The Econometrics of Unobservables - Identification, Estimation, and Empirical Applications

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Economic theory vs. econometric model: An example

- Economic theory: Permanent income hypothesis
- Econometric model: Measurement error model

$$
\begin{aligned}
Y &= \beta X^* + e \\
X &= X^* + v
\end{aligned}
$$

$$
\begin{cases}
Y: & \text{observed consumption} \\
X: & \text{observed income} \\
X^*: & \text{latent permanent income} \\
v: & \text{latent transitory income} \\
\beta: & \text{marginal propensity to consume}
\end{cases}
$$

- Maybe the most famous application of measurement error models

A canonical model of income dynamics: An example

- Permanent income: a random walk process
- Transitory income: an ARMA process

$$\begin{aligned} X_t &= X_t^* + v_t \\ X_t^* &= X_{t-1}^* + \eta_t \\ v_t &= \rho_t v_{t-1} + \lambda_t \epsilon_{t-1} + \epsilon_t \end{aligned}$$

$$\begin{cases} \eta_t : & \text{permanent income shock in period } t \\ \epsilon_t : & \text{transitory income shock} \\ X_t^* : & \text{latent permanent income} \\ v_t : & \text{latent transitory income} \end{cases}$$

- Can a sample of $\{X_t\}_{t=1,\ldots,T}$ uniquely determine distributions of latent variables η_t, ϵ_t, X_t^*, and v_t?

Road map

1. Empirical evidence on measurement error
2. Measurement models: observables vs unobservables
    - Definition of measurement and general framework
    - 2-measurement model
    - 2.1-measurement model
    - 3-measurement model
    - Dynamic measurement model
    - Estimation (closed-form, extremum, semiparametric)
    - Revealing unobservables by deep learning
3. Empirical applications with latent variables
    - Auctions with unobserved heterogeneity
    - Multiple equilibria in incomplete information games
    - Dynamic learning models
    - Effort and type in contract models
    - Unemployment and labor market participation
    - Cognitive and noncognitive skill formation
    - Matching models with latent indices
    - Income dynamics
4. Summary

Measurement error: empirical evidence and assumptions

- Kane, Rouse, and Staiger (1999): Self-reported education X conditional on true education X^*. (Data source: National Longitudinal Class of 1972 and Transcript data)

$f_{X\|X^*}(x_i\|x_j)$	X^* — true education level		
X — self-reported education	x_1–no college	x_2–some college	x_3–BA$^+$
x_1–no college	0.876	0.111	0.000
x_2–some college	0.112	0.772	0.020
x_3–BA$^+$	0.012	0.117	0.980

- Finding I: more likely to tell the truth than any other possible values

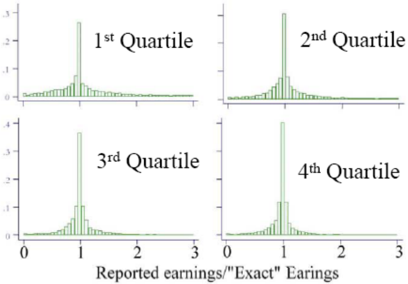
$$f_{X|X^*}(x^*|x^*) > f_{X|X^*}(x_i|x^*) \text{ for } x_i \neq x^*.$$

$\Longrightarrow$ error equals zero at the mode of $f_{X|X^*}(\cdot|x^*)$.

- Finding II: more likely to tell the truth than to lie. $f_{X|X^*}(x^*|x^*) > 0.5$.
$\Longrightarrow$invertibility of the matrix $\left[f_{X|X^*}(x_i|x_j)\right]_{i,j}$ in the table above.

Measurement error: empirical evidence and assumptions

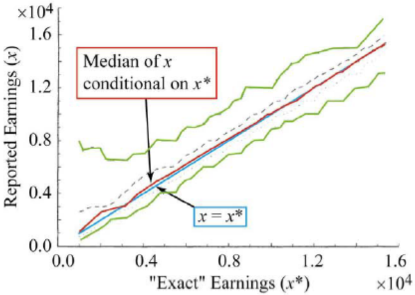
- Chen, Hong & Tarozzi (2005): ratio of self-reported earnings X vs. true earnings X^* by quartiles of true earnings. (Data source: 1978 CPS/SS Exact Match File)



- Finding I: distribution of measurement error depends on X^*.
- Finding II: distribution of measurement error has a zero mode.

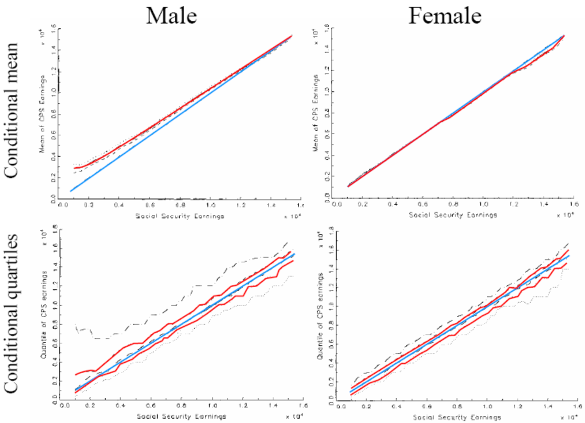
Measurement error: empirical evidence and assumptions

- Bollinger (1998, page 591): percentiles of self-reported earnings X given true earnings X^* for males. (Data source: 1978 CPS/SS Exact Match File)



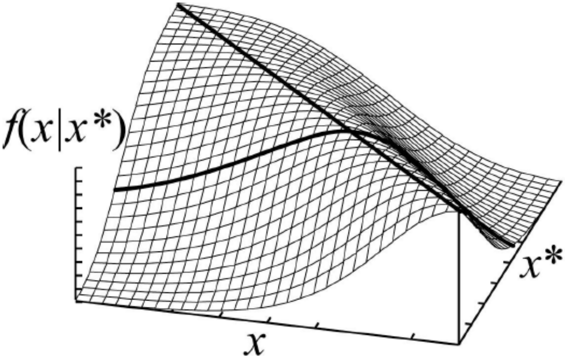
- Finding I: distribution of measurement error depends on X^*.
- Finding II: distribution of measurement error has a zero median.

- Self-reporting errors by gender



Source: Bollinger (1998) with data from Bound & Krueger (1991)

Graphical illustration of zero-mode measurement error



Latent variables in microeconomic models

empirical models	unobservables	observables
measurement error	true earnings	self-reported earnings
consumption function	permanent income	observed income
production function	productivity	output, input
wage function	ability	test scores
learning model	belief	choices, proxy
auction model	unobserved heterogeneity	bids
contract model	effort, type	outcome, state var.
...	...	...

Our definition of measurement

- X is defined as a measurement of X^* if

$$\text{cardinality of support}(X) \geq \text{cardinality of support}(X^*).$$

- there exists an injective function from support(X^*) into support(X).
- equality holds if there exists a bijective function between two supports.
- number of possible values of X is not smaller than that of X^*

X	X^*	
discrete $\{x_1, x_2, ..., x_L\}$	discrete $\{x_1^*, x_2^*, ..., x_K^*\}$	$L \geq K$
continuous	discrete $\{x_1^*, x_2^*, ..., x_K^*\}$	
continuous	continuous	

- $X - X^*$: measurement error (classical if independent of X^*)

A general framework

- observed & unobserved variables

X	measurement	observables
X^*	latent true variable	unobservables

- economic models described by distribution function f_{X^*}

$$f_X(x) = \int_{\mathcal{X}^*} f_{X|X^*}(x|x^*) f_{X^*}(x^*) dx^*$$

f_{X^*} : latent distribution
f_X : observed distribution
$f_{X|X^*}$: relationship between observables & unobservables

- identification: Does observed distribution f_X uniquely determine model of interest f_{X^*}?

