Lecture: Dynamic Discrete Choice

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1 Overview

- We considered one time choices between a finite number of alternatives.
- We will consider repeated choices between a finite number of alternatives in a sequence of time periods.
- We distinguish between static problems where choices are essentially unrelated and dynamic choices where there is interdependence between choices.
- We derive the optimal sequence of choices in the latter case.
- We discuss econometric models for optimal dynamic discrete choice.

2 Static and dynamic choice problems

2.1 Setup

- Consider an agent who chooses for periods $t = 1, \ldots, T$ between 2 alternatives, 1 or 2.
- The utilities are $u_{jt}, t = 1, \ldots, T$ with $j = 1, 2$ (for now we omit the subscript for the individual i).
- Define

$$\begin{aligned} y_t &= 1 \quad \text{if 2 is chosen in period } t \\ &= 0 \quad \text{if 1 is chosen in period t} \end{aligned}$$

- We choose an ARUM for $u_{jt}, t = 1, \dots, T$

$$u_{jt} = v_{jt} + \varepsilon_{jt}$$

with ε_{jt} random variables with $\mathrm{E}(\varepsilon_{jt}) = 0$.

- Obviously $y_t, t = 1, \dots, T$ is determined by $u_{jt}, t = 1, \dots, T, j = 1, 2$, but how?

2.2 Independent and interdependent choices

Independent choices Choices are independent if the choice in period t does not affect the utility of the choice in period $s > t$, nor restricts the choice in period $s > t$.

Example: Choice of mode of transportation.

- On any day $t = 1, \dots, T$ one can choose to commute by car or public transportation.
- Utility of the two alternatives depends on variables that are common knowledge, e.g. cost of commute by car or public transit, travel time, etc. This is summarized by v_{jt}.
- It also depends on variables that are private knowledge of the agent, e.g. variation in taste over time, assessment of weather on day t etc. This is summarized by ε_{jt}.
- Assume that choice on day t has no effect on the utilities on nor restricts the choice on days $s = t + 1, \dots, T$.
- Then

$$\Pr(y_t = 1) = Pr(u_{2t} > u_{1t}) = \Pr(\varepsilon_{1t} - \varepsilon_{2t} < v_{2t} - v_{1t})$$

i.e. the agent maximizes period t utility.

- Note that we assume that current choice does not affect future utility nor restricts future choice, but utilities can be correlated over time, i.e. ε_{jt} may be correlated. For $T = 2$ we have, e.g.

$$\Pr(y_1 = 1, y_2 = 1) = \Pr(u_{21} > u_{11}, u_{22} > u_{12}) =$$

$$\Pr(\varepsilon_{11} - \varepsilon_{21} < v_{21} - v_{11}, \varepsilon_{12} - \varepsilon_{22} < v_{22} - v_{12})$$

This is an orthant probability for the joint distribution of $\eta_t, t = 1, \ldots, T$ with $\eta_t = \varepsilon_{1t} - \varepsilon_{2t}$, e.g. a multivariate normal distribution.

- Hence correlation between $y_t, t = 1, \ldots, T$ does not imply that choices are interdependent.

Interdependent choices. Choices are interdependent if the choice in period t affects the utility of the choice in period $s > t$, or restricts the choice in period $s > t$.

Examples

- Learning or habit formation: Choosing an alternative changes the future utility of that alternative.
- Irreversibility or switching costs: If alternative 1 is chosen in period t it may be impossible or costly to to choose alternative 2 in period $t+1$. For instance
  - It is costly to switch cars soon after you have bought one due to steep initial depreciation in the second-hand market.
  - It is costly for a firm to reverse an investment decision. The choice can be made only once in $t = 1, ..., T$, e.g. job search by unemployed (if jobs are permanent).

Interdependence and choice

- If choices are interdependent, then a rational agent takes this into account in making his/her choice, i.e. the agent will be *forward looking*.
- This means that utilities and/or restrictions in future periods must be predicted.
- Because the future is not perfectly predictable, the forward-looking agent makes a choice under uncertainty.
- In the ARUM model,
$$u_{jt} = v_{jt} + \varepsilon_{jt}$$

uncertainty can be introduced by assuming that at t the agent has private knowledge of ε_{jt}, but not of $\varepsilon_{js}, s = t+1, \ldots, T$, i.e. the outcomes of these random variables are not known to the agent at time t when the choice is made.

3 Example of dynamic discrete choice: Optimal cake eating

- You have a piece of cake that must be eaten on day $t = 1, \ldots, T$.
- On any day t you may eat the cake, giving utility $u_t = v_t + \varepsilon_t$ (with ε_t private knowledge to you), or you may delay to day $t+1, \ldots, T$ which gives utility 0 at day t (but a positive utility at some later day).
- Interpretation of ε_t-s: Utility shocks or mood swings.
- Because you cannot have your cake and eat it, eating is irreversible: if you eat at day t you cannot eat on days $t+1, \ldots, T$.
- What is optimal choice, i.e. on which day the cake should be eaten?
- Because future utilities are uncertain, we can not maximize utility, but we can maximize expected utility.

3.1 Maximizing utility and expected utility

Maximizing utility

- Assume that $\varepsilon_1, \ldots, \varepsilon_T$ are known to you at time of decision.
- Utility maximizing choice is

$$t = \text{argmax}_{t=1,\ldots,T} u_t = \text{argmax}_{t=1,\ldots,T} v_t + \varepsilon_t$$

- This is the usual static discrete choice problem, and if the ε_t are e.g. i.i.d. with an extreme value distribution, then the choice probabilities are MNL.
- We can find t by sorting the u_t-s or by a recursive method called *dynamic programming*.

3.1.1 Dynamic programming: known utilities.

- Define V_t = maximal utility if cake is consumed on one of days $t, \ldots, T$, i.e. if cake is still uneaten at start of day t.
- Of course,
$$V_t = \max_{s=t,\ldots,T} u_s$$
- We compute the $V_t, t = 1, \ldots, T$ recursively, starting at day T.
- If cake is not eaten by day T there is only one choice: eat on that day. Hence, $V_T = u_T$.
- If cake is not eaten by day $T-1$, there are 2 options on day $T-1$
  - Eat with utility u_{T-1}.
  - Delay to day T with maximal utility V_T
- Hence $V_{T-1} = \max\{u_{T-1}, V_T\}$.
- Also the optimal decision at day $T-1$ is to eat if $u_{T-1} > V_T$ and wait if $u_{T-1} < V_T$.
- By same reasoning at any day t, if the cake is not eaten by that day, we can
  - Eat with utility t
  - Delay with (maximal) utility V_{t+1}
- Hence $V_t = \max\{u_t, V_{t+1}\}$ and we eat if $u_t > V_{t+1}$ and wait if $u_t < V_{t+1}$.
- General method is to compute the V_t-s with the recursive equation (note we go backwards in time beginning on the final day T)
$$V_t = \max\{u_t, V_{t+1}\} \quad , \; V_T = u_T \tag{1}$$
and to eat on the first day that $u_t > V_{t+1}$ (note that now we go forwards in time, beginning on day 1).
- The recursive equation for the maximal utilities V_t in 1 is called the Bellman equation. The maximal utility V_t is called the value function at time t. This recursive method to maximize utility is called dynamic programming.

Numerical example

- $u_1 = 2, u_2 = 3, u_3 = 1$
- Then

$$\begin{aligned} V_3 &= 1 \\ V_2 &= \max\{u_2, V_3\} = 3 \\ V_1 &= \max\{u_1, V_2\} = 3 \end{aligned}$$

- Optimal choice

$$u_1 = 2 < 3 = V_2 \quad \text{delay}$$

$$u_2 = 3 > 1 = V_3 \quad \text{eat}$$

3.1.2 Dynamic programming: Uncertain utilities

- At day t $\varepsilon_1, \ldots, \varepsilon_t$ are known, but $\varepsilon_{t+1}, \ldots, \varepsilon_T$ are not known. Note ε_t is known at time of decision on day t.
- $\varepsilon_1, \ldots, \varepsilon_T$ are independent.
- Define $\mathrm{E}_s(V_t)$ = *expected* maximal utility if cake is consumed on one of days $t, \ldots, T$, i.e. if cake is still uneaten at start of day t, where expectation is taken with information available at day s, i.e. $\varepsilon_1, \ldots, \varepsilon_s$, i.e. over the joint distribution of $\varepsilon_{s+1}, \ldots, \varepsilon_T$ given $\varepsilon_1, \ldots, \varepsilon_s$. Given independence assumption expectation is over $\varepsilon_{s+1}, \ldots, \varepsilon_T$.
- If cake is still uneaten at day T, it is optimal to eat on that day, i.e. $\mathrm{E}_T(V_T) = u_T = v_T + \varepsilon_T$.
- If delayed until day $T-1$, then options
  - Eat with utility $u_{T-1} = v_{T-1} + \varepsilon_{T-1}$
  - Delay with expected maximal utility $\mathrm{E}_{T-1}(V_T) = v_T$.
  - Hence $\mathrm{E}_{T-1}(V_{T-1}) = \max\{v_{T-1} + \varepsilon_{T-1}, E_{T-1}(V_T)\}$
- Hence

$$\mathrm{E}_{T-2}(V_{T-1}) = \mathrm{E}[\max\{v_{T-1} + \varepsilon_{T-1}, \mathrm{E}_{T-1}(V_T)\}]$$

  with the expectation taken over ε_{T-1}.

