is to report $\underline{r}(E) = \underline{\pi}(E)$ and $\bar{r}(E) = \bar{\pi}(E)$.*

Theorem 1 implies that this scheme elicits the measure, $m_b(E; \succ)$, of incompleteness of the subject's beliefs. Moreover, the elicitation procedure does not depend on the values of x and y or the decision maker's utility function.

4.2 Elicitation of the measure of overall incompleteness

Fix a bet $x_E y$ on E, and recall that $M(x_E y; \succ) = \bar{c}(x_E y; \succ) - \underline{c}(x_E y; \succ)$. At time $t = 0$, the subject is asked to report two numbers, $\underline{z}, \bar{z} \in [\underline{x}, \bar{x}] \supset [x, y]$ such that $\underline{z} < \bar{z}$. Then a random number, z, is drawn from a uniform distribution on $[\underline{x}, \bar{x}]$. In the interim period, $t = 1$, the subject is awarded the bet $x_E y$ if $z \leq \underline{z}$ and the outcome z if $z \geq \bar{z}$. If $z \in (\underline{z}, \bar{z})$, then the subject is allowed to choose between the bet $(x-\theta)_E(y-\theta)$ and the outcome $z-\theta$, where $\theta > 0$. In the last period, $t = 2$, after it is verified whether or not the event E obtained, all payments are made. Denote this mechanism $\mathcal{M}_o$.

Theorem 7 *Given $\mathcal{M}_o$, in the limit, as $\theta \to 0$, the subject's unique dominant strategy is to report $\underline{z} = \underline{c}(x_E y; \succ)$ and $\bar{z} = \bar{c}(x_E y; \succ)$.*

4.3 Elicitation of the measure of incomplete risk attitudes

Given $\succ$ on ΔX and $p = (x_1, p_1; ..., x_n, p_n) \in \Delta X$, recall that $m_t(p, \succ) = \bar{c}^{\succ}(p) - \underline{c}^{\succ}(p)$. The mechanism requires the subject to report, at time $t = 0$, two numbers, $\underline{z}, \bar{z} \in [\underline{x}, \bar{x}] \supset \{x_1, ...x_n\}$ such that $\underline{z} \leq \bar{z}$. A random number, z, is drawn from a uniform distribution on $[\underline{x}, \bar{x}]$. In the interim period, $t = 1$, the subject is awarded the lottery p if $z \leq \underline{z}$ and the outcome z if $z \geq \bar{z}$. If $\underline{z} < \bar{z}$ and $z \in (\underline{z}, \bar{z})$, then the subject is allowed to choose between the lottery $p' = (x_1 - \theta, p_1; ..., x_n - \theta, p_n)$ and the outcome $z - \theta$, where $\min\{x_1, ...x_n\} > \theta > 0$. In the last period, the outcome of the lottery is revealed, and all payments are made. Denote this mechanism $\mathcal{M}_t$.

Theorem 8 *Given $\mathcal{M}_t$, in the limit, as $\theta \to 0$, the subject's unique dominant strategy is to report $\underline{z} = \underline{c}(p)$ and $\bar{z} = \bar{c}(p)$.*

The proof is by the same argument as the proof of the preceding theorem.

5 Concluding Remarks

Whether it is belief, taste, or overall incompleteness, our characterizations of the relation "more incomplete than" are preference-based. Invoking these measures, the simple portfolio choice problem in Section 3.3 illustrates the usefulness of the strong measures in deriving comparative statics implications. The behavioral implications of the general greater incompleteness measures in subsection 3.1 are somewhat weaker. For instance, in the case of portfolio selection, under Knightian uncertainty, the level of inertia and unpredictability displayed by $\succ_1$ exceeds that displayed by $\succ_2$ but not necessarily in response to the same price variations or over the same price range, respectively. Similar observations apply to risk-attitude and overall incompleteness. Greater incompleteness according to our general measures imply higher levels of inertia and unpredictability, but not necessarily over the same price range. One advantage of the general measures of comparative incompleteness is that, for a given event, bet, or lottery, the general "more incomplete than" relation is itself a complete relation, as opposed to the corresponding strong sense relation, which is incomplete.

An important question that is beyond the scope of this work is how information affects the level of incompleteness. Here, the preference relations being compared are the prior and posterior preference relations. Addressing this issue requires a procedure of updating the set of priors. The example below illustrates that, updating all the priors in the set using Bayes' rule, becoming better informed about an event E makes the beliefs at E more complete but may or may not make the beliefs at some other event become more complete.

Example: An urn contains balls that come in three colors, blue, green, and yellow denoted B, G, and Y, respectively. Consider a decision maker who displays Knightian uncertainty and suppose that she holds the following set of beliefs: $\{(\pi_B, \pi_G, \pi_Y) | \pi_B \in [\frac{1}{3}, \frac{2}{3}], \pi_G \in [0, \frac{1}{3}], \pi_Y \in [\frac{1}{3}, \frac{2}{3}], \pi_B + \pi_G + \pi_Y = 1\}$. The measure of belief incompleteness of the decision maker's prior preferences at event B is $m_b(B, \succ) = \frac{1}{3}$. Assume that when she receives new information, the decision maker updates her beliefs prior by prior, using Bayes' rule.