Relationship between observables and unobservables

- discrete $X \in \{x_1, x_2, ..., x_L\}$ and $X^* \in \mathcal{X}^* = \{x_1^*, x_2^*, ..., x_K^*\}$

$$f_X(x) = \sum_{x^* \in \mathcal{X}^*} f_{X|X^*}(x|x^*) f_{X^*}(x^*),$$

- matrix expression

$$\begin{aligned} \overrightarrow{p}_X &= [f_X(x_1), f_X(x_2), ..., f_X(x_L)]^T \\ \overrightarrow{p}_{X^*} &= [f_{X^*}(x_1^*), f_{X^*}(x_2^*), ..., f_{X^*}(x_K^*)]^T \\ M_{X|X^*} &= \left[f_{X|X^*}(x_l|x_k^*)\right]_{l=1,2,...,L;k=1,2,...,K}. \end{aligned}$$

$$\overrightarrow{p}_X = M_{X|X^*} \overrightarrow{p}_{X^*}.$$

- given $M_{X|X^*}$, observed distribution f_X uniquely determine f_{X^*} if

$$Rank\left(M_{X|X^*}\right) = Cardinality\left(\mathcal{X}^*\right)$$

Identification and observational equivalence

- two possible marginal distributions $\overrightarrow{p}^{a}_{X^*}$ and $\overrightarrow{p}^{b}_{X^*}$ are observationally equivalent, i.e.,

$$\overrightarrow{p}_X = M_{X|X^*}\overrightarrow{p}^{a}_{X^*} = M_{X|X^*}\overrightarrow{p}^{b}_{X^*}$$

- that is, different unobserved distributions lead to the same observed distribution

$$M_{X|X^*}h = 0 \text{ with } h := \overrightarrow{p}^{a}_{X^*} - \overrightarrow{p}^{b}_{X^*}$$

- identification of f_{X^*} requires

$$M_{X|X^*}h = 0 \text{ implies } h = 0$$

that is, two observationally equivalent distributions are the same. This condition can be generalized to the continuous case.

Identification in the continuous case

- define a set of bounded and integrable functions containing f_{X^*}

$$\mathcal{L}^1_{bnd}(\mathcal{X}^*) = \left\{ h : \int_{\mathcal{X}^*} |h(x^*)|\, dx^* < \infty \text{ and } \sup_{x^* \in \mathcal{X}^*} |h(x^*)| < \infty \right\}$$

- define a linear operator

$$\begin{aligned} L_{X|X^*} &: \mathcal{L}^1_{bnd}(\mathcal{X}^*) \to \mathcal{L}^1_{bnd}(\mathcal{X}) \\ \left(L_{X|X^*} h\right)(x) &= \int_{\mathcal{X}^*} f_{X|X^*}(x|x^*) h(x^*) dx^* \end{aligned}$$

- operator equation

$$f_X = L_{X|X^*} f_{X^*}$$

- identification requires injectivity of $L_{X|X^*}$, i.e.,

$$L_{X|X^*} h = 0 \text{ implies } h = 0 \text{ for any } h \in \mathcal{L}^1_{bnd}(\mathcal{X}^*)$$

A 2-measurement model

- definition: two measurements X and Z satisfy

$$X \perp Z \mid X^*$$

- two measurements are independent conditional on the latent variable

$$f_{X,Z}(x,z) = \sum_{x^* \in \mathcal{X}^*} f_{X|X^*}(x|x^*) f_{Z|X^*}(z|x^*) f_{X^*}(x^*)$$

- matrix expression

$$\begin{aligned} M_{X,Z} &= \left[f_{X,Z}(x_l, z_j) \right]_{l=1,2,\ldots,L; j=1,2,\ldots,J} \\ M_{Z|X^*} &= \left[f_{Z|X^*}(z_j|x_k^*) \right]_{j=1,2,\ldots,J; k=1,2,\ldots,K} \\ D_{X^*} &= diag \left\{ f_{X^*}(x_1^*), f_{X^*}(x_2^*), \ldots, f_{X^*}(x_K^*) \right\} \end{aligned}$$

$$M_{X,Z} = M_{X|X^*} D_{X^*} M_{Z|X^*}^T$$

- suppose that matrices $M_{X|X^*}$ and $M_{Z|X^*}$ have a full rank, then

$$Rank\left(M_{X,Z}\right) = Cardinality\left(\mathcal{X}^*\right)$$

2-measurement model: binary case

- a binary latent regressor

$$
\begin{aligned}
Y &= \beta X^* + \eta \\
(X, X^*) &\perp \eta \\
X,\ X^* &\in \{0, 1\}
\end{aligned}
$$

- measurement error $X - X^*$ is correlated with X^* in general
- $f(y|x)$ is a mixture of $f_\eta(y)$ and $f_\eta(y-\beta)$

$$
\begin{aligned}
f(y|x) &= \sum_{x^*=0}^{1} f(y|x^*) f_{X^*|X}(x^*|x) \\
&= f_\eta(y) f_{X^*|X}(0|x) + f_\eta(y-\beta) f_{X^*|X}(1|x) \\
&\equiv f_\eta(y) P_x + f_\eta(y-\beta)(1-P_x)
\end{aligned}
$$

2-measurement model: binary case

- observed distributions $f(y|x=1)$ and $f(y|x=0)$ are mixtures of $f(y|x^*=1)$ and $f(y|x^*=0)$
  with different weights P_1 and P_2
- $$f(y|x=1) - f(y|x=0) = [f_\eta(y-\beta) - f_\eta(y)](P_0 - P_1)$$
- if $|P_0 - P_1| \leq 1$, then
  $$|f(y|x=1) - f(y|x=0)| \leq |f(y|x^*=1) - f(y|x^*=0)|$$
- leads to partial identification

2-measurement model: binary case

- parameter of interest

$$\beta = E(y|x^* = 1) - E(y|x^* = 0)$$

- bounds

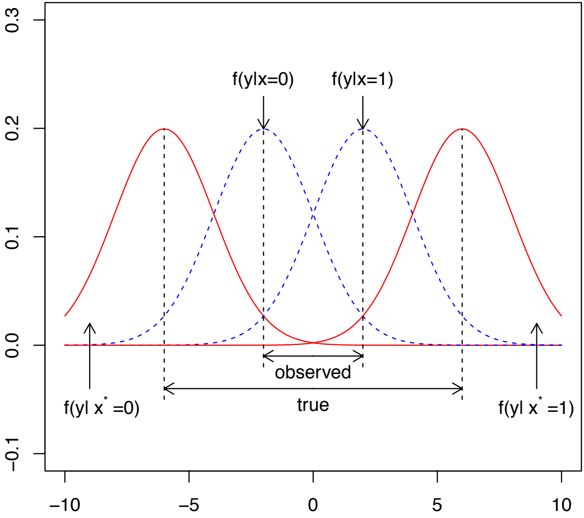
$$|\beta| \geq |E(y|x = 1) - E(y|x = 0)|$$

- If $\Pr(x^* = 0|x = 0) > \Pr(x^* = 0|x = 1)$, i.e., $P_0 - P_1 > 0$, then

$$sign\{\beta\} = sign\{E(y|x = 1) - E(y|x = 0)\}$$

2-measurement model: binary case

- measurement error causes attenuation



2-measurement model: discrete case

- a discrete latent regressor

$$
\begin{aligned}
Y &= m(X^*) + \eta \\
(X, X^*) &\perp \eta \\
X,\ X^* &\in \{x_1^*, x_2^*, ..., x_K^*\}
\end{aligned}
$$

- Chen Hu & Lewbel (2009): point identification generally holds
- general models without $(X, X^*) \perp \eta$: partial identification see Bollinger (1996) and Molinari (2008)

2-measurement model: linear model with classical error

- a simple linear regression model with zero means

$$\begin{aligned} Y &= \beta X^* + \eta \\ X &= X^* + \varepsilon \\ X^* &\perp \varepsilon \perp \eta \end{aligned}$$