- Also we eat on day $T-1$ iff

$$u_{T-1} > \mathrm{E}_{T-1}(V_T) \Leftrightarrow \varepsilon_{T-1} > \mathrm{E}_{T-1}(V_T) - v_{T-1}$$

- In general, if cake uneaten on day t, then options
  - Eat with utility $u_t = v_t + \varepsilon_t$.
  - Delay with expected maximal utility $\mathrm{E}_t(V_{t+1})$
- Hence $\mathrm{E}_t(V_t) = \max\{v_t + \varepsilon_t, \mathrm{E}_t(V_{t+1})\}$ and

$$\mathrm{E}_{t-1}(V_t) = \mathrm{E}_{t-1}[\max\{v_t + \varepsilon_t, \mathrm{E}_t(V_{t+1})\}]$$

- Also we eat on day t iff

$$\varepsilon_t > E_t(V_{t+1}) - v_t$$

- Recursive equation for value function

$$\mathrm{E}_{t-1}(V_t) = \mathrm{E}_{t-1}[\max\{v_t + \varepsilon_t, \mathrm{E}_t(V_{t+1})\}] =$$

$$= \int_{-\infty}^{\mathrm{E}_t(V_{t+1})-v_t} \mathrm{E}_t(V_{t+1}) f(\varepsilon_t) \mathrm{d}\varepsilon_t + \int_{\mathrm{E}_t(V_{t+1})-v_t}^{\infty} (v_t + \varepsilon_t) f(\varepsilon_t) \mathrm{d}\varepsilon_t =$$

$$= \mathrm{E}_t(V_{t+1}) F(\mathrm{E}_t(V_{t+1}) - v_t) + v_t(1 - F(\mathrm{E}_t(V_{t+1}) - v_t)) + \int_{\mathrm{E}_t(V_{t+1})-v_t}^{\infty} \varepsilon_t f(\varepsilon_t) \mathrm{d}\varepsilon_t$$

  Hence

$$\mathrm{E}_{t-1}(V_t) = \mathrm{E}_t(V_{t+1}) F(\mathrm{E}_t(V_{t+1}) - v_t) + \mathrm{E}(v_t + \varepsilon_t | \varepsilon_t > \mathrm{E}_t(V_{t+1}) - v_t)(1 - F(\mathrm{E}_t(V_{t+1} - v_t))) \tag{2}$$

- We find the choice that maximizes the expected utility in two steps
  - Solve (2) recursively from beginning at T (backwards).
  - Use this in decision rule: eat on day t iff

$$\varepsilon_s \leq \mathrm{E}_s(V_{s+1}) - v_s \quad s = 1, \ldots, t-1$$

$$\varepsilon_t > \mathrm{E}_t(V_{t+1}) - v_t$$

- This strategy solves

$$\max_{d_1,\dots,d_T} \mathrm{E}\left[\sum_{t=1}^{T} d_t u_t\right] \quad \text{s.t.} \quad \sum_{t=1}^{T} d_t = 1$$

3.2 Estimation

- At time t, $\varepsilon_1, \dots, \varepsilon_t$ are known to the agent, but not to the econometrician.
- Hence, the econometrician can only compute the probability the person eats the cake on day t

$$P(\text{cake eaten on day } t) =$$

$$= \Pr(\varepsilon_1 \leq \mathrm{E}_1(V_2) - v_1, \dots, \varepsilon_{t-1} \leq \mathrm{E}_{t-1}(V_t) - v_{t-1}, \varepsilon_t > \mathrm{E}_t(V_{t+1}) - v_t) =$$

$$= F(\mathrm{E}_1(V_2) - v_1) \cdots F(\mathrm{E}_{t-1}(V_t) - v_{t-1})(1 - F(\mathrm{E}_t(V_{t+1}) - v_t)) \equiv p_t(v)$$

- If $d_{it} = 1$ if i eats the cake on day t and we have a random sample of cake eaters $d_{i1}, \dots, d_{iT}, i = 1, \dots, n$, the likelihood function is

$$L(v) = \prod_{i=1}^{n} \prod_{t=1}^{T} p_t(v)^{d_{it}}$$

which can be used to estimate the average utilities $v_1, \dots, v_T$.

- Note that to compute the likelihood we must use the recursion to compute $\mathrm{E}_t(V_{t+1})$.

3.3 Complications

- Until now we assume that $\varepsilon_1, \dots, \varepsilon_T$ are independent. What happens if we allow for arbitrary correlation?
- Note

$$\mathrm{E}_{T-1}(V_T) = \mathrm{E}(v_T + \varepsilon_T | \varepsilon_1, \dots, \varepsilon_{T-1}) = v_T + \mathrm{E}(\varepsilon_T | \varepsilon_1, \dots, \varepsilon_{T-1})$$

Further for day t

$$\mathrm{E}_{t-1}(V_t) = \mathrm{E}[\max\{v_t + \varepsilon_t, \mathrm{E}(V_{t+1} | \varepsilon_1, \dots, \varepsilon_t)\} | \varepsilon_1, \dots, \varepsilon_{t-1}]$$

with the expectation over the distribution of ε_t given $\varepsilon_1, \ldots, \varepsilon_{t-1}$.

- This needs to be solved for all values of $\varepsilon_1, \ldots, \varepsilon_{t-1}$, i.e. we need the function
$$\mathrm{E}_{t-1}(V_t) = \mathrm{E}(V_t|\varepsilon_1, \ldots, \varepsilon_{t-1})$$
At best we can do the recursion on a finite set or grid of values, but has to be dense because we need the integral of the function.

- To keep things computationally feasible we need to limit the dependence, e.g. by assuming an AR(1) model
$$\varepsilon_t = \rho\varepsilon_{t-1} + \eta_t$$
with η_t i.i.d. In that case we have
$$\mathrm{E}_{t-1}(V_t) = \mathrm{E}(V_t|\varepsilon_{t-1})$$

4 Dynamic Discrete Choice

4.1 Notation and setup

- Choice between alternatives $i = 1, \ldots, I_t$ in periods $t = 1, \ldots, T$ (I omit subscript for agent).

- $d_i(t) = 1$ if i is chosen, and is 0 otherwise.

- Only one alternative can be chosen, i.e. $\sum_{i=1}^{I} d_i(t) = 1$. The vector of choice indicators is $d(t) = (d_1(t) \ldots d_{I_t}(t))'$.

- We denote
$$R_i(t) = \text{utility of alternative } i \text{ in period } t$$
$$\Omega(t) = \text{information available before decision in period } t \text{ is made}$$
It usually is a vector of variables, which are called the state variables of the problem.

- Agent maximizes expected discounted utility
$$\mathrm{E}\left[\sum_{s=t}^{T} \beta^{s-t} \sum_{i=1}^{I_s} R_i(s) d_i(s) |\Omega(t)\right] \quad \text{s.t.} \sum_{i=1}^{I_t} d_i(t) = 1 \quad , t = 1, \ldots, T$$

with $0 < \beta < 1$ the discount factor.

- A game-theory perspective.

  It is standard practice to formally model an economic agent as a sequence of temporal selves making choices in a dynamic game. Hence, a T-period utility maximization problem translates into a T-period game, with T players, or "selves," indexed by their respective periods of control over the choice decision. It forms a dynamic Stackelberg (or leader-follower) game. We may then look for subgame perfect equilibrium (SPE) strategies of this game.

4.2 Solution

- $V(\Omega(t))$ = maximal expected utility in period t.
- The Bellman equation is

$$V(\Omega(t)) = \max_{i=1,\ldots,I_t} \{R_i(t) + \beta \mathrm{E}[V(\Omega(t+1))|d_i(t) = 1, \Omega(t)]\}$$

- In period t $R_i(t)$ is known. The reward may depend on $\Omega(t)$, so that the future reward may be random (as in the cake eating example). The choice in period t may affect the uncertainty in period $t+1$, and this is expressed in the conditioning.
- The Bellman equation must be solved recursively for $V(\Omega(t))$ starting from

$$V(\Omega(T)) = \max_{i=1,\ldots,I_t} R_i(T)$$

- Special cases

  We distinguish between

  (i) Specification of the utility function $R_i(t), i = 1, \ldots, I_t$.

  (ii) Specification of choice set I_t.

  (iii) Distribution of rewards $R_i(t), i = 1, \ldots, I_t$ and in particular correlation between $R_i(t), R_j(s)$ for $s > t$.

  (iv) Choice of state variables in $\Omega(t)$, i.e. the history that has to be considered.

- A choice can be a change of state, i.e. if i is chosen the agent enters state i, e.g. in the cake eating example you move from state "cake" to state "no cake".
- State i may be absorbing, i.e. agent cannot leave this state, or not. In example state "no cake" is absorbing. Absorbing states simplify the analysis.

4.3 DDC with an absorbing state: Job search

Job search model

- Consider unemployed person looking for a job.
- In each period there is a job offer.
- Income while unemployed is b per period.
- Job offer $w(t)$ is draw from wage offer distribution with pdf f and cdf F; subsequent draws are independent.
- Upon receipt job offer must be accepted or rejected.
- If accepted job is held until T.
- If rejected job offer is lost (no recall).
- The unemployed maximizes expected wealth.
- Two alternatives with

$$\begin{aligned} d_1(t) &= 1 \text{ if search in } t \\ &= 0 \text{ if not} \\ d_2(t) &= 1 \text{ if job in } t \\ &= 0 \text{ if not} \end{aligned}$$

- Utility function

$$\begin{aligned} R_1(t) &= b \\ R_2(t) &= d_1(t-1)w(t) + d_2(t-1)R_2(t-1) \end{aligned}$$

so that if $d_1(t-1) = 1, d_2(t) = 1$, i.e. job accepted in period t, $R_2(s) = w(t),\ t \leq s \leq T$.