Suppose now that the decision maker is informed that the urn contains no yellow balls. Applying Bayesian updating, the posterior range of probabilities of the event B is $[\frac{1}{2}, 1]$. Hence, $m_b(B, \succ') = \frac{1}{2}$, where $\succ'$ denotes the updated beliefs given the information $\neg Y$. If, instead the decision maker learns that the urn contains no green balls, the range of her posterior probabilities that a ball is blue is $[\frac{1}{3}, \frac{2}{3}]$, so $m_b(B, \succ'') = \frac{1}{3}$, where $\succ''$ denotes the updated beliefs given the information $\neg G$. Hence, according to our measure of belief incompleteness, information that the ball is not yellow makes the decision maker's beliefs more incomplete at B, while information that the ball is not green does not change her belief incompleteness at B. In either case, however, the information about an event makes the beliefs about that event more complete.

As the example above and the asset market application in Section 3.3 show, both the general and strong measures of comparative incompleteness have their merits. Also, when preferences have MPEMU representations, our measures of incompleteness have intuitive interpretations in terms of the decision makers' beliefs and risk attitudes. To the exent to which measurement paves the road to knowledge as expressed by Lord Kelvin, "When you can measure what you are speaking about, and express it in numbers, you know something about it, when you cannot express it in numbers, your knowledge is of a meager and unsatisfactory kind," this paper, by suggesting measures of incompleteness, is a contribution towards the analysis of a variety of questions that have to do with the behavioral implications of incomplete preferences in a manner analogous to the use of measures of risk aversion.

6 Appendix

6.1 Proof of Theorem 1:

Applied to bets and the constant lottery acts, the representation in (2) implies that,

$$x_E y \succ \ell(r; x, y)$$

if and only if

$$U(\delta_x)\pi(E) + U(\delta_y)(1 - \pi(E)) > U(\delta_x)r + U(\delta_y)(1 - r), \forall\, (\pi, U) \in \Pi \times \mathcal{U}.$$

By definition of the set $R^{\succ}(E)$ in (3), it is the case that for any $r \in R^{\succ}(E)$,

$$\exists\, (\tilde{\pi}, U) \in \Pi \times \mathcal{U} \text{ such that } (\tilde{\pi}(E) - r)\,[U(\delta_x) - U(\delta_y)] \leq 0, \tag{13}$$

and

$$\exists\, (\hat{\pi}, U) \in \Pi \times \mathcal{U} \text{ such that } (\hat{\pi}(E) - r)\,[U(\delta_x) - U(\delta_y)] \geq 0. \tag{14}$$

But $x > y$. Hence, monotonicity with respect to first-order stochastic dominance implies that $U(\delta_x) - U(\delta_y) > 0$. Thus, the expression in (13) is equivalent to

$$\exists \tilde{\pi} \in \Pi \text{ such that } \tilde{\pi}(E) \leq r, \tag{15}$$

while the expression in (16) is equivalent to

$$\exists \hat{\pi} \in \Pi \text{ such that } \hat{\pi}(E) \geq r. \tag{16}$$

Since (15) holds for all $r \in R^{\succ}(E)$, $\underline{\pi}(E) \leq \underline{r}^{\succ}(E)$. Suppose $\underline{\pi}(E) < \underline{r}^{\succ}(E)$. Then $(\underline{\pi}(E) - \underline{r}^{\succ}(E))\,[U(\delta_x) - U(\delta_y)] < 0$, for all $U \in \mathcal{U}$. This contradicts that $x_E y \succ \ell(r; x, y)$ for $r < \underline{r}^{\succ}(E)$. It follows that $\underline{\pi}(E) = \underline{r}^{\succ}(E)$. A similar argument shows that $\bar{\pi}(E) = \bar{r}^{\succ}(E)$. Therefore,

$$[\underline{\pi}(E), \bar{\pi}(E)] = [\underline{r}^{\succ}, \bar{r}^{\succ}]. \tag{17}$$

It follows that $m_b(E; \succ) = \bar{\pi}(E) - \underline{\pi}(E)$. Since x, y, and U do not figure in this expression, $m_b(E; \succ)$ is independent of x, y, and U. ■

6.2 Proof of Theorem 2:

The proof follows the idea of Pratt (1964). Let $\tilde{U}$ be the utility function in $\mathcal{U}$ associated with the smallest risk premium $\underline{\xi}^{\succ}(\ell(r; x, y)) \equiv \underline{\xi}^{\succ}_{r;x,y}$ and let $\tilde{u}$ be the corresponding Bernoulli utility function. For ease of notation, we suppress the dependency of $\mu_r(x, y)$ on x and y in the intermediate steps below and simply write μ_r. By definition of the risk premium,

$$\tilde{U}(\delta_{\mu_r - \underline{\xi}^{\succ}_{r;x,y}}) = \tilde{U}(\delta_{\bar{c}^{\succ}(\ell(r;x,y))}) = \tilde{U}(\ell(r; x, y)).$$

