Probabilistically Sophisticated Choice: An Alternative Axiomatization

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Abstract

This paper provides an alterantive axiomatization of the probabilistically sophisticated choice model of Machina and Schmeidler (1995).

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1 Introduction

The idea of probability first emerged in the second half of the 17th century.[1] Right from its inception the idea of probability took two distinct forms: *Objective probability* describing numerically the relative frequency of different outcomes in repeated trials conducted under identical conditions, and *subjective probability* quantifying the 'degree of belief' a decision maker holds regarding the likely realization of events that may not be replicated, or the truth of propositions.

That subjective probabilities may be inferred from the odds a decision maker is willing to offer when betting on the realization of events or the truth of propositions is an idea that was first proposed by Borel (1924), Ramsey (1931) and de Finetti (1937) and found its ultimate expression in the seminal works of Savage (1954) and Anscombe and Aumann (1963). All of these works presume that decision makers choice behavior abides by expected utility theory. Casting the idea in these terms confounds the independent notions of representing of a decision maker's beliefs by subjective probabilities and her choice behavior by expected utility.

Machina and Schmeidler (1992, 1995) proposed a model, dubbed probabilistic sophistication, in which choice-based subjective probabilities are defined without requiring that the decision maker's preferences respect the strictures of expected utility theory. According to Machina and Schmeidler subjective probabilities transform acts (that is, random variables on state space that take their values in the set of consequences) into lotteries (that is, the corresponding probability distributions on the set of consequences) and preferences are represented by a utility function over the set of lotteries.

The main purpose of this paper is to propose an alternative axiomatization of the Machina and Schmeidler (1995) model. It provides a new insight into the preference structure underlying probabilistically sophisticated choice behavior and a new understanding of the nature of the uniqueness of the subjective probabilities.

[1] See Hacking (1984) for an historical review.

2 The Model

2.1 Analytical framework

Let $S = \{s_1, ..., s_n\}$ be a finite set of *states.* Subsets of S are *events.* Denote by $X = \{x_1, ..., x_m\}$ a set of *outcomes* and, without essential loss of generality, assume that $m \geq n$.[2] Let ΔX denote the set of simple probability distributions on X. Elements of ΔX are *lotteries*. Mappings on S to ΔX are referred to as *acts*.[3] Acts represent courses of action. Let H denote the set of all acts and identify constant acts with elements of ΔX, thus, $\Delta X \subset H$. Let $\succcurlyeq$ be a binary relation on H referred to as *preference relation.* A preference relation $\succcurlyeq$ is bounded on X if there are $\bar{x}$ and $\underline{x}$ in X such that $\delta_{\bar{x}} \succ \delta_x \succ \delta_{\underline{x}}$, for all $x \in X \backslash \{\bar{x}, \underline{x}\}$, where $\delta_x \in \Delta X$ is the degenerate lottery that assigns the unit probability mass to the outcome x. The *strict preference relation*, $\succ$, and the *indifference relation,* $\sim$, are the asymmetric and the symmetric parts of $\succcurlyeq$, respectively.

2.2 The axiomatic structure

The following axioms depict the preference structure. The first three axioms are part of the Machina and Schmeidler (1995) model.

(A.1) (**Weak order**) The preference relation $\succcurlyeq$ is complete and transitive.

(A.2) (**Mixture Continuity**) For all $f, g, h \in H$, if $f \succ g$ and $g \succ h$ then there exist $\beta \in (0, 1)$ such that $\beta f + (1 - \beta) h \sim g$.

The statement of the next two axioms invokes the following additional notations and definitions. For every event E and $f, g \in H$, let $f_E g \in H$ be the a act that coincides with f on E and with g on $S \backslash E$. An event E is *null* if $\neg(\delta^{\bar{x}}_E f \succ \delta^{\underline{x}}_E f)$ for all $f \in H$, and is *nonnull* otherwise. Following Machina and Schmeidler (1995) the lottery p is said to *dominates the lottery q according to first-order stochastic dominance*, denoted $p >^1 q$, if $\Sigma_{\{x_s \succ x\}} p(x_s) \geq \Sigma_{\{x_s \succ x\}} q(x_s)$ for all $x \in X$, with strict inequality for some $x \in X$. The

[2] If $m < n$ then what follows can be applied to arbitrary partitions of S into $k \leq m$ events. The subjective probability on the state space can be constructed from the probabilites of the events in these partitions in a unique way.