- Because job is held until T

$$\begin{aligned} I_t &= \{1,2\} \quad \text{if } d_1(t-1) = 1 \\ &= \{2\} \quad \text{if } d_2(t-1) = 1 \end{aligned}$$

- The state variables are $\Omega(t) = \{d_1(t-1), \sum_{s=1}^{t-1} d_1(s-1)d_2(s)w(s)\}$ where $d_1(0) = 1$. In sequel I denote the accepted wage by $w_a(t-1) = \sum_{s=1}^{t-1} d_1(s-1)d_2(s)w(s)$.

- Because offers are independent, only the accepted offer has to be included. When do all past offers have to be included?

- Let $V_i(\Omega(t))$ be the expected utility of choice i in period t

$$V_i(\Omega(t)) = R_i(t) + \beta \mathrm{E}[V(\Omega(t+1))|d_i(t) = 1, \Omega(t)]$$

- Because state 2 (job) is absorbing

$$d_2(t) = 1 \Rightarrow d_2(t+1) = d_2(t+2) = \ldots = d_2(T) = 1$$

and $w(s) = w_a(t), s = t, \ldots, T$, we have

$$V_2(\Omega(t)) = \sum_{s=t}^{T} \beta^{s-t} w_a(t)$$

Without causing confusion we can denote this by $V_2(t)$.

- For an absorbing state the conditional valuation function is easy to compute and we only need a recursion for $V_1(\Omega(t))$ that is only defined if $d_1(t-1) = 1$ so that $w_a(t-1) = 0$. We denote

$$V_1(t) = V_1(d_1(t-1) = 1, w_a(t-1) = 0)$$

- In final period

$$V_i(\Omega(T)) = R_i(T)$$

- In job search model, if $d_1(T-1) = 1$ (and $w_a(T-1) = 0$), then $I_T = \{1, 2\}$, and

$$\begin{aligned} V_1(T) &= V_1(d_1(T-1) = 1, w_a(T-1) = 0) = b \\ V_2(T) &= w_T \end{aligned}$$

Hence if we denote $V(t)$ as the maximal utility if $d_1(t-1) = 1, w_a(t-1) = 0$, i.e. if no job was found until period t

$$d_2(T) = 1 \Leftrightarrow w(T) \geq b$$

$$V(T) = \max\{b, w(T)\}$$

- For period t (as above $d_1(t-1) = 1$)

$$\begin{aligned} V_1(t) &= b + \beta \mathrm{E}(V(t+1)|d_1(t) = 1) \\ V_2(t) &= \sum_{s=t}^{T} \beta^{s-t} w(t) \end{aligned}$$

- A job offer is accepted iff

$$w(t) \geq \frac{V_1(t)}{\sum_{s=t}^{T} \beta^{s-t}} = w_r(t)$$

i.e. if the offer exceeds the reservation wage $w_r(t)$.

- At the reservation wage the unemployed is indifferent between working and searching.

- We derive a recursive equation for $V_1(t)$ and hence for the reservation wage. Note

$$\mathrm{E}(V(t+1)|d_1(t) = 1) = \mathrm{E}\left[\max\left\{V_1(t+1), \sum_{s=t+1}^{T} \beta^{s-t-1} w(t+1)\right\}\right] =$$

$$= V_1(t+1)\Pr(w(t+1) < w_r(t+1)) +$$

$$+ \sum_{s=t+1}^{T} \beta^{s-t-1} \mathrm{E}(w(t+1)|w(t+1) \geq w_r(t+1)) \Pr(w(t+1) \geq w_r(t+1))$$

and this gives the recursion

$$V_1(t) = b + \beta V_1(t+1)F(w_r(t+1)) +$$
$$+\beta\left(\sum_{s=t+1}^{T}\beta^{s-t-1}\right)\mathrm{E}(w(t+1)|w(t+1) \geq w_r(t+1))(1 - F(w_r(t+1)))$$
$$V_1(T) = b$$

- This can be rewritten as a recursion in the reservation wage

$$w_r(t)\sum_{s=t}^{T}\beta^{s-t} = b + \beta\left(\sum_{s=t+1}^{T}\beta^{s-t-1}\right)w_r(t+1)F(w_r(t+1)) +$$
$$+\beta\left(\sum_{s=t+1}^{T}\beta^{s-t-1}\right)\mathrm{E}(w(t+1)|w(t+1) \geq w_r(t+1))(1 - F(w_r(t+1)))$$
$$w_r(T) = b$$

- The expressions simplify if we consider an infinite horizon problem, i.e. if $T \to \infty$. Under our assumptions and there is no change over time, i.e. the environment is stationary, so that

$$V_1(t) = V_1(t+1) \qquad V_2(t) = V_2(t+1)$$

because the choice is the same every period.

- Substitute $w_r(t) = w_r(t+1) = w_r$ in the recursion

$$\frac{w_r}{1-\beta} = b + \frac{\beta}{1-\beta}w_r F(w_r) + \frac{\beta}{1-\beta}\mathrm{E}(w|w \geq w_r)(1 - F(w_r))$$

and rewrite (subtract $\frac{\beta}{1-\beta}w_r$)

$$w_r = b + \frac{\beta}{1-\beta}\int_{w_r}^{\infty} f(w)\mathrm{d}w$$

Note $w_r > b$.

- This is an asset price valuation equation: reservation wage (per period return) is yield (b) plus expected appreciation (note $\frac{\beta}{1-\beta} = \rho$ with ρ the discount rate).

4.3.1 Estimation of search model

Data

(i) Unemployment duration: $d_1(t), t = 1, \ldots, T$

(ii) Unemployment duration plus accepted wage: $d_1(t), t = 1, \ldots, T$ and $w_a = \sum_{t=1}^{T} d_1(t-1)d_2(t)w(t)$

(iii) Unemployment duration and all offers $w(t), d_1(t), t = 1, \ldots, T$

Likelihoods for these cases

(i) Unemployment duration: $d_1(t), t = 1, \ldots, T$

$$L_1 = \prod_{i=1}^{n}\prod_{t=1}^{T}(1 - F(w_r(t)))^{d_{i1}(t-1)d_{i2}(t)} F(w_r(t))^{d_{i1}(t-1)d_{i1}(t)}$$

(ii) Unemployment duration plus accepted wage: $d_1(t), t = 1, \ldots, T$ and $w_a = \sum_{t=1}^{T} d_1(t-1)d_2(t)w(t)$

$$L_2 = L_1 . \prod_{i=1}^{n} \frac{f(w_{ai})}{1 - F(w_r(t))}$$

(iii) Unemployment duration and all offers $w(t), d_1(t), t = 1, \ldots, T$

$$L_3 = L_2 \cdot \prod_{i=1}^{n}\prod_{t=1}^{T}\left(\frac{f(w_i(t))}{F(w_r(t))}\right)^{d_{i1}(t-1)d_{i1}(t)}$$

- Parameters to be estimated: β, b and parameters of wage offer distribution.
- Computation:
  1. Choose starting values of parameters.
  2. For these parameters, compute $w_r(t), t = 1, \ldots, T$ using the recursion.
  3. Evaluate likelihood and update parameters. Go back to 2.
- Identification (stationary case). From accepted wages we can obtain the truncated wage offer density

$$g(w) = \frac{f(w)}{1 - F(w_r)} \qquad w \geq w_r$$

- Consider
$$\frac{f_1(w)}{1-F_1(w_r)} = g(w) \quad , \quad w \geq w_r$$
If for any $0 < p < 1$
$$f_2(w) = pg(w) \quad , \quad w \geq w_r$$
then if we choose $f_2(w), w < w_r$ such that
$$\int_0^{w_r} f_2(w)\mathrm{d}w = 1 - p$$
For instance
$$f_2(w) = \frac{1-p}{w_r} \quad , \quad 0 \leq w < w_r$$
Then f_2 is a proper density and $f_2(w|w \geq w_r) = g(w)$, i.e. f_1 and f_2 are observationally equivalent.

- Conclusion: The wage offer distribution is not nonparametrically identified from accepted wages.

4.4 DDC: General case

- We generalize MNL model to intertemporal choice.

- As before for $i = 1, \ldots, I$
$$\begin{aligned} d_i(t) &= 1 \quad \text{if alternative } i \text{ is chosen} \\ &= 0 \quad \text{if not} \end{aligned}$$
with $d(t) = (d_1(t), \ldots, d_I(t))'$.

- Choices can be reversed, i.e. no absorbing state.

- Utility function
$$R_i(t) = u_i(x(t), d_i(t-1)) + \varepsilon_i(t) \quad i = 1, \ldots, I_t$$
with $x_i(t)$ characteristics of agent and/or environment. Some of these depend on $d(t-1)$.

- Random variables are $\varepsilon_i(t)$ and $x(t)$.