- β is generally identified (from observed $f_{Y,X}$) except when X^* is normal (Reiersol 1950)

2-measurement model: Kotlarski's identity

- a useful special case: $\beta = 1$

$$
\begin{aligned}
Y &= X^* + \eta \\
X &= X^* + \varepsilon
\end{aligned}
$$

2-measurement model: Kotlarski's identity

- a useful special case: $\beta = 1$

$$\begin{aligned} Y &= X^* + \eta \\ X &= X^* + \varepsilon \end{aligned}$$

- distribution function & characteristic function of X^* $(i = \sqrt{-1})$

$$f_{X^*}(x^*) = \frac{1}{2\pi} \int e^{-ix^* t} \Phi_{X^*}(t) dt \qquad \Phi_{X^*} = E\left[e^{itX^*}\right]$$

2-measurement model: Kotlarski's identity

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- Kotlarski's identity (1966)

$$\Phi_{X^*}(t) = \exp\left[\int_0^t \frac{iE\left[Ye^{isX}\right]}{Ee^{isX}} ds\right]$$

2-measurement model: Kotlarski's identity

- a useful special case: $\beta = 1$

$$\begin{aligned} Y &= X^* + \eta \\ X &= X^* + \varepsilon \end{aligned}$$

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- Kotlarski's identity (1966)

$$\Phi_{X^*}(t) = \exp\left[\int_0^t \frac{iE\left[Ye^{isX}\right]}{Ee^{isX}} ds\right]$$

- latent distribution f_{X^*} (and f_η,f_ε) is uniquely determined by observed distribution $f_{Y,X}$ with a closed form

2-measurement model: Kotlarski's identity

- Kotlarski's identity (1966)

$$\Phi_{X^*}(t) = \exp\left[\int_0^t \frac{iE\left[Ye^{isX}\right]}{Ee^{isX}} ds\right]$$

- intuition:

$$Var(X^*) = Cov(Y, X)$$

- All the moments of X^* may be written as a function of joint moments of Y and X with a closed form.
- The joint distribution of two variables can uniquely determine the joint distribution of three variables

$$f_{X,Y} \text{ uniquely determines } f_{X,Y,X^*}$$
$$f_{X,Y,X^*}(y, x, x^*) = f_\eta(y - x^*) f_\varepsilon(x - x^*) f_{X^*}(x^*)$$

2-measurement model: nonlinear model with classical error

- a nonparametric regression model

$$\begin{aligned} Y &= g(X^*) + \eta \\ X &= X^* + \varepsilon \\ X^* &\perp \varepsilon \perp \eta \end{aligned}$$

- Schennach & Hu (2013 JASA): $g(\cdot)$ is generally identified except some parametric cases of g or f_{X^*}
- Under sufficient assumptions, observed density $f_{Y,X}$ uniquely determines f_{Y,X,X^*} through

$$f_{Y,X}(y,x) = \int_{\mathcal{X}^*} f_\eta(y - g(x^*)) f_\varepsilon(x - x^*) f_{X^*}(x^*) dx^*$$

  with a unique solution $(g, f_\eta, f_\varepsilon, f_{X^*})$.
- a generalization of Reiersol (1950, ECMA)
- 2-measurement model needs strong specification assumptions for nonparametric identification: additivity, independence

2-measurement model: nonlinear model with nonclassical error

- a nonparametric regression model

$$
\begin{aligned}
Y &= g(x^*) + \eta, \text{ with } X^* \perp \eta \\
X &\leftarrow X^* \\
X &\perp \eta \mid X^*
\end{aligned}
$$

- key assumption: $L_{X|X^*}$ is bijective.
- discrete X^* - Chen Hu & Lewbel (2009, Statistica Sinica). There are interesting results in the binary case (Chen et al, 2008)
- continuous X^* - Hu, Schennach, & Shiu (2021, JE): $g(\cdot)$ is generally identified
- Under sufficient assumptions, observed density $f_{Y,X}$ uniquely determines f_{Y,X,X^*} through

$$
f_{Y,X}(y,x) = \int_{\mathcal{X}^*} f_\eta(y - g(x^*)) f_{X|X^*}(x|x^*) f_{X^*}(x^*) dx^*
$$

with a unique solution $(g, f_\eta, f_{X|X^*}, f_{X^*})$.

2.1-measurement model

- "0.1 measurement" refers to a 0-1 dochotomous indicator Y of X^*
- definition of 2.1-measurement model:
  two measurements X and Z and a 0-1 indicator Y satisfy

$$f_{X,Y,Z}(x,y,z) = \sum_{x^* \in \mathcal{X}^*} f_{X|X^*}(x|x^*) f_{Y|X^*}(y|x^*) f_{Z|X^*}(z|x^*) f_{X^*}(x^*)$$

- an important message: adding "0.1 measurement" in a 2-measurement model is enough for nonparametric identification, i.e., under mild conditions,

$$f_{X,Y,Z} \text{ uniquely determines } f_{X,Y,Z,X^*}$$

- a global nonparametric point identification
  (exact identification if $J = K = L$)
- extension: identification results still hold with a non-binary Y

Identification: discrete case (Hu, 2008, JE)

- Let $X, X^* \in \{x_1, x_2, x_3\}$ and $Z \in \{z_1, z_2, z_3\}$, e.g., education levels.

$$M_{X|X^*} = \begin{pmatrix} f_{X|X^*}(x_1|x_1) & f_{X|X^*}(x_1|x_2) & f_{X|X^*}(x_1|x_3) \\ f_{X|X^*}(x_2|x_1) & f_{X|X^*}(x_2|x_2) & f_{X|X^*}(x_2|x_3) \\ f_{X|X^*}(x_3|x_1) & f_{X|X^*}(x_3|x_2) & f_{X|X^*}(x_3|x_3) \end{pmatrix} \Longleftarrow \text{error structure}$$

$$M_{X^*|Z} = \begin{pmatrix} f_{X^*|Z}(x_1|z_1) & f_{X^*|Z}(x_1|z_2) & f_{X^*|Z}(x_1|z_3) \\ f_{X^*|Z}(x_2|z_1) & f_{X^*|Z}(x_2|z_2) & f_{X^*|Z}(x_2|z_3) \\ f_{X^*|Z}(x_3|z_1) & f_{X^*|Z}(x_3|z_2) & f_{X^*|Z}(x_3|z_3) \end{pmatrix} \Longleftarrow \text{IV structure}$$

$$D_{y|X^*} = \begin{pmatrix} f_{Y|X^*}(y|x_1) & 0 & 0 \\ 0 & f_{Y|X^*}(y|x_2) & 0 \\ 0 & 0 & f_{Y|X^*}(y|x_3) \end{pmatrix} \Longleftarrow \text{latent model}$$

$$M_{y;X|Z} = \begin{pmatrix} f_{Y,X|Z}(y, x_1|z_1) & f_{Y,X|Z}(y, x_1|z_2) & f_{Y,X|Z}(y, x_1|z_3) \\ f_{Y,X|Z}(y, x_2|z_1) & f_{Y,X|Z}(y, x_2|z_2) & f_{Y,X|Z}(y, x_2|z_3) \\ f_{Y,X|Z}(y, x_3|z_1) & f_{Y,X|Z}(y, x_3|z_2) & f_{Y,X|Z}(y, x_3|z_3) \end{pmatrix} \Longleftarrow \text{observed info.}$$

- $M_{y;X|Z}$ contains the same information as $f_{Y,X|Z}$.