[3] These acts are define by Anscombe and Aumann (1963) and are sometimes referred to as AA-acts.

next axiom requires that first-order stochastically dominating lotteries be preferred.[4]

(A.3) (**First-Order Stochastic Dominance Preference**) For all $p, q \in \Delta X$, if $p >^1 q$ then $p_E h \succ q_E h$, for all nonnull $E \subset S$ and all $h \in H$.

The next axiom replaces the Horse/Roulette Replacement axiom of Machina and Schmeidler (1995).[5] To state the axiom I introduce the following additional notations: Denote by Δ^n the simplex in $\mathbb{R}^n$ and for all $h \in H$ and $\alpha \in \Delta^n$ define $h^\alpha = \Sigma_{i=1}^n \alpha_i h(s_i)$.

(A.4) (**Reduction Equivalence**) For any $f, g \in H$ such that $f \sim g$, if $f \sim f^\alpha$ and $g \sim g^\alpha$ for some $\alpha \in \Delta^n$ then $h \sim h^\alpha$, for all $h \in H$.

Note that this axiom asserts neither the existence nor the uniqueness of α that satisfies its exigencies. Indeed, the proof of the following proposition, asserting that there are $f, g \in H$ and $\alpha \in \Delta^n$ that satisfy the conditions in the axiom, is the main theoretical challenge of this work.

Proposition: *Let $\succcurlyeq$ be a binary relation on H satisfying $\succcurlyeq$ boundedness on X and (A.1) - (A.3) then there are $f, g \in H$ and a unique $\alpha^* \in \Delta^n$ such that $f \sim g$, $f \sim f^{\alpha^*}$ and $g \sim g^{\alpha^*}$.*

3 The Main Result

To state the main result I invoke the following definitions: A function V is *mixture continuous* if $V(\alpha p + (1-\alpha) q)$ is continuous in α, for all $p, q \in \Delta X$. It is *strictly monotonic* if $V(p) \geq V(q)$ whenever p dominates q according to first-order stochastic dominance, with strict inequality in the case of strict dominance.

Theorem. *Let $\succcurlyeq$ be a binary relation on H then the following two conditions are equivalent:*

(*i*) *$\succcurlyeq$ is bounded on X and satisfies (A.1) - (A.4).*

(*ii*) *There exist a real-valued, mixture continuous, strictly monotonic function, V on ΔX, and a probability measure π on S such that, for all $f, g \in H$,*

$$f \succcurlyeq g \Leftrightarrow V\left(\Sigma_{s \in S} \pi(s) f(s)\right) \geq V\left(\Sigma_{s \in S} \pi(s) g(s)\right). \tag{1}$$

[4] This is Axiom 5 in Machina and Schmeidelr (1995). Grant (1995) characterized probabilistically sophisticated preferences without monotonicity with respect to first-order stochastic dominance.

[5] This is Axiom 6 in Machina and Schmeidler (1995).

and, for all $f \in H$,

$$V\left(\delta^{\bar{x}}\right) > V\left(\Sigma_{s\in S}\pi\left(s\right)f\left(s\right)\right) > V\left(\delta^{\underline{x}}\right). \tag{2}$$

Moreover, V *is unique up to monotonic increasing transformation and* π *is unique.*

4 Concluding Remarks

The replacement of the Horse/Roulette Replacement axiom of Machina and Schmeidler (1995) by the simpler axiom of Reduction Equivalence has the advantage of simplifying the proof of the representation theorem and, in particular, the proof of the uniqueness of the subjective probabilities.