- $\varepsilon_i(t)$ and $x(t)$ revealed at the start of period t and then choice is made.
- $\varepsilon_i(t)$ are assumed i.i.d. over alternatives and time with extreme value distribution

$$f(\varepsilon_i(t)) = e^{-\varepsilon_i(t)} e^{-e^{-\varepsilon_i(t)}} \quad , \ -\infty < \varepsilon_i(t) < \infty$$

- Usually it is assumed that the distribution of $x(t)$ is given by the transition density

$$f(x(t)|x(t-1), d(t-1))$$

  i.e. the transition density has the Markov property, and depends on choice.
- Note that in this case we assume that $x(t), \varepsilon_1(t), \ldots, \varepsilon_I(t)$ are independent conditional on $x(t-1)$.
- State variables are (why are the random utility components not included?)

$$\Omega(t) = \{x(t-1), d(t-1)\}$$

- **Properties of the Type I extreme value distribution**

  We assume ε_i are i.i.d. with Type I extreme value distribution

$$f(\varepsilon) = e^{-\varepsilon} e^{-e^{-\varepsilon}} \quad , \ -\infty < \varepsilon < \infty$$

  Consider

$$\eta = \max_{i=1,\ldots,I} \{\varepsilon_i + c_i\}$$

  with constants c_i. Then

$$F(\eta) = \Pr(\varepsilon_1 + c_1 \leq \eta, \ldots, \varepsilon_I + c_I \leq \eta) = \prod_{i=1}^{I} e^{-e^{-\eta + c_i}} = e^{-e^{-\eta + \ln(\sum_{i=1}^{I} e^{c_i})}}$$

  so that

$$\begin{aligned} \mathrm{E}[\max_{i=1,\ldots,I} \{\varepsilon_i + c_i\}] &= \mathrm{E}(\eta) \\ &= \gamma + \ln\left(\sum_{i=1}^{I} e^{c_i}\right) \end{aligned}$$

  with $\gamma = \mathrm{E}(\varepsilon) = 0.57722$.

Let

$$d^{opt} \equiv \arg \max_{i=1,\ldots,I} \{\varepsilon_i + c_i\}.$$

The choice probability equals

$$\Pr(d^{opt} = j) = \frac{e^{c_j}}{\sum_{i=1}^{I} e^{c_i}},$$

with

$$\mathrm{E}(\varepsilon_j + c_j | d^{opt} = j) \quad = \quad \gamma + \ln \left(\sum_{i=1}^{I} e^{c_i} \right).$$

Therefore,

$$\begin{aligned} \mathrm{E}(\varepsilon_j | d^{opt} = j) &= \gamma + \ln \left(\sum_{i=1}^{I} e^{c_i} \right) - c_j \\ &= \gamma - \ln \Pr(d^{opt} = j). \end{aligned}$$

This relationship is particularly convenient for the derivation of the value function from the optimal path.

4.4.1 Solution

- Bellman equation

$$V(\Omega(t)) = \max_{i=1,\ldots,I_t} u_i(x(t), d(t-1)) + \varepsilon_i(t) + \beta \mathrm{E}[V(\Omega(t+1)) | x(t), d_i(t) = 1]$$

- Note

$$\mathrm{E}[V(\Omega(t+1)) | x(t), d_i(t) = 1] = \mathrm{E}\left[\max_{j=1,\ldots,I_{t+1}} \{u_j(x(t+1), d_i(t) = 1) + \varepsilon_j(t+1) + \right.$$

$$\left. + \beta \mathrm{E}(V(\Omega(t+2) | d_j(t+1) = 1, x(t+1))\} | x(t), d_i(t) = 1 \right]$$

- In period $T-1$

$$V_i(\Omega(T-1)) = u_i(x(T-1), d(T-2)) + \varepsilon_i(T-1) +$$

$$+ \beta \mathrm{E}\left[\max_{j=1,\ldots,I_T} \{u_j(x(T), d_i(T-1) = 1) + \varepsilon_j(T)\} | x(T-1), d_i(T-1) = 1 \right]$$

- From this we find if $x(T)$ is known

$$\mathrm{E}\left[\max_{j=1,\ldots,I_T}\{u_j(x(T),d_i(T-1)=1)+\varepsilon_j(T)\}|d_i(T-1)=1\right]=$$

$$=\gamma+\ln\left(\sum_{j=1}^{I_T}e^{u_j(x(T),d_i(T-1)=1)}\right)=E(V(\Omega(T))|d_i(T-1)=1)$$

or if $x(T)$ is not known

$$\mathrm{E}\left[\max_{j=1,\ldots,I_T}\{u_j(x(T),d_i(T-1)=1)+\varepsilon_j(T)\}|x(T-1),d_i(T-1)=1\right]=$$

$$=\gamma+\mathrm{E}\left[\ln\left(\sum_{j=1}^{I_T}e^{u_j(x(T),d_i(T-1)=1)}\right)|x(T-1),d_i(T-1)=1\right]=$$

$$=\mathrm{E}(V(\Omega(T))|d_i(T-1)=1,x(T-1))$$

- The choice probabilities in $T-1$ are for $x(T)$ known (with obvious change for the other case)

$$p_i(T-1|d(T-2),x(T-1))=$$

$$=\frac{e^{u_i(x(T-1),d(T-2))+\beta\ln\left(\sum_{j=1}^{I_T}e^{u_j(x(T),d_i(T-1)=1)}\right)}}{\sum_{k=1}^{I_{T-1}}e^{u_k(x(T-1),d(T-2))+\beta\ln\left(\sum_{j=1}^{I_T}e^{u_j(x(T),d_k(T-1)=1)}\right)}}$$

Note that this resembles the nested MNL model.

- The choice probabilities in period T are

$$p_j(T|d_i(T-1)=1,x(T))=\frac{e^{u_j(x(T),d_i(T-1)=1)}}{\sum_{k=1}^{I_T}e^{u_k(x(T),d_k(T-1)=1)}}$$

- Note that if both $x(T-1)$ and $x(T)$ are known

$$\mathrm{E}(V(\Omega(T-1))|d(T-2))=$$

$$=\mathrm{E}\left[\max_{i=1,\ldots,I_{T-1}}\{u_i(x(T-1),d(T-2))+\varepsilon_i(T-1)+\right.$$

$$+\beta\mathrm{E}(V(\Omega(T))|d_i(T-1)=1)\}]=$$

$$\gamma + \ln \left[\sum_{j=1}^{I_{T-1}} e^{u_j(x(T-1),d(T-2)) + \beta \mathrm{E}(V(\Omega(T))|d_j(T-1)=1)} \right] =$$

$$2\gamma + \ln \left[\sum_{j=1}^{I_{T-1}} e^{u_j(x(T-1),d(T-2)) + \beta \ln\left(\sum_{k=1}^{I_{T-1}} e^{u_k(x(T),d_j(T-1)=1)}\right)} \right]$$

or if $x(T-1)$ and $x(T)$ are not known (omit 2γ)

$$\mathrm{E}(V(\Omega(T-1))|d(T-2), x(T-2)) =$$

$$\mathrm{E}\left[\ln \left(\sum_{j=1}^{I_{T-1}} e^{u_j(x(T-1),d(T-2)) + \beta \mathrm{E}\left[\ln\left(\sum_{k=1}^{I_{T-1}} e^{u_k(x(T),d_j(T-1)=1)}\right)|x(T-1),d_j(T-1)=1\right]} \right) |x(T-2) \right]$$

These are the additional terms in the choice probabilities for $T-2$.

- The choice probabilities have a closed form expression, but they are rather complicated. The complexity derives from the additional terms in the MNL model that can be computed recursively as

$$\mathrm{E}(V(\Omega(t))|d(t-1)) = \mathrm{E}\left[\max_{i=1,\ldots,I_t} \{u_i(x(t), d(t-1)) + \varepsilon_i(t) + \right.$$

$$\left. + \beta \mathrm{E}(V(\Omega(t+1))|d_i(t) = 1)\} \right] =$$

$$= \gamma + \ln \left(\sum_{j=1}^{I_t} e^{u_j(x(t),d(t-1)) + \beta \mathrm{E}(V((t+1))|d_j(t)=1)} \right)$$

if $x(t)$ is known or as

$$\mathrm{E}(V(\Omega(t))|d(t-1), x(t-1)) = \mathrm{E}\left[\max_{i=1,\ldots,I_t} \{u_i(x(t), d(t-1)) + \varepsilon_i(t) + \right.$$

$$\left. + \beta \mathrm{E}(V(\Omega(t+1))|d_i(t) = 1, x(t))\} |d(t-1), x(t-1) \right] =$$

$$= \gamma + \mathrm{E}\left[\ln \left(\sum_{j=1}^{I_t} e^{u_j(x(t),d(t-1)) + \beta \mathrm{E}(V(\Omega(t+1))|x(t),d_j(t)=1)} \right) |d(t-1), x(t-1) \right]$$

if $x(t)$ is not known.

- Note that the coefficient of the additional term is the discount factor.

4.5 DDC and Conditional Choice Probability

Using choices by others to avoid recursions: Hotz and Miller's Conditional Choice Probability estimator

- The CCP estimator avoids computationally expensive recursions by observing that the conditional valuation functions of the alternatives that summarize the effect of the current choice on future utility can be expressed of functions of the choice probabilities in future periods. These choice probabilities that depend on state variables on the relevant date, can be estimated using data on choices made by individuals and their state variables in those future periods. Because in any model choice probabilities are complicated functions of state variables these choice probabilities must be estimated nonparametrically.

- Consider a T period discrete choice problem with two alternatives indexed by $j = 1, 2$. The utilities of the two alternatives are

$$\begin{aligned} u_{t1} &= \alpha_1 + \beta_1 x_t + \varepsilon_{t1} \\ u_{t2} &= \alpha_2 + \beta_2 x_t + \varepsilon_{t2} \end{aligned}$$

- The dummy d_t is 1 if alternative 2 is chosen in period t. The variable x_t is observed and we assume

$$x_t = d_{t-1}\rho_2 x_{t-1} + (1 - d_{t-1})\rho_1 x_{t-1} + \eta_t$$

  The state variables are x_t, d_{t-1}. The current choice affects the future through x_{t+1}.