Matrix equivalence

- The main equation for a given y

$$\boxed{f_{Y,X|Z}\left(y,x|z\right)=\sum_{x^*} f_{X|X^*}(x|x^*)f_{Y|X^*}(y|x^*)f_{X^*|Z}(x^*|z)}$$

$$\Updownarrow$$

$$\boxed{M_{y;X|Z}=M_{X|X^*}D_{y|X^*}M_{X^*|Z}}$$

Matrix equivalence

- The main equation for a given y

$$\boxed{f_{Y,X|Z}\left(y,x|z\right)=\sum_{x^*}f_{X|X^*}(x|x^*)f_{Y|X^*}(y|x^*)f_{X^*|Z}(x^*|z)}$$

$$\Updownarrow$$

$$\boxed{M_{y;X|Z}=M_{X|X^*}D_{y|X^*}M_{X^*|Z}}$$

- Similarly,

$$\boxed{f_{X|Z}\left(x|z\right)=\sum_{x^*}f_{X|X^*}(x|x^*)f_{X^*|Z}(x^*|z)}$$

$$\Updownarrow$$

$$\boxed{M_{X|Z}=M_{X|X^*}M_{X^*|Z}}$$

Matrix equivalence

- The main equation for a given y

$$\boxed{f_{Y,X|Z}(y,x|z) = \sum_{x^*} f_{X|X^*}(x|x^*) f_{Y|X^*}(y|x^*) f_{X^*|Z}(x^*|z)}$$

$$\Updownarrow$$

$$\boxed{M_{y;X|Z} = M_{X|X^*} D_{y|X^*} M_{X^*|Z}}$$

- Similarly,

$$\boxed{f_{X|Z}(x|z) = \sum_{x^*} f_{X|X^*}(x|x^*) f_{X^*|Z}(x^*|z)}$$

$$\Updownarrow$$

$$\boxed{M_{X|Z} = M_{X|X^*} M_{X^*|Z}}$$

- Eliminate $M_{X^*|Z}$

$$\begin{aligned} M_{y;X|Z} M_{X|Z}^{-1} &= \left(M_{X|X^*} D_{y|X^*} M_{X^*|Z}\right) \times \left(M_{X^*|Z}^{-1} M_{X|X^*}^{-1}\right) \\ &= M_{X|X^*} D_{y|X^*} M_{X|X^*}^{-1}. \end{aligned}$$

An inherent matrix diagonalization

- An eigenvalue-eigenvector decomposition:

$$
\begin{aligned}
M_{y;X|Z} M_{X|Z}^{-1} &= M_{X|X^*} D_{y|X^*} M_{X|X^*}^{-1} \\
&= \begin{pmatrix} f_{X|X^*}(x_1|x_1) & f_{X|X^*}(x_1|x_2) & f_{X|X^*}(x_1|x_3) \\ f_{X|X^*}(x_2|x_1) & f_{X|X^*}(x_2|x_2) & f_{X|X^*}(x_2|x_3) \\ f_{X|X^*}(x_3|x_1) & f_{X|X^*}(x_3|x_2) & f_{X|X^*}(x_3|x_3) \end{pmatrix} \\
&\quad \times \begin{pmatrix} f_{Y|X^*}(y|x_1) & 0 & 0 \\ 0 & f_{Y|X^*}(y|x_2) & 0 \\ 0 & 0 & f_{Y|X^*}(y|x_3) \end{pmatrix} \\
&\quad \times \begin{pmatrix} f_{X|X^*}(x_1|x_1) & f_{X|X^*}(x_1|x_2) & f_{X|X^*}(x_1|x_3) \\ f_{X|X^*}(x_2|x_1) & f_{X|X^*}(x_2|x_2) & f_{X|X^*}(x_2|x_3) \\ f_{X|X^*}(x_3|x_1) & f_{X|X^*}(x_3|x_2) & f_{X|X^*}(x_3|x_3) \end{pmatrix}^{-1}
\end{aligned}
$$

- For $\clubsuit \in \{x_1, x_2, x_3\}$, i.e., an index of eigenvalues and eigenvectors:
  - eigenvalues: $f_{Y|X^*}(y|\clubsuit)$
  - eigenvectors: $\left[f_{X|X^*}(x_1|\clubsuit),\ f_{X|X^*}(x_2|\clubsuit),\ f_{X|X^*}(x_3|\clubsuit) \right]^T$

Ambiguity Inside the decomposition

- Ambiguity in indexing eigenvalues and eigenvectors, i.e.,

$$\{\clubsuit, \heartsuit, \spadesuit\} \overset{\text{1-to-1}}{\Longleftrightarrow} \{x_1, x_2, x_3\}$$

- Decompositions with different indexing are observationally equivalent,

$$\begin{aligned}
M_{y;X|Z} M_{X|Z}^{-1} &= M_{X|X^*} D_{y|X^*} M_{X|X^*}^{-1} \\
&= \begin{pmatrix} f_{X|X^*}(x_1|\clubsuit) & f_{X|X^*}(x_1|\heartsuit) & f_{X|X^*}(x_1|\spadesuit) \\ f_{X|X^*}(x_2|\clubsuit) & f_{X|X^*}(x_2|\heartsuit) & f_{X|X^*}(x_2|\spadesuit) \\ f_{X|X^*}(x_3|\clubsuit) & f_{X|X^*}(x_3|\heartsuit) & f_{X|X^*}(x_3|\spadesuit) \end{pmatrix} \\
&\quad \times \begin{pmatrix} f_{Y|X^*}(y|\clubsuit) & 0 & 0 \\ 0 & f_{Y|X^*}(y|\heartsuit) & 0 \\ 0 & 0 & f_{Y|X^*}(y|\spadesuit) \end{pmatrix} \\
&\quad \times \begin{pmatrix} f_{X|X^*}(x_1|\clubsuit) & f_{X|X^*}(x_1|\heartsuit) & f_{X|X^*}(x_1|\spadesuit) \\ f_{X|X^*}(x_2|\clubsuit) & f_{X|X^*}(x_2|\heartsuit) & f_{X|X^*}(x_2|\spadesuit) \\ f_{X|X^*}(x_3|\clubsuit) & f_{X|X^*}(x_3|\heartsuit) & f_{X|X^*}(x_3|\spadesuit) \end{pmatrix}^{-1}
\end{aligned}$$

- Identification of $f_{X|X^*}$ boils down to identification of symbols $\clubsuit$, $\heartsuit$, $\spadesuit$.

Restrictions on eigenvalues and eigenvectors

- Eigenvalues are distinct if x^* is relevant, i.e.,
  – $f_{Y|X^*}(y|x_i) \neq f_{Y|X^*}(y|x_j)$ with $x_i \neq x_j$ for some y.
- Symbols $\clubsuit$, $\heartsuit$, $\spadesuit$ are identified under zero-mode assumption.

– For example, error distribution $f_{X|X^*}$ is the same as in Kane et al (1999).

$$\begin{array}{r} \text{no clg.} - x_1: \\ \text{some clg.} - x_2: \\ \text{BA}^+ - x_3: \end{array} \begin{pmatrix} f_{X|X^*}(x_1|\clubsuit) \\ f_{X|X^*}(x_2|\clubsuit) \\ f_{X|X^*}(x_3|\clubsuit) \end{pmatrix} = \begin{pmatrix} 0.111 \\ 0.772 \\ 0.117 \end{pmatrix}$$

$\Downarrow$

$x_2 = \arg\max_{x_i} f_{X|X^*}(x_i|\clubsuit)$
"x_2 is the mode"

zero-mode assumption

$\Downarrow$

$\arg\max_{x_i} f_{X|X^*}(x_i|\clubsuit) = \clubsuit$
"truth at the mode"

$$\clubsuit = x_2 \text{ (some college)}$$

- Similarly, we can identify $\heartsuit$ and $\spadesuit$.
  $\Longrightarrow$ The model $f_{Y|X^*}$ and the error structure $f_{X|X^*}$ are identified.