Written in terms of the Bernoulli utility function $\tilde{u}$, we have that

$$\tilde{u}(\mu_r - \underline{\xi}^{\succ}_{r;x,y}) = E_{\ell(r;x,y))}[\tilde{u}(z)], \tag{18}$$

where $E_{\ell(r;x,y))}$ denotes the expectation w.r.t. the distribution $\ell(r; x, y))$. Expanding the left-hand-side of (18) around μ_r gives

$$\tilde{u}\left(\mu_r - \underline{\xi}^{\succ}_{r;x,y}\right) = \tilde{u}(\mu_r) - \tilde{u}'(\mu_r)\underline{\xi}^{\succ}_{r;x,y} + O((\underline{\xi}^{\succ}_{r;x,y})^2) \tag{19}$$

while expanding the right-hand-side of (18) around μ_r gives

$$E_{\ell(r;x,y))}[\tilde{u}(z)] = E_{\ell(r;x,y))}\left[\tilde{u}(\mu_r) + \tilde{u}'(\mu_r)(z - \mu_r) + \frac{1}{2}\tilde{u}''(\mu_r)(z - \mu_r)^2\right] + o(\sigma_r^2(x, y)) \tag{20}$$

By (18), the right-hand-sides of (19) and (20) are equal, which results in

$$\underline{\xi}^{\succ}(\ell(r; x, y)) = -\frac{1}{2}\frac{\tilde{u}''(\mu_r(x, y))}{\tilde{u}'(\mu_r(x, y))}\sigma_r^2(x, y) + o(\sigma_r^2(x, y)) \tag{21}$$

Now, let $\hat{U}$ be the utility function in $\mathcal{U}$ associated with the largest risk premium $\bar{\xi}^{\succ}(\ell(r;x,y))$ and let $\hat{u}$ be the corresponding Bernoulli utility function. By steps similar to those for $\underline{\xi}^{\succ}(\ell(r;x,y))$, we obtain

$$\bar{\xi}^{\succ}(\ell(r;x,y)) = -\frac{1}{2}\frac{\hat{u}''(\mu_r(x,y))}{\hat{u}'(\mu_r(x,y))}\sigma_r^2(x,y) + o(\sigma_r^2(x,y)) \tag{22}$$

Note that we must have that

$$\hat{U} = \arg\max_{U\in\mathcal{U}} -\frac{u''(\mu_r(x,y))}{u'(\mu_r(x,y))} \text{ and } \tilde{U} = \arg\min_{U\in\mathcal{U}} -\frac{u''(\mu_r(x,y))}{u'(\mu_r(x,y))}.$$

Hence, using the expressions in (21) and (22) the definition of $m_t(p;\succ)$ gives that for small $x-y$, the measure of taste incompleteness of $\succ$ at $\ell(r;x,y)$ satisfies

$$\begin{aligned} m_t(\ell(r;x,y);\succ) &= \left[\max_{U\in\mathcal{U}}\left\{-\frac{u''(\mu_r(x,y))}{u'(\mu_r(x,y))}\right\} - \min_{U\in\mathcal{U}}\left\{-\frac{u''(\mu_r(x,y))}{u'(\mu_r(x,y))}\right\}\right]\frac{\sigma_r^2(x,y)}{2} \\ &+ o(\sigma_r^2(x,y)). \end{aligned}$$

■

6.3 Proof of Theorem 3:

By definition, $\bar{c}^{\succ}(\ell(r;x,y) \geq \underline{c}^{\succ}(\ell(r;x,y)))$ for any $r \in [0,1]$, and $\bar{r}^{\succ}(E) \geq \underline{r}^{\succ}(E)$ for any E. By first order stochastic dominance, $\bar{c}^{\succ}(\ell(\bar{r}^{\succ}(E);x,y)) \geq \bar{c}^{\succ}(\ell(\underline{r}^{\succ}(E);x,y))$ and $\underline{c}^{\succ}(\ell(\bar{r}^{\succ}(E);x,y)) \geq \underline{c}^{\succ}(\ell(\underline{r}^{\succ}(E);x,y))$. Therefore, we must have that

$$\bar{c}(x_E y;\succ) = \bar{c}^{\succ}(\ell(\bar{r}^{\succ}(E);x,y))$$

and

$$\underline{c}(x_E y;\succ) = \underline{c}^{\succ}(\ell(\underline{r}^{\succ}(E);x,y)).$$

To ease notation in the derivations below, let $\underline{\mu} = \underline{r}^{\succ}(E)x + (1-\underline{r}^{\succ}(E))y$, that is, the expected value of the bet according to the least favourable distribution in $R^{\succ}(E)$ and let $\bar{\mu} = \bar{r}^{\succ}(E)x + (1-\bar{r}^{\succ}(E))y$, that is, the expected value of the bet according to the most favourable distribution in $R^{\succ}(E)$.

Let $\tilde{U}$ be the utility function in $\mathcal{U}$ associated with the smallest risk premium at $\ell(\bar{r}^{\succ}(E);x,y))$ and let $\tilde{u}$ be the corresponding Bernoulli utility function. By definition of the risk premium,

$$\tilde{U}(\delta_{\bar{\mu}-\underline{\xi}^{\succ}_{\bar{r}^{\succ}(E);x,y}}) = \tilde{U}(\delta_{\bar{c}^{\succ}(\ell(\bar{r}^{\succ}(E);x,y))}) = \tilde{U}(\ell(\bar{r}^{\succ}(E);x,y)).$$