- The random $\varepsilon_{t1}, \varepsilon_{t2}$ are independent of η_t and have a type I extreme value distribution.

- The maximal utility over time periods $t, \ldots, T$ given information up to period t is (include $\varepsilon_{t1}, \varepsilon_{t2}$)

$$V(x_t, \varepsilon_{t1}, \varepsilon_{t2}) = \max\{V_1(x_t, \varepsilon_{t1}), V_2(x_t, \varepsilon_{t2})\}$$

with

$$V_j(x_t, \varepsilon_{tj}) = u_{tj} + v_{t+1,j}(x_t) = \alpha_j + \beta_j x_t + v_{t+1,j}(x_t) + \varepsilon_{tj}$$

and $v_{t+1,j}(x_t)$ the expected maximal utility over periods $t+1, \ldots, T$ given that j is chosen in t and x_t, i.e. the conditional valuation function.

- The choice probabilities in period t are

$$p_j(x_t) = \frac{e^{\alpha_j + \beta_j x_t + v_{t+1,j}(x_t)}}{e^{\alpha_1 + \beta_1 x_t + v_{t+1,1}(x_t)} + e^{\alpha_2 + \beta_2 x_t + v_{t+1,2}(x_t)}} \qquad j = 1, 2$$

Note that

$$p_2(x_t) = \frac{e^{\alpha_2 + \beta_2 x_t + v_{t+1,2}(x_t)}}{e^{\alpha_1 + \beta_1 x_t + v_{t+1,1}(x_t)} + e^{\alpha_2 + \beta_2 x_t + v_{t+1,2}(x_t)}} =$$

$$\frac{e^{\alpha_2 - \alpha_1 + (\beta_2 - \beta_1) x_t + v_{t+1,2}(x_t) - v_{t+1,1}(x_t)}}{1 + e^{\alpha_2 - \alpha_1 + (\beta_2 - \beta_1) x_t + v_{t+1,2}(x_t) - v_{t+1,1}(x_t)}}$$

which depends only on the difference of the conditional valuation functions.

- The next step is to express $v_{t+1,j}(x_t)$ as a function of $p_j(x_{t+1})$. We have

$$v_{t+1,j}(x_t) = \mathrm{E}\left[\max\left\{V_1(x_{t+1}, \varepsilon_{t+1,1}), V_2(x_{t+1}, \varepsilon_{t+1,2})\right\} | x_t, d_t = j - 1\right]$$

where the expectation is over the conditional distribution of $\varepsilon_{t+1,1}, \varepsilon_{t+1,2}, x_{t+1}$ given x_t. By assumption $\varepsilon_{t+1,1}, \varepsilon_{t+1,2}$ are independent of x_{t+1} , so that we can compute the expectation by first conditioning on x_{t+1} and taking the expected value of the result with respect to the conditional distribution of x_{t+1} given x_t.

- Doing this we find

$$\mathrm{E}\left[\max\left\{V_1(x_{t+1}, \varepsilon_{t+1,1}), V_2(x_{t+1}, \varepsilon_{t+1,2})\right\} | x_{t+1}, x_t, d_t = j - 1\right] =$$

$$p_1(x_{t+1})\mathrm{E}\left[V_1(x_{t+1}, \varepsilon_{t+1,1}) | d_{t+1} = 0\right] + p_2(x_{t+1})\mathrm{E}\left[V_2(x_{t+1}, \varepsilon_{t+1,2}) | d_{t+1} = 1\right] =$$

$$p_1(x_{t+1})\mathrm{E}\left[\alpha_1 + \beta_1 x_{t+1} + v_{t+2,1}(x_{t+1}) + \varepsilon_{t+1,1} \right|$$

$$\left. \alpha_1 + \beta_1 x_{t+1} + v_{t+2,1}(x_{t+1}) + \varepsilon_{t+1,1} > \alpha_2 + \beta_2 x_{t+1} + v_{t+1,2}(x_{t+1}) + \varepsilon_{t+1,2}\right]$$

$$+ p_2(x_{t+1})\mathrm{E}\left[\alpha_2 + \beta_2 x_{t+1} + v_{t+2,2}(x_{t+1}) + \varepsilon_{t+1,2} \right|$$

$$\alpha_1 + \beta_1 x_{t+1} + v_{t+2,1}(x_{t+1}) + \varepsilon_{t+1,1} < \alpha_2 + \beta_2 x_{t+1} + v_{t+2,2}(x_{t+1}) + \varepsilon_{t+1,2}] =$$

$$p_1(x_{t+1}) \left[\alpha_1 + \beta_1 x_{t+1} + v_{t+2,1}(x_{t+1}) + \gamma + \right.$$
$$\left. \ln\left(1 + e^{\alpha_2+\beta_2 x_{t+1}+v_{t+2,2}(x_{t+1})-\alpha_1-\beta_1 x_{t+1}-v_{t+2,1}(x_{t+1})}\right)\right] +$$
$$p_2(x_{t+1}) \left[\alpha_2 + \beta_2 x_{t+1} + v_{t+2,2}(x_{t+1}) + \gamma + \right.$$
$$\left. \ln\left(1 + e^{\alpha_1+\beta_1 x_{t+1}+v_{t+2,1}(x_{t+1})-\alpha_2-\beta_2 x_{t+1}-v_{t+2,2}(x_{t+1})}\right)\right]$$

- Here we use: If $\varepsilon_1, \varepsilon_2$ independent type I extreme value then

  $$\mathbb{E}[\varepsilon_1+c_1|\varepsilon_1+c_1 > \varepsilon_2+c_2] = c_1+\mathbb{E}[\varepsilon_1|\varepsilon_1+c_1 > \varepsilon_2+c_2] = c_1+\gamma+\ln\left(1 + e^{c_2-c_1}\right)$$

  with $\gamma = \mathrm{E}(\varepsilon)$.

- Relation future choice probability and future conditional valuation function

  $$v_{t+2,2}(x_{t+1}) - v_{t+2,1}(x_{t+1}) = \ln\left(\frac{p_2(x_{t+1})}{p_1(x_{t+1})}\right) - (\alpha_2 - \alpha_1) - (\beta_2 - \beta_1)x_t$$

- Upon substitution

  $$\mathrm{E}\left[\max\left\{V_1(x_{t+1}, \varepsilon_{t+1,1}), V_2(x_{t+1}, \varepsilon_{t+1,2})\right\}|x_{t+1}, x_t, d_t = j-1\right] =$$

  $$\gamma + \alpha_1 + \beta_1 x_{t+1} + v_{t+2,1}(x_{t+1}) - \ln p_1(x_{t+1})$$

- This expression does not directly depend on d_t. However, we must take the expectation over the conditional distribution of x_{t+1} given x_t and d_{t-1}:

  $$v_{t+1,1}(x_t) = \gamma + \alpha_1 + \beta_1 \mathrm{E}[x_{t+1}|x_t, d_t = 0] + \mathrm{E}[v_{t+2,1}(x_{t+1})|x_t, d_t = 0]$$

  $$-\mathrm{E}\left[\ln p_1(x_{t+1})|x_t, d_t = 0\right]$$

  and

  $$v_{t+1,2}(x_t) = \gamma + \alpha_1 + \beta_1 \mathrm{E}[x_{t+1}|x_t, d_t = 0] + \mathrm{E}[v_{t+2,1}(x_{t+1})|x_t, d_t = 0]$$

  $$-\mathrm{E}\left[\ln p_1(x_{t+1})|x_t, d_t = 1\right]$$

  Now

  $$\mathrm{E}[v_{t+2,1}(x_{t+1})|x_t, d_t = 1] - \mathrm{E}[v_{t+2,1}(x_{t+1})|x_t, d_t = 0]$$

is not a function of $p_j(x_{t+1})$ alone, which is the problem.

4.5.1 CCP Special case: State 1 is absorbing

- A solution is to assume that 1 is an absorbing state. In that case the conditional valuation function is simple if $d_t = 0$

$$v_{t+1,1}(x_t) = (T-t)\alpha_1 + \beta_1\rho_1\frac{1-\rho_1^{T-t}}{1-\rho_1}x_t$$

  Hence we only need to consider $v_{t+1,2}(x_t)$.

- We obtain

$$\mathrm{E}\left[\max\left\{V_1(x_{t+1},\varepsilon_{t+1,1}),V_2(x_{t+1},\varepsilon_{t+1,2})\right\}|x_{t+1},x_t,d_t=1\right] =$$

$$\gamma+\alpha_1+\beta_2 x_{t+1}+(T-t-1)\alpha_1+\beta_1\rho_1\frac{1-\rho_1^{T-t-1}}{1-\rho_1}x_{t+1}-\ln p_1(x_{t+1})$$

  so that

$$v_{t+1,2}(x_t) = \gamma+\alpha_1+\beta_2\rho_2 x_t+(T-t-1)\alpha_1+\beta_1\rho_1\frac{1-\rho_1^{T-t-1}}{1-\rho_1}\rho_2 x_t$$

$$-\mathrm{E}[\ln p_1(x_{t+1})|x_t,d_t=1]$$

  and this is a function of the choice probability in $t+1$ only.