Uniqueness of the eigen decomposition

- uniqueness of the eigenvalue-eigenvector decomposition (Hu 2008 JE)
  1. distinct eigenvalues: $\exists$ a nonempty set of y, s.t.,
  $f_{Y|X^*}(y|x_1^*) \neq f_{Y|X^*}(y|x_2^*)$ for any $x_1^* \neq x_2^*$
  2. eigenvectors are colums in $M_{X|X^*}$, i.e., $f_{X|X^*}(\cdot|x^*)$. A natural normalization is $\sum_x f_{X|X^*}(x|x^*) = 1$ for all x^*
  3. ordering of the eigenvalues or eigenvectors
  That is to reveal the value of x^* for either $f_{X|X^*}(\cdot|x^*)$ or $f_{Y|X^*}(y|x^*)$ from one of below
     a. x^* is the mode of $f_{X|X^*}(\cdot|x^*)$: very intuitive, people are more likely to tell the truth; consistent with validation study
     b. x^* is a quantile of $f_{X|X^*}(\cdot|x^*)$: useful in some applications
     c. x^* is the mean of $f_{X|X^*}(\cdot|x^*)$: useful when x^* is continuous
     d. $E(g(y)|x^*)$ is increasing in x^* for a known g, say $\Pr(y > 0|x^*)$

Nonparametric identification in Hu (2008)

Assumptions:

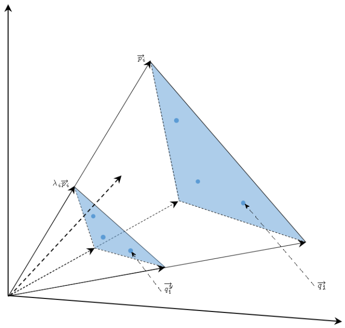
- (conditional independence)

$$f_{X,Y,Z,X^*} = f_{X|X^*} f_{Y|X^*} f_{Z|X^*} f_{X^*}.$$

- (measurements) X, Z, X^* share same number of possible values, K
- (full rank) matrix $M_{X,Z}$ has rank K.
- (distinct eigenvalues) $\exists\ \omega(\cdot)$ such that $E\left[\omega(Y)\,|X^* = x_j^*\right]$ is different for different j.
- (ordering condition) one of the following conditions holds:
  1) $f_{X|X^*}(x^*|x^*) > f_{X|X^*}(\widetilde{x}|x^*)$ for any $\widetilde{x} \neq x^* \in \mathcal{X}^*$;
  2) $f_{X|X^*}\left(x_1|x_j^*\right) > f_{X|X^*}\left(x_1|x_{j+1}^*\right)$ for some x_1;
  3) $\exists\ \omega(\cdot)$ such that $E\left[\omega(Y)\,|X^* = x_j^*\right]$ is increasing in j.

Assumptions above imply

$$f_{X,Y,Z} \text{ uniquely determines } f_{X,Y,Z,X^*}$$



Eigen-decomposition in the 2.1-measurement model

- Eigenvalue: $\lambda_i = f_{Y|X^*}(1|x_i^*)$
- Eigenvector: $\vec{p_i} = \vec{p}_{X|x_i^*} = \left[f_{X|X^*}(x_1|x_i^*), f_{X|X^*}(x_2|x_i^*), f_{X|X^*}(x_3|x_i^*)\right]^T$
- Observed distribution in the whole sample: $\vec{q}_1 = \vec{p}_{X|z_1} = \left[f_{X|Z}(x_1|z_1), f_{X|Z}(x_2|z_1), f_{X|Z}(x_3|z_1)\right]^T$
- Observed distribution in the subsample with $Y = 1$:
  $\vec{q}_1^{\,y} = \vec{p}_{y_1,X|z_1} = \left[f_{Y,X|Z}(1, x_1|z_1), f_{Y,X|Z}(1, x_2|z_1), f_{Y,X|Z}(1, x_3|z_1)\right]^T$

Discrete case without ordering conditions: finite mixture

- conditional independence with general discrete X, Y, Z, and X^* (Allman, Matias and Rhodes, 2009, Ann Stat)
- advantages:
  1. cardinality of X^* can be larger than that of X or Z or both
  2. a lower bound on the so-called Kruskal rank is sufficient for identification up to permutation. (but ordering is innocuous)
- disadvantages:
  1. Kruskal rank is hard to interpret in economic models, not testable as regular rank
  2. not clear how to extend to the continuous case
- cf. classic local parametric identification condition:
  Number of restrictions $\geqslant$ Number of unknowns
- cf. 2.1 measurement model:
  1. reach the lower bound on the Kruskal rank: $2 \times Cardinality\left(\mathcal{X}^*\right) + 2$
  2. directly extend to the continuous case
  3. values of X^* may have economic meaning

2.1-measurement model: continuous case

- X, Z, and X^* are continuous

$$f_{Y,X,Z}(y,x,z) = \int f_{X|X^*}(x|x^*) f_{Y|X^*}(y|x^*) f_{X^*,Z}(x^*,z) dx^*$$

- share the same idea as the discrete case in Hu (2008)
- from matrix to integral operator

diagonal matrix	$\Rightarrow$	"diagonal" operator (multiplication)
matrix diagonalization	$\Rightarrow$	spectral decomposition
eigenvector	$\Rightarrow$	eigenfunction

- nontrivial extension, highly technical
- Hu & Schennach (2008, ECMA)

From conditional density to integral operator

- From 2-variable function to an integral operator

$$f_{X|X^*}(\cdot|\cdot)$$
$$\Downarrow$$
$$\left(L_{X|X^*}g\right)(x) = \int f_{X|X^*}(x|x^*)\, g(x^*)\, dx^* \quad \text{for any } g.$$

- Operator $L_{X|X^*}$ transforms unobserved f_{X^*} to observed f_X, i.e., $f_X = L_{X|X^*} f_{X^*}$.

$$\left(\begin{array}{c} f_{X^*}(x^*) \\ \text{distribution of } x^* \end{array} \right) \overset{L_{X|X^*}}{\Longrightarrow} \left(\begin{array}{c} f_X(x) \\ \text{distribution of } x \end{array} \right)$$

- $f_{X|X^*}(\cdot|\cdot)$ is called the *kernel* function of $L_{X|X^*}$.

Identification: from matrix to integral operator

- From matrix to integral operator

$$\begin{aligned} L_{y;X|Z}g &= \int f_{Y,X|Z}(y,\cdot|z)\, g(z)\, dz \\ L_{X|Z}g &= \int f_{X|Z}(\cdot|z)\, g(z)\, dz \\ L_{X|X^*}g &= \int f_{X|X^*}(\cdot|x^*)\, g(x^*)\, dx^* \\ L_{X^*|Z}g &= \int f_{X^*|Z}(\cdot|z)\, g(z)\, dz \\ D_{y;X^*|X^*}g &= f_{Y|X^*}(y|\cdot)\, g(\cdot)\ . \end{aligned}$$

- $L_{y;X|Z}$: y viewed as a fixed parameter.
- $D_{y;X^*|X^*}$: "diagonal" operator (multiplication by a function).

Identification: operator equivalence

- The main equation

$$L_{y;X|Z} = L_{X|X^*} D_{y;X^*|X^*} L_{X^*|Z}.$$

– for a function g,

$$\begin{aligned}
\left[L_{y;X|Z}g\right](x) &= \int f_{Y,X|Z}(y,x|z)\, g(z)\, dz \\
&= \int\int f_{X|X^*}(x|x^*)\, f_{Y|X^*}(y|x^*)\, f_{X^*|Z}(x^*|z)\, dx^* g(z)\, dz \\
&= \int f_{X|X^*}(x|x^*)\, f_{Y|X^*}(y|x^*) \int f_{X^*|Z}(x^*|z)\, g(z)\, dz dx^* \\
&= \int f_{X|X^*}(x|x^*)\, f_{Y|X^*}(y|x^*) \left[L_{X^*|Z}g\right](x^*)\, dx^* \\
&= \int f_{X|X^*}(x|x^*) \left[D_{y;X^*|X^*} L_{X^*|Z}g\right](x^*)\, dx^* \\
&= \left[L_{X|X^*} D_{y;X^*|X^*} L_{X^*|Z}g\right](x).
\end{aligned}$$

- Similarly,

$$L_{X|Z} = L_{X|X^*} L_{X^*|Z}.$$

Identification: a necessary condition on error distribution

- Intuition: if $f_{X|X^*}$ is known, we want f_{X^*} to be identifiable from f_X.