Similar to expression (18) in the proof of Theorem 2, we can rewrite the expression in terms of $\tilde{u}$. Expanding around $\bar{\mu}$ and following the steps as in (19) through (21) we obtain that

$$\underline{\xi}^{\succ}(\ell(\bar{r}^{\succ}(E);x,y)) = -\frac{1}{2}\frac{\tilde{u}''(\bar{\mu})}{\tilde{u}'(\bar{\mu})}\sigma^2_{\bar{r}^{\succ}(E)}(x,y) + o(\sigma^2_{\bar{r}^{\succ}(E)}(x,y)) \tag{23}$$

Now, let $\hat{U}$ be the utility function in $\mathcal{U}$ associated with the largest risk premium at $\ell(\underline{r}^{\succ}(E);x,y))$ and let $\hat{u}$ be the corresponding Bernoulli utility function. By expanding around $\underline{\mu}$ and equating terms as in the steps above, we obtain

$$\bar{\xi}^{\succ}(\ell(\underline{r}^{\succ}(E);x,y)) = -\frac{1}{2}\frac{\hat{u}''(\underline{\mu})}{\hat{u}'(\underline{\mu})}\sigma^2_{\underline{r}^{\succ}(E)}(x,y) + o(\sigma^2_{\underline{r}^{\succ}(E)}(x,y)). \tag{24}$$

Note that we must have that

$$\hat{U} = \arg\max_{U\in\mathcal{U}} -\frac{u''(\underline{\mu})}{u'(\underline{\mu})} \textit{ and } \tilde{U} = \arg\min_{U\in\mathcal{U}} -\frac{u''(\bar{\mu})}{u'(\bar{\mu})}.$$

By definition,

$$M(x_E y;\succ) = \bar{c}(x_E y;\succ) - \underline{c}(x_E y;\succ) = \bar{\mu} - \underline{\mu} + \bar{\xi}^{\succ}(\ell(\underline{r}^{\succ}(E);x,y)) - \underline{\xi}^{\succ}(\ell(\bar{r}^{\succ}(E);x,y)).$$

Note that $\sigma_r^2(x,y) = r(1-r)(x-y)^2$, so $o(\sigma_r^2(x,y)) = o((x-y)^2)$.

By Theorem 1, $\underline{\pi}(E) = \underline{r}^{\succ}(E)$ and $\bar{\pi}(E) = \bar{r}^{\succ}(E)$. Hence, plugging in expressions (23) and (24) gives that for small $x-y$, the measure of overall incompleteness of $\succ$ at $x_E y$ satisfies

$$\begin{aligned} & M(x_E y;\succ) = (\bar{\pi}(E) - \underline{\pi}(E))(x-y) \\ + \ & \frac{1}{2}\left[\max_{U\in\mathcal{U}}\left\{-\frac{u''(\mu_{\underline{\pi}(E)}(x,y))}{u'(\mu_{\underline{\pi}(E)}(x,y))}\right\}\sigma^2_{\underline{\pi}(E)}(x,y) - \min_{U\in\mathcal{U}}\left\{-\frac{u''(\mu_{\bar{\pi}(E)}(x,y))}{u'(\mu_{\bar{\pi}(E)}(x,y))}\right\}\sigma^2_{\bar{\pi}(E)}(x,y)\right] \\ + \ & o((x-y)^2). \end{aligned} \tag{25}$$

■

6.4 Proof of Theorem 4:

For a lottery $\ell(r;x,y)$, the variance $\sigma_r^2(x,y) = r(1-r)(x-y)^2$. Plugging this into (25) from the proof of Theorem 3 gives

$$\begin{aligned} M(x_E y; \succ) &= (\bar{\pi}(E) - \underline{\pi}(E))(x-y) \\ &+ \frac{1}{2}\left[\max_{U\in\mathcal{U}}\left\{-\frac{u''(\mu_{\underline{\pi}(E)}(x,y))}{u'(\mu_{\underline{\pi}(E)}(x,y))}\right\}\underline{\pi}(E)(1-\underline{\pi}(E))(x-y)^2\right. \\ &- \left.\min_{U\in\mathcal{U}}\left\{-\frac{u''(\mu_{\bar{\pi}(E)}(x,y))}{u'(\mu_{\bar{\pi}(E)}(x,y))}\right\}\bar{\pi}(E)(1-\bar{\pi}(E))(x-y)^2\right] \\ &+ o((x-y)^2). \end{aligned} \tag{26}$$

Dividing by $(x-y)$ on both sides of (26) and taking the limit as $y \to x$ gives that

$$\lim_{y\to x}\frac{M(x_E y;\succ)}{x-y} = \bar{r}^{\succ}(E) - \underline{r}^{\succ}(E) = m_b(E;\succ).$$