- For difference in conditional valuation functions

$$v_{t+1,2}(x_t)-v_{t+1,1}(x_t) = \gamma+\beta_1(\rho_2-\rho_1)\frac{1-\rho_1^{T-t}}{1-\rho_1}x_t-\mathrm{E}[\ln p_1(x_{t+1})|x_t,d_t=1]$$

  and the choice probability

$$p_2(x_t) = \frac{e^{\alpha_2-\alpha_1+(\beta_2-\beta_1)x_t+\gamma+\beta_1(\rho_2-\rho_1)\frac{1-\rho_1^{T-t}}{1-\rho_1}x_t-\mathrm{E}[\ln p_1(x_{t+1})|x_t,d_t=1]}}{1+e^{\alpha_2-\alpha_1+(\beta_2-\beta_1)x_t+\gamma+\beta_1(\rho_2-\rho_1)\frac{1-\rho_1^{T-t}}{1-\rho_1}x_t-\mathrm{E}[\ln p_1(x_{t+1})|x_t,d_t=1]}}$$

- Even if we know ρ_1,ρ_2 we can only estimate $\alpha_2-\alpha_1$. We can also not separately identify β_1,β_2, but that is due to the fact that the utility functions are linear in x_t.

4.5.2 CCP general case without absorbing state

- In this case we use another approach to compute $v_{t+1,j}(x_t)$. We have

$$v_{t+1,j}(x_t) =$$

$$\mathrm{E}\left[\sum_{s=t+1}^{T}(1-d_s^{opt})\left(\alpha_1+\beta_1 x_s+\varepsilon_{s1}\right)+d_s^{opt}\left(\alpha_2+\beta_2 x_s+\varepsilon_{s2}\right)\middle| x_t, d_t = j-1\right]$$

with

$$d_s^{opt} = I\left(\alpha_1+\beta_1 x_s+v_{s+1,1}(x_s)+\varepsilon_{s,1} < \alpha_2+\beta_2 x_s+v_{s+1,2}(x_s)+\varepsilon_{s,2}\right)$$

- We compute the expectation term by term. For $t+1$ x_{t+1} is independent of $\varepsilon_{t+1,1},\varepsilon_{t+1,2}$ given d_t, x_t so that we first condition on x_{t+1} and then average over its (conditional) distribution

$$\mathrm{E}\left[(1-d_{t+1}^{opt})\left(\alpha_1+\beta_1 x_{t+1}+\varepsilon_{t+1,1}\right)+d_{t+1}^{opt}\left(\alpha_2+\beta_2 x_{t+1}+\varepsilon_{t+1,2}\right)\middle| x_t, d_t = j-1\right] =$$

$$\mathrm{E}\left[p_1(x_{t+1})\left(\alpha_1+\beta_1 x_{t+1}+\gamma-\ln p_1(x_{t+1})\right)+\right.$$

$$\left. p_2(x_{t+1})\left(\alpha_2+\beta_2 x_{t+1}+\gamma-\ln p_2(x_{t+1})\right)\right| x_t, d_t = j-1]$$

where the expectation is over the distribution with pdf $f(x_{t+1}|x_t, d_t)$.

- Note that the extreme value distribution implies that, for given c_1 and c_2, the choice probability equals

$$\begin{aligned} p(c_1,c_2) &\equiv \mathbb{E}[d^{opt}=1|c_1,c_2] \\ &= \mathbb{E}[I(\varepsilon_1+c_1>\varepsilon_2+c_2)] \\ &= \frac{e^{c_1}}{e^{c_1}+e^{c_2}}. \\ &= \frac{1}{1+e^{c_2-c_1}}. \end{aligned}$$

and with $\gamma = \mathrm{E}(\varepsilon_1)$,

$$\begin{aligned} \mathbb{E}[\varepsilon_1|d^{opt}=1,c_1,c_2] &= \mathbb{E}[\varepsilon_1|\varepsilon_1+c_1>\varepsilon_2+c_2] \\ &= \gamma+\ln\left(1+e^{c_2-c_1}\right) \\ &= \gamma-\ln p(c_1,c_2). \end{aligned}$$

That means the conditional mean of the random utility ε_1 is associated with the conditional choice probability as follows,

$$\mathbb{E}[\varepsilon_1|d^{opt}=1,c_1,c_2] \quad = \quad \gamma - \ln p(c_1,c_2).$$

- For $t+2$ we first obtain the conditional distribution of x_{t+2} given x_{t+1}

  $$f(x_{t+2}|x_{t+1}) = f(x_{t+2}|x_{t+1},d_{t+1}=0)p_1(x_{t+1}) + f(x_{t+2}|x_{t+1},d_{t+1}=1)p_2(x_{t+1})$$

  so that

  $$f(x_{t+2}|x_t,d_t) =$$

  $$\int [f(x_{t+2}|x_{t+1},d_{t+1}=0)p_1(x_{t+1}) + f(x_{t+2}|x_{t+1},d_{t+1}=1)p_2(x_{t+1})] f(x_{t+1}|x_t,d_t)\mathrm{d}x_{t+1}$$

- Given d_t, x_t, x_{t+1} x_{t+2} is independent of $\varepsilon_{t+2,1}, \varepsilon_{t+2,2}$ so that

  $$\mathrm{E}\left[(1-d_{t+2}^{opt})(\alpha_1+\beta_1 x_{t+2}+\varepsilon_{t+2,1}) + d_{t+2}^{opt}(\alpha_2+\beta_2 x_{t+2}+\varepsilon_{t+2,2})\big|\, x_t, d_t=j-1\right] =$$

  $$\mathrm{E}\left[p_1(x_{t+2})(\alpha_1+\beta_1 x_{t+2}+\gamma-\ln p_1(x_{t+2})) + \right.$$

  $$\left. p_2(x_{t+2})(\alpha_2+\beta_2 x_{t+2}+\gamma-\ln p_2(x_{t+2}))\right|\, x_t, d_t=j-1]$$

  where the expectation is over the distribution with pdf $f(x_{t+2}|x_t,d_t)$.

- The conditional valuation function

  $$v_{t+1,j}(x_t) =$$

  $$\sum_{s=t+1}^{T} \mathrm{E}\left[p_1(x_s)(\alpha_1+\beta_1 x_s+\gamma-\ln p_1(x_s)) + \right.$$

  $$\left. p_2(x_s)(\alpha_2+\beta_2 x_s+\gamma-\ln p_2(x_s))\right|\, x_t, d_t=j-1]$$

  with

  $$f(x_s|x_t,d_t=j-1) = \left(\prod_{r=t+2}^{s} f(x_r|x_{r-1})\right) f(x_{t+1}|x_t,d_t=j-1)$$

  and hence the conditional valuation function can be expressed as a function of the choice probabilities $p_1(x_s), s=t+1,\ldots,T$ (but in general not just as an expression in $p_1(x_{t+1})$).

4.5.3 The CCP estimator

Special case: Model with an absorbing state

- The data is a single cohort that is observed for $t = 1, \dots, T$. In that case the data are $d_{it}, x_{it}, t = 1, \dots, T, i = 1, \dots, N$. The observation period may be individual specific T_i. It is possible that we observe multiple cohorts, in which case we can allow for calendar time effects in the utility function and/or the transition densities. Subscript i only added if needed.

- The choice probabilities have the logit form and we have for all x_t the conditional moment
$$\mathbb{E}[d_t - p_2(x_t)|x_t] = 0$$

- This suggests the unconditional moments
$$\mathbb{E}[x_t(d_t - p_2(x_t))] = 0$$
$$\mathbb{E}[d_t - p_2(x_t)] = 0$$

- If we replace $p_1(x_{t+1})$ by a nonparametric estimator and we denote the resulting choice probability in period t by $\hat{p}_2(x_t)$, we have the sample moment conditions
$$\frac{1}{N}\sum_{i=1}^{N}(d_{it} - \hat{p}_2(x_t)) = 0$$
$$\frac{1}{N}\sum_{i=1}^{N}x_{it}(d_{it} - \hat{p}_2(x_t)) = 0$$

- These can be solved for an estimate of $\alpha_2 - \alpha_1$ and $(\beta_2 - \beta_1) + \beta_1(\rho_2 - \rho_1)\frac{1-\rho_1^{T-t}}{1-\rho_1}$.

- Because
$$\mathbb{E}(x_{T+1}|d_T = 1, x_T) = \rho_2 x_T$$
$$\mathbb{E}(x_{T+1}|d_T = 0, x_T) = \rho_1 x_T$$

  ρ_1, ρ_2 can be estimated by OLS and we can add the corresponding sample moment conditions to those derived from the logit model.