An important feature of both the probabilistic sophistication model and subjective expected utility theory is state-independent preferences. This feature has two aspects: ordinal state-independence and cardinal state-independence.[6] In the probabilistic sophistication model ordinal state-independence is implied by Monotonicity (A.3) which presumes that the preference ordering of the outcomes in the definition of first-order stochastic is independent of the underlying state. Cardinal state-independence is implied by the Reduction Equivalence axiom (A.4).[7] To grasp this claim suffices it to note that (A.4) implies that if two events are assigned the same probability, $\alpha_i = \alpha_j$ then two acts that agree on the event $S\backslash\{s_i, s_j\}$, that is $f = p_{\{s_i\}}q_{\{s_j\}}h$ and $g = q_{\{s_i\}}p_{\{s_j\}}h$, $p, q \in \Delta X$, induce, by reduction, the same lotteries (that is, $f^{\alpha} = g^{\alpha}$). Hence, by (A.4), f is indifferent g, regardless of the risks represented by lotteries p and q. In other words, the risk associated with the lotteries p and q is treated in the same way independently of the states in which they obtain.

5 Proofs

5.1 Proof of the Proposition

Without loss of generality rearrange the outcomes in an ascending order of preference, that is, let $\delta_{x_n} \succ \delta_{x_{n-1}} \succ ... \succ \delta_{x_1}$. Consider the act $f = \left(\delta_{x_n} \text{ on } s_n, \delta_{x_{n-1}} \text{ on } s_{n-1}, ..., \delta_{x_1} \text{ on } s_1\right)$.

[6] In Savage's subjective expected utility theory these aspects are captured, respectively, by postulates P3 and P4.

[7] In Machina and Schmeidler (1995) cadinal state-independence is implicit in the Horse/Roulette Replacement axiom.

Then, by boundedness, $\delta_{x_n} \succcurlyeq f \succcurlyeq \delta_{x_1}$. By (A.2) there is $\gamma \in [0,1]$ such that $\gamma\delta_{x_n} + (1-\gamma)\,\delta_{x_1} \sim f$. Let $I\left(\gamma\delta_{x_n} + (1-\gamma)\,\delta_{x_1}\right) = \{p \in \Delta X \mid p \sim \gamma\delta_{x_n} + (1-\gamma)\,\delta_{x_1}\}$, then $f^{\alpha} \in I\left(\gamma\delta_{x_n} + (1-\gamma)\,\delta_{x_1}\right) \cap int\,(\Delta X)\,,$ for some $\alpha \in \Delta^n$.

Next we construct $g \in H$ such that $g \sim f$. Suppose, without loss of generality, that s_n is a nonnull state. Let $g_1 = p^1_{\{s_1\}} p^{n,1}_{\{s_n\}} f$, where $p^1 = (1-\varepsilon_1, \varepsilon_1, 0, ..., 0)\,,$ $p^{n,1} = (\varepsilon_{n_1}, 0, ..., 0, 1-\varepsilon_{n_1})\,,$ for small positive ε_1, such that $g_1 \sim f$. That this kind of "compensating variation" is possible follows from (A.3) and (A.2). Let $g_2 = p^1_{\{s_1\}} p^2_{\{s_2\}} p^{n,2}_{\{s_n\}} f$, where $p^2 = (0, 1-\varepsilon_2, \varepsilon_2, 0, ..., 0)$ and $p^{n,2} = (\varepsilon_{n_1} + \varepsilon_{n_2}, 0, ..., 0, 1-\varepsilon_{n_2} - \varepsilon_{n_1})\,,$ such that $g_2 \sim g_1$. Continuing in this manner we get $g_n = p^1_{\{s_1\}} p^2_{\{s_2\}} ... p^{n-1}_{\{s_{n-1}\}} p^n = g$, where $p^n = (\Sigma^n_{i=1}\varepsilon_{n_i}, 0, 0, 0..., 1-\Sigma^n_{i=1}\varepsilon_{n_i})\,.$ By transitivity, $g \sim f$. Let $\varepsilon_n = \Sigma^n_{i=1}\varepsilon_{n_i}$, then g is depicted in the matrix below.