■

6.5 Proof of Theorem 5:

Observe that the proof of Theorem 2 does not hinge on the support of the lottery being binary, with the understanding that for a general p the local requirement is that we let all values in the support be close to the mean. We therefore have that

$$m_t(p;\succ_i) = \left[\max_{U\in\mathcal{U}_i}\left\{-\frac{u''(\mu(p))}{u'(\mu(p))}\right\} - \min_{U\in\mathcal{U}_i}\left\{-\frac{u''(\mu(p))}{u'(\mu(p))}\right\}\right]\frac{\sigma^2(p)}{2} + o(\sigma^2(p)), \tag{27}$$

for $i = 1,2$. Suppose now that $m_t(p;\succ_1) > m_t(p;\succ_2)$. By (27),

$$\begin{aligned} &\left[\max_{U\in\mathcal{U}_1}\left\{-\frac{u''(\mu(p))}{u'(\mu(p))}\right\} - \min_{U\in\mathcal{U}_1}\left\{-\frac{u''(\mu(p))}{u'(\mu(p))}\right\}\right]\frac{\sigma^2(p)}{2} \\ - &\left[\max_{U\in\mathcal{U}_2}\left\{-\frac{u''(\mu(p))}{u'(\mu(p))}\right\} - \min_{U\in\mathcal{U}_2}\left\{-\frac{u''(\mu(p))}{u'(\mu(p))}\right\}\right]\frac{\sigma^2(p)}{2} \\ = &\; m_t(p;\succ_1) - m_t(p;\succ_2) + o(\sigma^2(p)) \end{aligned} \tag{28}$$

Therefore, for any positive value of $m_t(p;\succ_1) - m_t(p;\succ_2)$, there exists $\epsilon > 0$ such that for all $0 < \sigma^2(p) < \epsilon$, $o(\sigma^2(p)) < m_t(p;\succ_1) - m_t(p;\succ_2)$. Then (28) implies that

$$\left[\left[\max_{U\in\mathcal{U}_1}\left\{-\frac{u''(\mu(p))}{u'(\mu(p))}\right\} - \min_{U\in\mathcal{U}_1}\left\{-\frac{u''(\mu(p))}{u'(\mu(p))}\right\}\right] - \left[\max_{U\in\mathcal{U}_2}\left\{-\frac{u''(\mu(p))}{u'(\mu(p))}\right\} - \min_{U\in\mathcal{U}_2}\left\{-\frac{u''(\mu(p))}{u'(\mu(p))}\right\}\right]\right]\frac{\sigma^2(p)}{2}$$

is also positive. Since $\sigma^2(p) > 0$, it then follows that

$$\left[\max_{U\in\mathcal{U}_1}\left\{-\frac{u''(\mu(p))}{u'(\mu(p))}\right\} - \min_{U\in\mathcal{U}_1}\left\{-\frac{u''(\mu(p))}{u'(\mu(p))}\right\}\right] > \left[\max_{U\in\mathcal{U}_2}\left\{-\frac{u''(\mu(p))}{u'(\mu(p))}\right\} - \min_{U\in\mathcal{U}_2}\left\{-\frac{u''(\mu(p))}{u'(\mu(p))}\right\}\right].$$

A similar argument can be used to show the other direction as well. ■

6.6 Proof of Theorem 6:

Consider a bet $x_E y$. Since $\succ_1$ displays greater belief incompleteness in the strong sense than $\succ_2$, we have that for any E, $[\underline{\pi}_2(E), \bar{\pi}_2(E)] \subseteq [\underline{\pi}_1(E), \bar{\pi}_1(E)]$. Since $\succ_1$ displays greater taste incompleteness in the strong sense than $\succ_2$, we have that for any p, $[\underline{c}^{\succ_2}(p), \bar{c}^{\succ_2}(p)] \subseteq [\underline{c}^{\succ_1}(p), \bar{c}^{\succ_1}(p)]$. It follows that

$$\bar{c}^{\succ_1}(\bar{\pi}_1(E)) \geq \underline{c}^{\succ_2}(\underline{\pi}_2(E)). \tag{29}$$

As argued in the beginning of the proof of Theorem 3, it must be that for any $\succ$,

$$\bar{c}(x_E y; \succ) = \bar{c}^{\succ}(\ell(\bar{r}^{\succ}(E); x, y))$$

and

$$\underline{c}(x_E y; \succ) = \underline{c}^{\succ}(\ell(\underline{r}^{\succ}(E); x, y)).$$

The result then follows from (29). ■

6.7 Proof of Proposition 1:

Suppose that a preference relation $\succ_1$ displays greater belief and taste incompleteness in the strong sense than preference relation $\succ_2$. Greater belief incompleteness in the strong sense is equivalent to $\Pi_2 \subseteq \Pi_1$. Therefore,

$$\underline{\pi}_1 \leq \underline{\pi}_2 \leq \bar{\pi}_2 \leq \bar{\pi}_1. \tag{30}$$

For $i = 1, 2$, define $\bar{u}_i$ and $\underline{u}_i$, respectively by

$$\arg\max_{u\in\mathcal{U}_i} \frac{\bar{\pi}_i}{1-\bar{\pi}_i}\frac{u'(\bar{w}_1^i(w_0, q))}{u'(\underline{w}_2^i(w_0, q))} \tag{31}$$

and

$$\arg\min_{u\in\mathcal{U}_i} \frac{\underline{\pi}_i}{1-\underline{\pi}_i}\frac{u'(\underline{w}_1^i(w_0, q))}{u'(\bar{w}_2^i(w_0, q))}. \tag{32}$$

Greater taste incompleteness in the strong sense implies that $C^{\succ_2}(p) \subseteq C^{\succ_1}(p)$, for all $p \in \Delta(X)$. Therefore, $\underline{c}^{\succ_1}(p) \leq \underline{c}^{\succ_2}(p) \leq \bar{c}^{\succ_2}(p) \leq \bar{c}^{\succ_1}(p)$ for all $p \in \Delta(X)$. Thus, by definition, $\underline{\xi}^{\succ_1}(p) \leq \underline{\xi}^{\succ_2}(p)$ and $\bar{\xi}^{\succ_2}(p) \leq \bar{\xi}^{\succ_1}(p)$ for all $p \in \Delta(X)$. By Theorem 1 in Pratt (1964), there exist monotonic increasing and concave functions $\bar{T}$ and $\underline{T}$ such that $\bar{u}_2 = \bar{T} \circ \bar{u}_1$ and $\underline{u}_1 = \underline{T} \circ \underline{u}_2$.