The general case

- Data as before
- Choice probabilities

$$p_j(x_t) = \frac{e^{\alpha_j+\beta_j x_t+v_{t+1,j}(x_t)}}{e^{\alpha_1+\beta_1 x_t+v_{t+1,1}(x_t)} + e^{\alpha_2+\beta_2 x_t+v_{t+1,2}(x_t)}} \qquad j = 1, 2$$

with

$$v_{t+1,j}(x_t) =$$

$$\sum_{s=t+1}^{T} \mathrm{E}\left[p_1(x_s)\left(\alpha_1 + \beta_1 x_s + \gamma - \ln p_1(x_s)\right) + p_2(x_s)\left(\alpha_2 + \beta_2 x_s + \gamma - \ln p_2(x_s)\right)\right| x_t, d_t = j-1]$$

where the expectation for term s is over the density

$$f(x_s|x_t, d_t = j-1) = \left(\prod_{r=t+2}^{s} f(x_r|x_{r-1})\right) f(x_{t+1}|x_t, d_t = j-1)$$

with

$$f(x_r|x_{r-1}) = f(x_r|x_{r-1}, d_{r-1} = 0)p_1(x_{r-1}) + f(x_r|x_{r-1}, d_{r-1} = 1)p_2(x_{r-1})$$

- If we specify the conditional transition densities $f(x_r|x_{r-1}, d_{r-1} = 0), f(x_r|x_{r-1}, d_{r-1} = 1)$ the expressions depend on the parameters and choice probabilities that can be estimated. Alternatively we can use nonparametric estimates of the conditional transition densities as well.
- To find a moment condition note that for t we also have

$$v_{t+1,2}(x_t) - v_{t+1,1}(x_t) = \ln\frac{p_2(x_t)}{p_1(x_t)} - (\alpha_2 - \alpha_1) - (\beta_2 - \beta_1)x_t$$

- Hence there are two ways to obtain the difference of the conditional valuation functions and we define

$$m(x_t, p_1(x_t), p_2(x_t), \alpha_1, \alpha_2, \beta_1, \beta_2, \rho_1, \rho_2, \sigma_\eta^2) = \ln\frac{p_2(x_t)}{p_1(x_t)} - (\alpha_2-\alpha_1) - (\beta_2-\beta_1)x_t -$$

$$\sum_{s=t+1}^{T} \mathrm{E}\left[p_1(x_s)\left(\alpha_1 + \beta_1 x_s + \gamma - \ln p_1(x_s)\right) + p_2(x_s)\left(\alpha_2 + \beta_2 x_s + \gamma - \ln p_2(x_s)\right)\right| x_t, d_t = 1] +$$

$$\sum_{s=t+1}^{T} \mathrm{E}\left[p_1(x_s)\left(\alpha_1 + \beta_1 x_s + \gamma - \ln p_1(x_s)\right) + p_2(x_s)\left(\alpha_2 + \beta_2 x_s + \gamma - \ln p_2(x_s)\right)\right| x_t, d_t = 0]$$

- Hotz, Miller, Sanders and Smith (1994) propose to use the fact that $m(x_t, \alpha_1, \alpha_2, \beta_1, \beta_2, \rho_1, \rho_2, \sigma_\eta^2) = 0$ for all x_t (at the population value of the parameters) to derive a set of moment restrictions.

- Define for $j = 1, 2$

$$\hat{h}_j(x_t) = \sum_{s=t+1}^{T} \left(\hat{\mathbb{E}}\left[\hat{p}_j(x_s)|x_t, d_t = 1\right] - \hat{\mathbb{E}}\left[\hat{p}_j(x_s)|x_t, d_t = 0\right] \right) + (-1)^j$$

$$\hat{k}_j(x_t) = \sum_{s=t+1}^{T} \left(\hat{\mathbb{E}}\left[x_s \hat{p}_j(x_s)|x_t, d_t = 1\right] - \hat{\mathbb{E}}\left[x_s \hat{p}_j(x_s)|x_t, d_t = 0\right] \right) + (-1)^j x_t$$

$$\hat{g}(x_t) = \ln \frac{\hat{p}_2(x_t)}{\hat{p}_1(x_t)} +$$

$$\sum_{s=t+1}^{T} \left(\hat{\mathbb{E}}\left[\hat{p}_1(x_s) \ln \hat{p}_1(x_s) + \hat{p}_2(x_s) \ln \hat{p}_2(x_s) |\, x_t, d_t = 1\right] - \hat{\mathbb{E}}\left[\hat{p}_1(x_s) \ln \hat{p}_1(x_s) + \hat{p}_2(x_s) \ln \hat{p}_2(x_s) |\, x_t, d_t = 0\right] \right)$$

  where the hat on the expectation indicates that it is computed with an estimate of the transition density that depends on the estimated choice probabilities.

- If we estimate the transition densities nonparametrically we can use the same notation (and we do not estimate $\rho_1, \rho_2, \sigma_\eta^2$).

- At the population values we have

$$m(x_t, \alpha_1, \alpha_2, \beta_1, \beta_2, \rho_1, \rho_2, \sigma_\eta^2) = g(x_t) - \alpha_1 h_1(x_t) - \alpha_2 h_2(x_t) - \beta_1 k_1(x_t) - \beta_2 k_2(x_t)$$

- Because this is 0 for all x_t we have that

$$\sum_{i=1}^{N} m(x_{it}, \alpha_1, \alpha_2, \beta_1, \beta_2, \rho_1, \rho_2, \sigma_\eta^2)^2 = 0$$

  We obtain an estimator by replacing g, h_j, k_j by estimates and minimizing the resulting sum of squares.

- If $\rho_1, \rho_2, \sigma_\eta^2$ are known (or we use the nonparametric transition density estimator), this is just OLS of $\hat{g}$ on $\hat{h}_1, \hat{h}_2, \hat{k}_1, \hat{k}_2$. Note that the OLS estimator involves averages over the observations of nonparametric estimators of the choice probabilities.

- In general we need also to minimize over $\rho_1, \rho_2, \sigma_\eta^2$ so that we have a nonlinear least squares problem.

- Note that this by no means the only way to estimate the parameters. For instance, we can use the observed x_t to estimate $\rho_1, \rho_2, \sigma_\eta^2$. We have

$$\mathbb{E}(x_t | x_{t-1}, d_{t-1} = 1) = \rho_2 x_{t-1}$$

$$\mathbb{E}(x_t | x_{t-1}, d_{t-1} = 0) = \rho_1 x_{t-1}$$

  Hence we can estimate ρ_2 by OLS with the dependent variable x_t and independent variable x_{t-1} for the subsample with $d_{t-1} = 1$. The estimated variance of the error is an estimate of σ_η^2.

- Note that no assumptions on terminal conditions are needed. If we assume e.g. that $u_{t1} = u_{t2} = 0$ for $t \geq T+1$, then this could be used in estimation.

- In the final period T the choice probabilities are for $j = 1, 2$

$$p_j(x_T) = \frac{e^{\alpha_j + \beta_j x_T}}{e^{\alpha_1 + \beta_1 x_T} + e^{\alpha_2 + \beta_2 x_T}}$$

  This identifies $\alpha_2 - \alpha_1$ and $\beta_2 - \beta_1$.

- In general, if the average utility is $u_j(x_t)$, then we can identify the function $g(x_T) = u_2(x_T) - u_1(x_T)$

$$\ln \frac{p_2(x_T)}{p_1(x_T)} = g(x_T)$$

- If the average utilities only depend on x_t this determines the function g in all periods. Note however that in the dynamic discrete choice model $\alpha_j, \beta_j, j = 1, 2$ are identified, not just their differences.

- Data on d_{T-1}, x_{T-1} would be sufficient to identify all parameters. For $T - 1$ we have

$$v_{Tj}(x_{T-1}) = \mathrm{E}\left[p_1(x_T) \ln\left(e^{\alpha_1 + \beta_1 x_T} + e^{\alpha_2 + \beta_2 x_T}\right) + p_2(x_T) \ln\left(e^{\alpha_1 + \beta_1 x_T} + e^{\alpha_2 + \beta_2 x_T}\right) \middle| x_{T-1}, d_{T-1} = j - 1\right]$$

  Combined with

$$v_{T2}(x_t) - v_{T1}(x_t) = \ln \frac{p_2(x_{T-1})}{p_1(x_{T-1})} - (\alpha_2 - \alpha_1) - (\beta_2 - \beta_1) x_{T-1}$$

we again obtain a 'moment condition' that can be used in estimation.

- By writing this 'moment condition' in terms of $u_2(x_t)$ and $u_1(x_t)$, we can study the identification of these utility functions. This involves the unique solution to an integral equation. More periods will give better estimates.

4.6 The CCP estimator in a general discrete setting

In each discrete time period $t = 1, 2, \cdots, T$ (T can be finite or infinite), a single agent chooses an action a_t from a finite set of actions, $\mathcal{A} = \{1, \cdots, K\}, K \geq 2$, to maximize her expected lifetime utility. The utility-relevant state variables in period t consist of two parts, x_t and ϵ_t, where x_t is the state variable observed by the econometrician, and ϵ_t is a vector of unobserved choice-specific shocks, i.e., $\epsilon_t = (\epsilon_t(1), \cdots, \epsilon_t(K))$. We assume that the observed state variable x_t is discrete and takes values in $\mathcal{X} \equiv \{1, \cdots, J\}, J \geq 2$. Both state variables are known to the agent at the beginning of period t. The agent then makes choice a_t and obtains per-period utility $u(x_t, a_t, \epsilon_t)$. There is uncertainty regarding future states, which is governed by an exogenous mechanism and is assumed to be a Markov process, given the agent's choice. Specifically, given the current state (x, ϵ) and agent choice a, the state variables in the next period (x', ϵ') are determined by the transition function $f(x', \epsilon'|x, \epsilon, a)$. We suppress period subscripts and use prime to represent the next period for ease of notation. Following the existing literature, we impose the following assumption on the transition function.
(a). The observed and unobserved state variables evolve independently conditional on x and a. That is,

$$f(x', \epsilon'|x, a, \epsilon) = f(x'|x, a)f(\epsilon'|\epsilon, a).$$

(b). The unobserved state variables over time periods and actions are i.i.d. draws from the mean zero type-I extreme value distribution.[1]

The transition process can be simplified as

$$f(x', \epsilon'|x, a, \epsilon) = f(x'|x, a)f(\epsilon').$$

Because the agent is forward-looking and her choice involves intertemporal optimization, her beliefs about the state transition play an essential role in her

[1]The assumption of type-I extreme value distribution is for ease of illustration. As long as the distribution is known and absolutely continuous, our identification argument holds.

decision-making process. Let $s(x'|x,a)$ be the agent's subjective beliefs about the transition of the observed state. In the literature of DDC models, a ubiquitous assumption is that agents have perfect beliefs or rational expectations—i.e., $s(x'|x,a) = f(x'|x,a)$ for all x', x and a—and this is crucial for identifying and estimating DDC models. (Unfortunately, this assumption is very restrictive.)