  – That is, if f_{X^*} and $\widetilde{f}_{X^*}$ are observationally equivalent as follows:

$$f_X(x) = \int f_{X|X^*}(x|x^*) f_{X^*}(x^*)\, dx^* = \int f_{X|X^*}(x|x^*) \widetilde{f}_{X^*}(x^*)\, dx^*,$$

  then $f_{X^*} = \widetilde{f}_{X^*}$.

  – In other words, let $h = f_{X^*} - \widetilde{f}_{X^*}$, we want

$$\int f_{X|X^*}(x|x^*) h(x^*)\, dx^* = 0 \text{ for all } x \implies h = 0.$$

- An equivalent condition:

  – **Assumption 2(i)**: $L_{X|X^*}$ *is injective.*

- Implications:

  – Inverse $L^{-1}_{X|X^*}$ exists on its domain. $L^{-1}_{X|X^*} \times L_{X|X^*} = I_{X^*|X^*}$

  – Assumption 2(i) is implied by *bounded completeness* of $f_{X|X^*}$, e.g., exponential family.

A necessary condition on instrumental variable

- This is related to nonparametric identification with IV

$$\int f_{X^*|Z}(x^*|z)h\left(x^*\right)dx^* = 0 \text{ for all } z \implies h = 0$$

- Implications:
  – Used in Newey&Powell (2003), Darolles Florens&Renault (2005).
  – It is a necessary condition to achieve point identification using IV.
  – Implied by the bounded completeness of $f_{X^*|Z}$, e.g., exponential family.
- Here $L_{X|Z} = L_{X|X^*}L_{X^*|Z}$ and $L_{X|X^*}$ is injective, $L_{X^*|Z} = L^{-1}_{X|X^*}L_{X|Z}$.
- We will need the right inverse of $L_{X|Z}$, i.e., $L_{X|Z} \times L^{-1}_{X|Z} = I_{X|X}$, which is implied by:
  – **Assumption 2(ii)**: *$L_{z|x}$ is injective.*

An inherent spectral decomposition

- left inverse $L^{-1}_{X|X^*}$ and right inverse $L^{-1}_{X|Z}$ exist
  $\Longrightarrow$ an inherent spectral decomposition

$$\begin{aligned} L^{-1}_{X|X^*} L_{X|Z} &= L^{-1}_{X|X^*} \left(L_{X|X^*} L_{X^*|Z} \right) \\ &= L_{X^*|Z} \end{aligned}$$

$$\begin{aligned} L_{y;X|Z} L^{-1}_{X|Z} &= \left(L_{X|X^*} D_{y;X^*|X^*} L_{X^*|Z} \right) \times L^{-1}_{X|Z} \\ &= \left(L_{X|X^*} D_{y;X^*|X^*} \left(L^{-1}_{X|X^*} L_{X|Z} \right) \right) \times L^{-1}_{X|Z} \\ &= L_{X|X^*} D_{y;X^*|X^*} L^{-1}_{X|X^*}. \end{aligned}$$

- An eigenvalue-eigenfunction decomposition of an observed operator on LHS
  – Eigenvalues: $f_{Y|X^*}(y|x^*)$, kernel of $D_{y;X^*|X^*}$.
  – Eigenfunctions: $f_{X|X^*}(\cdot|x^*)$, kernel of $L_{X|X^*}$.

Identification: uniqueness of the decomposition

- **Assumption 3**: $\sup_{y\in\mathcal{Y}} \sup_{x^*\in\mathcal{X}^*} f_{Y|X^*}(y|x^*) < \infty$.
  $\Longrightarrow$ boundedness of $L_{y;X|Z} L_{X|Z}^{-1}$, the observed operator on the LHS.
- Theorem XV.4.5 in Dunford & Schwartz (1971):
  The representation of a bounded linear operator as a "weighted sum of projections" is unique.
- Each "eigenvalue" $\lambda = f_{Y|X^*}(y|x^*)$ is the weight assigned to the projection onto a linear subspace $S(\lambda)$ spanned by the corresponding "eigenfunction(s)" $f_{X|X^*}(\cdot|x^*)$.
- However, there are ambiguities inside "weighted sum of projections".
  $\Longrightarrow$ We need to "freeze" these degrees of freedom to show that $L_{X|X^*}$ and $D_{y;X^*|X^*}$ are uniquely determined by $L_{y;X|Z} L_{X|Z}^{-1}$.

A close look at weighted sum of projections

- Discrete case:

$$
\begin{aligned}
L_{y;X|Z}L_{X|Z}^{-1} &= L_{X|X^*}D_{y;X^*|X^*}L_{X|X^*}^{-1} \\
&= f_{Y|X^*}(y|x_1) \times L_{X|X^*}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} L_{X|X^*}^{-1} \\
&+ f_{Y|X^*}(y|x_2) \times L_{X|X^*}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} L_{X|X^*}^{-1} \\
&+ f_{Y|X^*}(y|x_3) \times L_{X|X^*}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} L_{X|X^*}^{-1}
\end{aligned}
$$

- Continuous case:

$$
L_{y;X|Z}L_{X|Z}^{-1} = \int_{\sigma} \lambda P(d\lambda)
$$

Identification: uniqueness of the decomposition

- **Ambiguity I**: Eigenfunctions $f_{X|X^*}(\cdot|x^*)$ are defined only up to a constant:
  – Solution: Constant determined by $\int f_{X|X^*}(x|x^*)\,dx = 1$.
  – Intuition: Eigenfunctions are conditional densities, therefore, are automatically normalized.
- **Ambiguity II**: If λ is a degenerate eigenvalue, more than one possible eigenfunctions.
  – Solution: **Assumption 4**: *for all* $x_1^*, x_2^* \in \mathcal{X}^*$, *the set*

$$\left\{y : f_{Y|X^*}(y|x_1^*) \neq f_{Y|X^*}(y|x_2^*)\right\}$$

  has positive probability whenever $x_1^* \neq x_2^*$.
  – Intuition: eigenvalues $f_{Y|X^*}(y_1|x^*)$ and $f_{Y|X^*}(y_2|x^*)$ share the same eigenfunction $f_{X|X^*}(\cdot|x^*)$. Therefore, y is helpful to distinguish eigenfunctions.
  – Note: this assumption is weaker than (or implied by) the monotonicity assumptions typically made in the nonseparable error literature

Identification: uniqueness of the decomposition

- **Ambiguity III**: Freedom in indexing eigenvalues: e.g., use x^* or $(x^*)^3$?
  – Solution: the zero "location" assumption, i.e., **Assumption 5:** *there exists a known functional M such that $x^* = M\left[f_{X|X^*}(\cdot|x^*)\right]$ for all x^*.*
- Examples of M

error has a zero mean:	$M[f] = \int xf(x)dx$ (thus, allow classical error)
error has a zero mode:	$M[f] = \arg\max_x f(x)$
error has a zero τ-th quantile:	$M[f] = \inf\left\{x^* : \int 1(x \leq x^*) f(x)dx \geq \tau\right\}$

- Importance: this assumption is based on the findings from validation studies.