Note that if $\bar{w}_1^i(w_{0,}q) \geq \underline{w}_2^i(w_{0,}q)$, then $(\bar{w}_1^i(w_{0,}q), \underline{w}_2^i(w_{0,}q))$ is a bet on state 1 and $\bar{c}(\bar{w}_1^i(w_{0,}q)_E \underline{w}_2^i(w_{0,}q); \succ_i) = \bar{c}(\bar{\pi}_i; \succ_i)$, while $\underline{c}(\bar{w}_1^i(w_{0,}q)_E \underline{w}_2^i(w_{0,}q); \succ_i) = \underline{c}(\underline{\pi}_i; \succ_i)$.

By definition of $(\bar{w}_1^i(w_{0,}q)), \underline{w}_2^i(w_{0,}q)))$, the expression in (31) equals q for $i = 1, 2$. Using this and that $\bar{u}_2 = \bar{T} \circ \bar{u}_1$, we have

$$\frac{\bar{\pi}_1}{1-\bar{\pi}_1} \frac{\bar{u}_1'(\bar{w}_1^1(w_{0,}q))}{\bar{u}_1'(\underline{w}_2^1(w_{0,}q))} = \frac{\bar{\pi}_2}{1-\bar{\pi}_2} \frac{\bar{u}_2'(\bar{w}_1^2(w_{0,}q))}{\bar{u}_2'(\underline{w}_2^2(w_{0,}q))} = \frac{\bar{\pi}_2}{1-\bar{\pi}_2} \frac{\bar{T}'(\bar{u}_1(\bar{w}_1^2(w_{0,}q)))\bar{u}_1'(\bar{w}_1^2(w_{0,}q))}{\bar{T}'(\bar{u}_1(\underline{w}_2^2(w_{0,}q)))\bar{u}_1'(\underline{w}_2^2(w_{0,}q))}. \tag{33}$$

By (30), $\frac{\bar{\pi}_2}{1-\bar{\pi}_2} \leq \frac{\bar{\pi}_1}{1-\bar{\pi}_1}$. If $\bar{w}_1^2(w_{0,}q) \geq \underline{w}_2^2(w_{0,}q)$, then the monotonicity of u_1 and the concavity of $\bar{T}$ imply that $\bar{T}'(\bar{u}_1(\bar{w}_1^2(w_{0,}q))) \leq T'(\bar{u}_1(\underline{w}_2^2(w_{0,}q)))$. Hence, the equality in (33) implies that

$$\frac{\bar{u}_1'(\bar{w}_1^1(w_{0,}q))}{\bar{u}_1'(\underline{w}_2^1(w_{0,}q))} \leq \frac{\bar{u}_1'(\bar{w}_1^2(w_{0,}q))}{\bar{u}_1'(\underline{w}_2^2(w_{0,}q))}.$$

That is, the marginal rate of substitution corresponding to $\bar{u}_1$ is larger at $(\bar{w}_1^2(w_{0,}q), \underline{w}_2^2(w_{0,}q))$ than it is at $(\bar{w}_1^1(w_{0,}q), \underline{w}_2^1(w_{0,}q))$. Hence, $\bar{w}_1^1(w_{0,}q) \geq \bar{w}_1^2(w_{0,}q)$ and $\underline{w}_2^1(w_{0,}q) \leq \underline{w}_2^2(w_{0,}q)$.

By definition of $(\underline{w}_1^i(w_{0,}q), \bar{w}_2^i(w_{0,}q))$, the expression in (32) equals q for $i = 1, 2$. Using this and that $\underline{u}_1 = \underline{T} \circ \underline{u}_2$, we have

$$\frac{\underline{\pi}_2}{1-\underline{\pi}_2} \frac{\underline{u}_2'(\underline{w}_1^2(w_{0,}q))}{\underline{u}_2'(\bar{w}_2^2(w_{0,}q))} = \frac{\underline{\pi}_1}{1-\underline{\pi}_1} \frac{\underline{u}_1'(\underline{w}_1^1(w_{0,}q))}{\underline{u}_1'(\bar{w}_2^1(w_{0,}q))} = \frac{\underline{\pi}_1}{1-\underline{\pi}_1} \frac{\underline{T}'(\underline{u}_2(\underline{w}_1^1(w_{0,}q)))\underline{u}_2'(\underline{w}_1^1(w_{0,}q))}{\underline{T}'(\underline{u}_2(\bar{w}_2^1(w_{0,}q)))\underline{u}_2'(\bar{w}_2^1(w_{0,}q))}. \tag{34}$$