$$G = \begin{pmatrix} & s_1 & s_2 & s_3 & s_4 & ... & s_n \\ x_1 & 1-\varepsilon_1 & 0 & 0 & 0 & ... & \varepsilon_n \\ x_2 & \varepsilon_1 & 1-\varepsilon_2 & 0 & 0 & ... & 0 \\ x_3 & 0 & \varepsilon_2 & 1-\varepsilon_3 & 0 & ... & 0 \\ x_4 & 0 & 0 & \varepsilon_3 & 1-\varepsilon_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_4 & 1-\varepsilon_5 & . \\ . & . & . & . & 0 & \varepsilon_5 & . \\ . & . & . & . & . & 0 & . \\ x_n & 0 & 0 & 0 & 0 & 0 & 1-\varepsilon_n \end{pmatrix}$$

Next consider the equations:

$$\begin{pmatrix} -\varepsilon_1 & 0 & 0 & 0 & ... & \varepsilon_n \\ \varepsilon_1 & -\varepsilon_2 & 0 & 0 & ... & 0 \\ 0 & \varepsilon_2 & -\varepsilon_3 & 0 & ... & 0 \\ 0 & 0 & \varepsilon_3 & -\varepsilon_4 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_4 & -\varepsilon_5 & . \\ . & . & . & 0 & \varepsilon_5 & . \\ . & . & . & . & 0 & . \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ . \\ . \\ . \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{pmatrix}$$

In matrix notation we write $M\alpha^t = e^n$, where $e^n = (0, 0..., 0, 1)\,.$ Now, M has full rank, hence, the system of equations has a unique solution. Denote the solution by $\hat{\alpha}$.

Denote by I the identity matrix. Then $G\hat{\alpha}^t = g^{\hat{\alpha}}$ and $I\hat{\alpha}^t = f^{\hat{\alpha}}$ and $g^{\hat{\alpha}} = f^{\hat{\alpha}}$. Thus, by transitivity, $g \sim g^{\hat{\alpha}}$ and $f \sim f^{\hat{\alpha}}$. ■

5.2 Proof of the Theorem

Necessity is immediate. Below I prove sufficiency. For each $f \in H$ let $\alpha_f \in [0,1]$ such that $f \sim \alpha_f \delta_{\bar{x}} + (1-\alpha_f)\,\delta_{\underline{x}}$. By (A.2) and (A.3) such α_f exists and is unique. Define $V : H \to \mathbb{R}$ by $V(f) = \alpha_f$. Then V is well-defined. By (A.3), for all $f, g \in H$, $f \succcurlyeq g$ if and only if $V(f) \geq V(g)$. By (A.2) V is mixture continuous and by (A.3) it is strictly monotonic.

By the proposition there is a unique $\hat{\alpha} = (\hat{\alpha}_1, ..., \hat{\alpha}_n)$ such that, for all $f \in H$, $f \sim f^{\hat{\alpha}}$. Thus, $V(f) = V\left(f^{\hat{\alpha}}\right) = V\left(\Sigma_{i=1}^n \hat{\alpha}_i f(s_i)\right)$. Let $\pi(s_i) = \hat{\alpha}_i$, $i = 1, ..., n$, then, for all $f, g \in H$,

$$f \succcurlyeq g \Leftrightarrow V\left(\Sigma_{i=1}^n \pi(s_i) f(s_i)\right) \geq V\left(\Sigma_{i=1}^n \pi(s_i) g(s_i)\right).$$

Moreover, since $\succcurlyeq$ is bounded we have $V\left(\delta^{\bar{x}}\right) > V\left(\Sigma_{s_i \in S} \pi(s_i) f(s_i)\right) > V\left(\delta^{\underline{x}}\right)$, for all $f \in H$.

The uniquness π follows form that of $\hat{\alpha}$ and the uniqueness of V is obvious. ■

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