The Hu-Schennach Theorem

- key identification conditions:
  1) (X, Y, Z) are independent conditional on X^*, i.e.,
  $$f_{X,Y,Z,X^*} = f_{X|X^*} f_{Y|X^*} f_{Z|X^*} f_{X^*}$$
  2) all densities are bounded. The operators $L_{X|X^*}$ and $L_{Z|X}$ are injective.
  3) for all $\overline{x}^* \neq \widetilde{x}^*$ in $\mathcal{X}^*$, the set $\left\{y : f_{Y|X^*}\left(y|\overline{x}^*\right) \neq f_{Y|X^*}\left(y|\widetilde{x}^*\right)\right\}$ has positive probability.
  4) there exists a known functional M such that $M\left[f_{X|X^*}\left(\cdot|x^*\right)\right] = x^*$ for all $x^* \in \mathcal{X}^*$.
- $$f_{X,Y,Z} \text{ uniquely determines } f_{X,Y,Z,X^*}$$
- a global nonparametric point identification
- 2.1-measurement model is identified even in the continuous case.
- extension: identification results still hold with a non-binary Y

3-measurement model

- definition: three measurements X, Y, and Z satisfy

$$X \perp Y \perp Z \mid X^*$$

- can always be reduced to a 2.1-measurement model.
  all the identification conditions remain with a general $\mathcal{Y}$.
- doesn't matter which is called dependent variable, measurement, or instrument.
- examples:
  Hausman Newey & Ichimura (1991)
  add $X^* = \gamma Z + u$, Z instrument, $g(\cdot)$ is a polynomial
  Schennach (2004): use a repeated measurement $X_2 = X^* + \varepsilon_2$
  general $g(\cdot)$, use ch.f. Kotlarski's identity
  Schennach (2007): use IV: $X^* = \gamma Z + u \quad u \perp Z$
  general $g(\cdot)$, use ch.f. similar to Kotlarski's identity

Hidden Markov model: a 3-measurement model

- an unobserved Markov process

$$X^*_{t+1} \perp \{X^*_s\}_{s \leq t-1} \mid X^*_t.$$

- a measurement X_t of the latent X^*_t satisfying

$$X_t \perp \{X_s, X^*_s\}_{s \neq t} \mid X^*_t.$$

- a hidden Markov model

$$\begin{array}{ccccccc} & X_{t-1} & & X_t & & X_{t+1} & \\ & \uparrow & & \uparrow & & \uparrow & \\ \longrightarrow & X^*_{t-1} & \longrightarrow & X^*_t & \longrightarrow & X^*_{t+1} & \longrightarrow \end{array}$$

- a 3-measurement model

$$X_{t-1} \perp X_t \perp X_{t+1} \mid X^*_t,$$

dynamic measurement model

- $\{X_t, X_t^*\}$ is a first-order Markov process satisfying

$$f_{X_t,X_t^*|X_{t-1},X_{t-1}^*} = f_{X_t|X_t^*,X_{t-1}} f_{X_t^*|X_{t-1},X_{t-1}^*}.$$

- Flow of chart

$$\begin{array}{ccccccccc}
\longrightarrow & X_{t-2} & \longrightarrow & X_{t-1} & \longrightarrow & X_t & \longrightarrow & X_{t+1} & \longrightarrow \\
\searrow & \updownarrow & \searrow & \updownarrow & \searrow & \updownarrow & \searrow & \updownarrow & \searrow \\
\longrightarrow & X_{t-2}^* & \longrightarrow & X_{t-1}^* & \longrightarrow & X_t^* & \longrightarrow & X_{t+1}^* & \longrightarrow
\end{array}$$

- Hu & Shum (2012, JE): nonparametric identification of the joint process
- Special case with $X_t^* = X_{t-1}^*$ needs 4 periods of data.
  cf. 6 periods with discrete X^* in Kasahara and Shimotsu (2009)

dynamic measurement model

- Hu & Shum (2012): nonparametric identification of the joint process. (use Carroll Chen & Hu (2010, JNPS))
- key identification assumptions:
  1) for any $x_{t-1} \in \mathcal{X}$, $M_{X_t|x_{t-1},X_{t-2}}$ is invertible.
  2) for any $x_t \in \mathcal{X}$, there exists a $(x_{t-1}, \overline{x}_{t-1}, \overline{x}_t)$ such that $M_{X_{t+1},x_t|x_{t-1},X_{t-2}}$, $M_{X_{t+1},x_t|\overline{x}_{t-1},X_{t-2}}$, $M_{X_{t+1},\overline{x}_t|x_{t-1},X_{t-2}}$, and $M_{X_{t+1},\overline{x}_t|\overline{x}_{t-1},X_{t-2}}$ are invertible and that for all $x_t^* \neq \widetilde{x}_t^*$ in $\mathcal{X}^*$

  $$\Delta_{x_t}\Delta_{x_{t-1}} \ln f_{X_t|X_t^*,X_{t-1}}\left(x_t^*\right) \neq \Delta_{x_t}\Delta_{x_{t-1}} \ln f_{X_t|X_t^*,X_{t-1}}\left(\widetilde{x}_t^*\right)$$

  3) for any $x_t \in \mathcal{X}$, $E\left[X_{t+1}|X_t = x_t, X_t^* = x_t^*\right]$ is increasing in x_t^*.
- joint distribution of five periods of data $f_{X_{t+1},X_t,X_{t-1},X_{t-2},X_{t-3}}$ uniquely determines Markov transition kernel $f_{X_t,X_t^*|X_{t-1},X_{t-1}^*}$

Other approaches: use a secondary sample

- $\{Y, X\}, \{X^*\}$ (administrative sample) Hu & Ridder (2012)
- $\{Y, X\}, \{X, X^*\}$ (validation sample) Chen, Hong & Tamer (2005) among many other papers in econometrics & statistics
- $\{Y, X, W\}, \{Y_a, X_a, W_a\}$ (auxiliary survey sample) Carroll, Chen & Hu (2010) with model of interest $f(Y|X^*, W) = f(Y_a|X_a^*, W_a)$
- also related to literature on missing data, where X^* can be considered as missing

Estimation: discrete case

- Estimate the matrices directly

$$L_{y;X,Z} = \begin{pmatrix} f_{Y,X,Z}(y,x_1,z_1) & f_{Y,X,Z}(y,x_1,z_2) & f_{Y,X,Z}(y,x_1,z_3) \\ f_{Y,X,Z}(y,x_2,z_1) & f_{Y,X,Z}(y,x_2,z_2) & f_{Y,X,Z}(y,x_2,z_3) \\ f_{Y,X,Z}(y,x_3,z_1) & f_{Y,X,Z}(y,x_3,z_2) & f_{Y,X,Z}(y,x_3,z_3) \end{pmatrix}$$

- Use sample proportion
- Use kernel density estimator with continuous covariates
- Identification is globe, nonparametric, and constructive
- Mimic identification procedure:

  a unique mapping from $f_{Y,X,Z}$ to $f_{Y|X^*}$, $f_{X|X^*}$, and $f_{X^*,Z}$
- Easy to compute without optimization or iteration
- May have problems with a small sample: estimated prob outside [0,1]

Estimation: discrete case

- Eigen decomposition holds after averaging over Y with a known $\omega(.)$

$$E[\omega(Y)|X=x,Z=z]f_{X,Z}(x,z)=\sum_{x^*\in\mathcal{X}^*}f_{X|X^*}(x|x^*)E[\omega(Y)|x^*]f_{Z|X^*}(z|x^*)f_{X^*}(x^*)$$

- Define

$$\begin{aligned} M_{X,\omega,Z} &= \left[E[\omega(Y)|X=x_k,Z=z_l]f_{X,Z}(x_k,z_l)\right]_{k=1,2,...,K;l=1,2,...,K} \\ D_{\omega|X^*} &= diag\left\{E[\omega(Y)|x_1^*],E[\omega(Y)|x_2^*],\ldots,E[\omega(Y)|x_K^*]\right\} \end{aligned}$$

- $$M_{X,\omega,Z}M_{X,Z}^{-1}=M_{X|X^*}D_{\omega|X^*}M_{X|X^*}^{-1}$$

- The matrix $M_{X,\omega,Z}$ can be directly estimated as

$$\widehat{M_{X,\omega,Z}}=\left[\frac{1}{N}\sum_{i=1}^{N}\omega(Y_i)\mathbf{1}(X_i=x_k,Z_i=z_l)\right]_{k=1,2,...,K;l=1,2,...,K}$$