By (30), $\frac{\underline{\pi}_2}{1-\underline{\pi}_2} \geq \frac{\underline{\pi}_1}{1-\underline{\pi}_1}$. If $\bar{w}_2^1(w_{0,}q) \leq \underline{w}_1^1(w_{0,}q)$, then the monotonicity of and the concavity of $\underline{T}$ imply that $\underline{T}'(\underline{u}_2(\underline{w}_1^1(w_{0,}q))) \leq \underline{T}'(\underline{u}_2(\bar{w}_2^1(w_{0,}q)))$. Hence, equality in (34) implies that

$$\frac{\underline{u}_2'(\underline{w}_1^2(w_{0,}q))}{\underline{u}_2'(\bar{w}_2^2(w_{0,}q))} \leq \frac{\underline{u}_2'(\underline{w}_1^1(w_{0,}q))}{\underline{u}_2'(\bar{w}_2^1(w_{0,}q))}.$$

That is, the marginal rate of substitution corresponding to $\bar{u}_1$ is larger at $(\underline{w}_1^1(w_{0,}q), \bar{w}_2^1(w_{0,}q))$ than at $(\underline{w}_1^2(w_{0,}q), \bar{w}_2^2(w_{0,}q))$. It follows that $\underline{w}_1^1(w_{0,}q) \leq \underline{w}_1^2(w_{0,}q)$ and $\bar{w}_2^1(w_{0,}q) \geq \bar{w}_2^2(w_{0,}q)$.

Apply the same logic to the case in which $\underline{w}_1^2(w_{0,}q) \leq \bar{w}_2^2(w_{0,}q)$ and $\underline{w}_2^1(w_{0,}q) \geq \bar{w}_1^1(w_{0,}q)$, and use that if $\bar{w}_1^i(w_{0,}q) \leq \underline{w}_2^i(w_{0,}q)$, then $(\bar{w}_1^i(w_{0,}q), \underline{w}_2^i(w_{0,}q))$ is a bet on state 2 so $\bar{c}(\bar{w}_1^i(w_{0,}q)_E\underline{w}_2^i(w_{0,}q); \succ_i) = \bar{c}(1-\underline{\pi}_i; \succ_i)$, while $\underline{c}(\bar{w}_1^i(w_{0,}q)_E\underline{w}_2^i(w_{0,}q); \succ_i) = \underline{c}(1-\bar{\pi}_i; \succ_i)$. Then we get $\underline{w}_1^1(w_{0,}q) \geq \underline{w}_1^2(w_{0,}q)$ and $\bar{w}_1^1(w_{0,}q) \geq \bar{w}_1^2(w_{0,}q)$. ■

6.8 Proof of proposition 2:

Suppose that $\succ_1$ displays greater incompleteness in the strong sense than $\succ_2$. Let $\bar{u}_i$ and $\underline{u}_i$ be given by (31) and (32), respectively, for $i = 1, 2$. By greater belief incompleteness in the strong sense, (30) holds.

Assume that $w_1^* > w_2^*$. Greater taste incompleteness in the strong sense implies that $C^{\succ_2}(p) \subseteq C^{\succ_1}(p)$, for all $p \in \Delta(X)$. Thus, $\underline{c}^{\succ_1}(p) \leq \underline{c}^{\succ_2}(p) \leq \bar{c}^{\succ_2}(p) \leq \bar{c}^{\succ_1}(p)$, for all $p \in \Delta(X)$. Hence, by definition, $\underline{\xi}^{\succ_1}(p) \leq \underline{\xi}^{\succ_2}(p)$ and $\bar{\xi}^{\succ_2}(p) \leq \bar{\xi}^{\succ_1}(p)$, for all $p \in \Delta(X)$. Therefore, by Theorem 1 in Pratt (1964), there exist monotonic increasing and concave functions $\bar{T}$ and $\underline{T}$ such that $\bar{u}_2 = \bar{T} \circ \bar{u}_1$ and $\underline{u}_1 = \underline{T} \circ \underline{u}_2$. Therefore,

$$\frac{\bar{\pi}_2}{1-\bar{\pi}_2}\frac{\bar{u}_2'(w_1^*)}{\bar{u}_2'(w_2^*)} = \frac{\bar{\pi}_2}{1-\bar{\pi}_2}\frac{\bar{T}'(\bar{u}_1(w_1^*)\bar{u}_1'(w_1^*)}{\bar{T}'(\bar{u}_1(w_2^*))\bar{u}_1'(w_2^*)} \leq \frac{\bar{\pi}_1}{1-\bar{\pi}_1}\frac{\bar{u}_1'(w_1^*)}{u_1'(w_2^*))}. \tag{35}$$

and

$$\frac{\underline{\pi}_1}{1-\underline{\pi}_1}\frac{\underline{u}_1'(w_1^*)}{\underline{u}_1'(w_2^*)} = \frac{\underline{\pi}_1}{1-\underline{\pi}_1}\frac{\underline{T}'(\underline{u}_2(w_1^*))\underline{u}_2'(w_1^*)}{\underline{T}'(\underline{u}_2(\bar{w}_2^*))\underline{u}_2'(\bar{w}_2^*)} \leq \frac{\underline{\pi}_2}{1-\underline{\pi}_2}\frac{\underline{u}_2'(w_1^*)}{\underline{u}_2'(\bar{w}_2^*)}, \tag{36}$$

where concavity of $\bar{T}$ and $\underline{T}$ is used to conclude that $\frac{\underline{T}'(\underline{u}_2(w_1^*))}{\underline{T}'(\underline{u}_2(\bar{w}_2^*))} \leq 1$ and $\frac{\bar{T}'(\bar{u}_1(w_1^*))}{\bar{T}'(\bar{u}_1(w_2^*))} \leq 1$ and using the relationships in (30). It follows from (35) and (36) that $I_{\succ_1}(w_1^*, w_2^*) \supseteq I_{\succ_2}(w_1^*, w_2^*)$.