In each discrete period, the agent's problem is to decide what action maximizes her expected life-time utility, based on her subjective beliefs about the future evolution of the state variable. The optimization problem is characterized as

$$\max_{a_t \in \mathcal{A}} \sum_{\tau=t,t+1,\ldots} \beta^{\tau-t} E\big[u(x_\tau, a_\tau, \epsilon_\tau)|x_t, a_t, \epsilon_t\big],$$

where $\beta \in [0,1)$ is the discount factor, $u(x_\tau, a_\tau, \epsilon_\tau)$ is the flow utility, and the expectation is taken over all future actions and states, based on the agent's rational beliefs $f(x'|x,a)$. We assume $f(x'|x,a)$ is time-invariant.

Next, we present an assumption about agent preferences following the existing literature: The flow utility is time-invariant. The unobserved state is assumed to enter the preference additively and separably, i.e., $u(x,a,\epsilon) = u(x,a) + \epsilon(a) \equiv u_a(x) + \epsilon(a)$ for any $a \in \mathcal{A}$.

The stationarity and additive separability of agent utility imposed are used widely in the literature. Consequently, we can represent the agent's optimal choice a_t, which depends on the state, x_t and ϵ_t, in period t as

$$\begin{aligned} a_t^{opt}(x) &= \arg\max_{a\in\mathcal{A}} \Big\{ u_a(x) + \epsilon_t(a) + \beta \sum\nolimits_{x' \in \mathcal{X}} V_{t+1}(x') f(x'|x,a) \Big\} \\ &\equiv \arg\max_{a\in\mathcal{A}} \{ v_{t,a}(x) + \epsilon_t(a) \}, \end{aligned}$$

where $v_{t,a}(x)$ is the choice-specific value function for action a conditional on state x,

$$v_{t,a}(x) \equiv u_a(x) + \beta \sum\nolimits_{x' \in \mathcal{X}} V_{t+1}(x') f(x'|x,a)$$

and $V_{t+1}(x)$ is the *ex-ante* or continuation value function in period $t+1$ defined below.

$$\begin{aligned}
V_t(x) &= E\left[\max_{a\in\mathcal{A}}\left\{u_a(x)+\epsilon_t(a)+\beta\sum\nolimits_{x'\in\mathcal{X}}V_{t+1}(x')f(x'|x,a)\right\}\right] && (4)\\
&\equiv E\left[\max_{a\in\mathcal{A}}\left\{v_{t,a}(x)+\epsilon_t(a)\right\}\right] && (5)\\
&= \int\sum\nolimits_{a\in\mathcal{A}}\mathbb{1}\{a=a_t^{opt}\}\left[v_t(x_t,a)+\epsilon_t(a)\right]g(\epsilon_t)d\epsilon_t && (6)\\
&= \ln\sum_{a\in\mathcal{A}}\exp(v_{t,a}(x)) && (7)\\
&= -\log p_{t,i}(x)+v_{t,i}(x) && (8)\\
&= -\log p_{t,K}(x)+v_{t,K}(x) && (9)
\end{aligned}$$

where the expectation is taken with respect to the distribution of ϵ and and $p_{t,i}(x)$ is the conditional choice probability (CCPs)

Following the existing literature, we characterize agents' optimal behavior using a whole set of probabilities (CCPs) that each action $i\in\mathcal{A}$ is chosen conditional on the observed state in period t, denoted as $p_{t,i}(x)$. Under Assumption 1(b), the agent's optimal behavior can be characterized as

$$\begin{aligned}
p_{t,i}(x) &= \Pr(a_t^{opt}=i|x) && (10)\\
&= \frac{\exp(v_{t,i}(x))}{\sum_{a\in\mathcal{A}}\exp(v_{t,a}(x))} && (11)\\
&= \frac{\exp(v_{t,i}(x))}{\exp(V_t(x))} && (12)
\end{aligned}$$

In the finite horizon setting, an agent can solve the model using backward induction, starting from the terminal period; this requires the continuation value at the terminal period is known to the agent. In the existing literature, the continuation value at the terminal period could be zero or nonzero, depending on the empirical context. In the infinite horizon setting, stationarity implies that the value function is a fixed point of a contraction mapping.

We derive the recursive relationship of value functions over time. We first express the *ex-ante* value function using the choice-specific value function $v_{t,i}(x)$ with an adjustment of $-\log p_{t,i}(x)$. Specifically, the *ex-ante* value function at t can be expressed as

$$
\begin{aligned}
V_t(x) &= -\log p_{t,i}(x) + v_{t,i}(x) \\
&= -\log p_{t,i}(x) + u_i(x) + \beta \sum\nolimits_{x'} V_{t+1}(x') f(x'|x, a=i) \\
&= -\log p_{t,i}(x) + u_i(x) + \beta F_i(x) \boldsymbol{V}_{t+1}.
\end{aligned}
$$

We stack the equation above for all state x and obtain the following matrix form.

$$
\begin{aligned}
\boldsymbol{V}_t &\equiv \begin{pmatrix} V_t(1) \\ V_t(2) \\ \dots \\ V_t(J) \end{pmatrix} \\
&= -\begin{pmatrix} \log \boldsymbol{p}_{t,i}(1) \\ \log \boldsymbol{p}_{t,i}(2) \\ \dots \\ \log \boldsymbol{p}_{t,i}(J) \end{pmatrix} + \begin{pmatrix} u_i(1) \\ u_i(2) \\ \dots \\ u_i(J) \end{pmatrix} + \beta \begin{pmatrix} F_i(1) \\ F_i(2) \\ \dots \\ F_i(J) \end{pmatrix} \boldsymbol{V}_{t+1} \\
&\equiv -\log \boldsymbol{p}_{t,i} + \boldsymbol{u}_i + \beta \boldsymbol{F}_i \boldsymbol{V}_{t+1},
\end{aligned}
$$

where $\log \boldsymbol{p}_{t,i}$ and $\boldsymbol{u}_i$ are defined similarly to $\boldsymbol{V}_t$, and $\boldsymbol{F}_i \equiv \left[F_i(1)', F_i(2)', \cdots, F_i(J)'\right]'$. Consequently, we obtain the recursive relationship of *ex-ante* value functions over time.

For the choice K

$$
\boldsymbol{V}_t = -\log \boldsymbol{p}_{t,K} + \boldsymbol{u}_K + \beta \boldsymbol{F}_K \boldsymbol{V}_{t+1}. \tag{13}
$$

Under a normalization condition

$$
\boldsymbol{u}_K = 0, \textit{ i.e.}, \; u(x_t, K) = 0 \quad \forall x_t
$$

and an assumption that $[1 - \beta F_K]$ is invertible, we have in the infinite horizon case with $\boldsymbol{V}_t = \boldsymbol{V}_{t+1}$

$$
\boldsymbol{V}_t = -(1 - \beta \boldsymbol{F}_K)^{-1} \log \boldsymbol{p}_{t,K} \tag{14}
$$

There is a direct mapping from CCPs to the value function. The utility function can them be identified from

$$\boldsymbol{V}_t = -\log \boldsymbol{p}_{t,i} + \boldsymbol{u}_i + \beta \boldsymbol{F}_i \boldsymbol{V}_{t+1}. \tag{15}$$

Therefore,

$$\begin{aligned}
\boldsymbol{u}_i &= \log \boldsymbol{p}_{t,i} + \boldsymbol{V}_t - \beta \boldsymbol{F}_i \boldsymbol{V}_{t+1} && (16)\\
&= \log \boldsymbol{p}_{t,i} + (1 - \beta \boldsymbol{F}_i)\boldsymbol{V}_t && (17)\\
&= \log \boldsymbol{p}_{t,i} - (1 - \beta \boldsymbol{F}_i)(1 - \beta \boldsymbol{F}_K)^{-1} \log \boldsymbol{p}_{t,K} && (18)
\end{aligned}$$

Notice that the last line is an equation with model primitives and directly estimable elements only and can be used as a moment equation for estimation. It also implies that the utility function and the discount factor β can't not be identified at the same time. There are also pseudo-likelihood estimators based on this derivation. Such derivation is based on a known discount factor β. The identification of β will need an extra restriction.

4.7 The CCP estimator and unobserved heterogeneity

- The CCP estimator as derived above cannot allow for permanent unobserved differences between individuals, e.g. assuming that intercepts α_{k1}, α_{k2} differ between types k where we only know the fraction π_k of type k in the population.

- This is an issue because we may need these permanent unobserved differences/heterogeneity to fit the data.

- We can estimate the choice probabilities $p_2(x_t), t = 1, \ldots, T$ nonparametrically. Under the assumptions on the transition density, we have that given x_{t-1}, d_{t-1} x_t is independent of x_{t-2}. This implies that if that given the type and x_{t-1}, d_{t-1} $p_2(x_t)$ is independent of x_{t-2}. However, type is correlated with choices so that $p_2(x_t)$ is correlated with x_{t-2} given d_{t-1}, x_{t-1}. Hence with panel data we can identify unobserved heterogeneity if we assume finite memory in the transition density.

- Use latent variable and measurement error models –
  The econometrics of unobservables