The proof for the case in which $w_1^* \leq w_2^*$ is by a similar argument, noting that when $w_1^* \leq w_2^*$, we are considering a bet on state 2.

To show that if $I_{\succ_1}(w_1^*, w_2^*) \supseteq I_{\succ_2}(w_1^*, w_2^*)$ for all (w_1^*, w_2^*) then $\succ_1$ displays greater belief incompleteness in the strong sense than $\succ_2$, suffices it to observe that $I_{\succ_i}(w_0, w_0) = \left[\frac{\underline{\pi}_i}{1-\underline{\pi}_i}, \frac{\bar{\pi}_i}{1-\bar{\pi}_i}\right]$ for $i = 1, 2$. Thus, if $(w_1^*, w_2^*) = (w_0, w_0)$, then $\neg\left(\left[\frac{\underline{\pi}_2}{1-\underline{\pi}_2}, \frac{\bar{\pi}_2}{1-\bar{\pi}_2}\right] \subseteq \left[\frac{\underline{\pi}_1}{1-\underline{\pi}_1}, \frac{\bar{\pi}_1}{1-\bar{\pi}_1}\right]\right)$ implies $\neg\left(I_{\succ_1}(w_0, w_0) \supseteq I_{\succ_2}(w_0, w_0)\right)$. ■

6.9 Proof of Theorem 7

Given $x_E y$ and $\theta > 0$, suppose that the subject reports $\bar{z} > \bar{c}(x_E y; \succ)$. If $r \leq \underline{c}(x_E y; \succ)$ or $r \geq \bar{z}$ then the subject's payoffs are the same regardless of whether he reports $\bar{z}$ or $\bar{c}(x_E y; \succ)$. If $r \in (\bar{c}(x_E y; \succ), \bar{z})$, the subject's payoff is a choice between the bet $(x-\theta)_E(y-\theta)$ and the outcome $r-\theta$; had he reported $\bar{c}(x_E y;\succ)$ instead of $\bar{z}$ his payoff would have been r. But $r>r-\theta$ implies that $\delta_r \succ \delta_{r-\theta}$ and, since $r>\bar{c}(x_E y;\succ)$, implies $\delta_r \succ x_E y \succ (x-\theta)_E(y-\theta)$, the subject is worse off reporting $\bar{z}$ instead of $\bar{c}(x_E y;\succ)$.

Suppose that the subject reports $\underline{z}<\underline{c}(x_E y;\succ)$. If $r\le\underline{z}$ or $r\ge\underline{c}(x_E y;\succ)$ the subject's payoffs are the same regardless of whether he reports $\underline{z}$ or $\underline{c}(x_E y;\succ)$. If $r\in(\underline{z},\underline{c}(x_E y;\succ))$, the subject's payoff is a choice between $(x-\theta)_E(y-\theta)$ and the outcome $r-\theta$; had he reported $\underline{c}(x_E y;\succ)$ instead of $\underline{z}$ his payoff would have been $x_E y$. By stochastic dominance, $x_E y\succ(x-\theta)_E(y-\theta)$, and $r<\underline{c}(x_E y;\succ)$ implies that $x_E y\succ\delta_r\succ\delta_{r-\theta}$. Thus the subject is worse off reporting $\underline{z}$ instead of $\underline{c}(x_E y;\succ)$.

Suppose that the subject reports $\bar{z}\in(\underline{c}(x_E y;\succ),\bar{c}(x_E y;\succ))$. If $r\in[\bar{z},\bar{c}(x_E y;\succ)]$, the subject's payoff is r, whereas had he reported $\bar{c}(x_E y;\succ)$ he would have the opportunity to choose between the bet $(x-\theta)_E(y-\theta)$ and the outcome $r-\theta$. If the signal, c, indicates that $(x-\theta)_E(y-\theta)\prec\delta_c$, where $c\le r-\theta$, the subject would choose the outcome $r-\theta$ and if the signal indicates that $(x-\theta)_E(y-\theta)\succ\delta_c$, $c\ge r-\theta$, indicating that the value of the bet $(x-\theta)_E(y-\theta)$ exceeds $r-\theta$, the subject would choose the bet. Thus, in the limit as $\theta\to 0$, the subject's subjective expected utility is:

$$\eta(r)u(r)+\int_{r}^{\bar{c}(x_E y;\succ)} u(c)\,d\eta(c)>u(r).$$

Thus, reporting $\bar{z}<\bar{c}(x_E y;\succ)$ is dominated by reporting truthfully, $\bar{z}=\bar{c}(x_E y;\succ)$. By similar argument, $\underline{z}\not\succ\underline{c}(x_E y;\succ)$. Hence, the dominant strategy is to report truthfully, that is, $\underline{z}=\underline{c}(x_E y;\succ)$. ■

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