- Estimation mimics identification procedure

Estimation: discrete case

- May also use extremum estimator with restrictions

$$\left(\widehat{M_{X|X^*}}, \widehat{D_{\omega|X^*}}\right) = \arg\min_{M,D} \left\| \widehat{M_{X,\omega,Z}} \left(\widehat{M_{X,Z}}\right)^{-1} M - M \times D \right\|$$

such that

1) each entry in M is in $[0, 1]$

2) each column sum of M equals 1

3) D is diagonal

4) entries in M satisfies the ordering Assumption

- See Bonhomme et al. (2015, 2016) for more extremum estimators

Closed-form estimators

- Global nonparametric identification
  elements of interest can be written as a function of observed distributions
  - continuous case: Kotlarski's identity
  - nonparametric regression with measurement error: Schennach (2004b, 2007), Hu and Sasaki (2015)
  - discrete case: eigen-decomposition in Hu (2008)
- Closed-form estimator
  - mimic identification procedure
  - don't need optimization or iteration
  - less nuisance parameters than semiparametric estimators
  - but may not be efficient

Closed-form estimators

- a 3-measurement model

$$\begin{aligned} X_1 &= g_1(X^*) + \epsilon_1 \\ X_2 &= g_2(X^*) + \epsilon_2 \\ X_3 &= g_3(X^*) + \epsilon_3 \end{aligned}$$

- normalization: $g_3(x^*) = x^*$
- Schennach (2004b): $g_2(x^*) = x^*$
- Hu and Sasaki (2015): g_2 is a polynomial
- Hu and Schennach (2008): g_1 and g_2 are nonparametrically identified
- Open question: Do closed-form estimators for g_1 and g_2 exist?

Estimation: a sieve semiparametric MLE

- Based on :

$$f_{Y,X|Z}(y,x|z) = \int f_{Y|X^*}(y|x^*) f_{X|X^*}(x|x^*) f_{X^*|Z}(x^*|z) dx^*$$

- Approximate ∞-dimensional parameters, e.g., $f_{X|X^*}$, by truncated series

$$\widehat{f}_1(x|x^*) = \sum_{i=0}^{i_n} \sum_{j=0}^{j_n} \widehat{\gamma}_{ij} p_i(x) p_j(x^*),$$

– where $p_k(\cdot)$ are a sequence of known univariate basis functions.

- Sieve Semiparametric MLE

$$\begin{aligned} \widehat{\alpha} &= \left(\widehat{\beta}, \widehat{\eta}, \widehat{f}_1, \widehat{f}_2\right) \\ &= \underset{(\beta,\eta,f_1,f_2)\in\mathcal{A}_n}{\arg\max} \frac{1}{n} \sum_{i=1}^{n} \ln \int f_{Y|X^*}(y_i|x^*;\beta,\eta) f_1(x_i|x^*) f_2(x^*|z_i) dx^* \end{aligned}$$

$$\begin{cases} \beta: & \text{parameter vector of interest} \\ \eta, f_1, f_2: & \infty\text{-dimensional nuisance parameters} \\ \mathcal{A}_n: & \text{space of series approximations} \end{cases}$$

Estimation: handling moment conditions

- Use η to handle moment conditions:
  – For parametric likelihoods: omit η.
  – For moment condition models: need η.
- Model defined by:

$$E\left[m\left(Y, X^{*}, \beta\right) | X^{*}\right] = 0.$$

- Method:
  – Define a family of densities $f_{Y|X^{*}}\left(y|x^{*}, \beta, \eta\right)$ such that

$$\int m\left(y, x^{*}, \beta\right) f_{Y|X^{*}}\left(y|x^{*}, \beta, \eta\right) dy = 0, \quad \forall x^{*}, \beta, \eta.$$

  – Use sieve MLE

$$\begin{aligned} \widehat{\alpha} &= \left(\widehat{\beta}, \widehat{\eta}, \widehat{f_1}, \widehat{f_2}\right) \\ &= \underset{(\beta,\eta,f_1,f_2)\in\mathcal{A}_n}{\arg\max} \frac{1}{n}\sum_{i=1}^{n} \ln \int f_{Y|X^{*}}(y_i|x^{*}; \beta, \eta) f_1(x_i|x^{*}) f_2(x^{*}|z_i) dx^{*}. \end{aligned}$$

Estimation: consistency and normality

- Consistency of $\widehat{\alpha}$
  - Conditions: too technical to show here.
  - **Theorem (consistency)**: *Under sufficient conditions, we have*

$$\left\|\widehat{\alpha} - \alpha_0\right\|_s = o_p(1).$$

  - Proof: use Theorem 4.1 in Newey and Powell (2003).

- Asymptotic normality of parameters of interest $\widehat{\beta}$.
  - Conditions: even more technical.
  - **Theorem (normality)**: *Under sufficient conditions, we have*

$$\sqrt{n}\left(\widehat{\beta} - \beta_0\right) \xrightarrow{d} N\left(0, J^{-1}\right).$$

  - Proof: use Theorem 1 in Shen (1997) and Chen and Shen (1998).

Revealing unobservables by deep learning

- Can we estimate the true values in each observation?
- From identification in distribution to identification in observation
- An ongoing research

Empirical applications with latent variables

- Auctions with unknown number of bidders
- Auctions with unobserved heterogeneity
- Auctions with heterogeneous beliefs
- Multiple equilibria in incomplete information games
- Dynamic learning models
- Effort and type in contract models
- Unemployment and labor market participation
- Cognitive and noncognitive skill formation
- Dynamic discrete choice with unobserved state variables
- Matching models with latent indices
- Income dynamics

First-price sealed-bid auctions

- Bidder i forms her own valuation of the object: x_i
  - Bidders' values are private and independent
  - Common knowledge: value distribution F, number of bidders N^*
- Bidder i chooses bid b_i to maximize her expected utility function

$$U_i = (x_i - b_i)\Pr(\max_{j \neq i} b_j < b_i)$$

  - Winning probability $\Pr(\max_{j \neq i} b_j < b_i)$ depends on bidder i's belief about her opponents' bidding behavior
- Perfectly correct beliefs about opponents' bidding behavior $\to$ Nash equilibrium

Auctions with unknown number of bidders

- An Hu & Shum (2010, JE):

$$\text{IPV auction model:} \begin{cases} N^*\text{: \# of potential bidders} \\ A\text{: \# of actual bidders} \\ b\text{: observed bids} \end{cases}$$

- bid function

$$b(x_i; N^*) = \begin{cases} x_i - \frac{\int_r^{x_i} F_{N^*}(s)^{N^*-1} ds}{F_{N^*}(x_i)^{N^*-1}} & \text{for } x_i \geq r \\ 0 & \text{for } x_i < r. \end{cases}$$

- conditional independence

$$\begin{aligned} & f\left(A_t, b_{1t}, b_{2t} | b_{1t} > r, b_{2t} > r\right) \\ = & \sum_{N^*} f\left(A_t | A_t \geq 2, N^*\right) f\left(b_{1t} | b_{1t} > r, N^*\right) f\left(b_{2t} | b_{2t} > r, N^*\right) \times \\ & \times f\left(N^* | b_{1t} > r, b_{2t} > r\right) \end{aligned}$$

Auctions with unobserved heterogeneity

- s_t^* is an auction-specific state or unobserved heterogeneity

$$b_{it} = s_t^* \times a_i(x_i)$$

- 2-measurement model

$$b_{1t} \perp b_{2t} \mid s_t^*$$

  and

$$\begin{aligned} \ln b_{1t} &= \ln s_t^* + \ln a_1 \\ \ln b_{2t} &= \ln s_t^* + \ln a_2 \end{aligned}$$

- in general

$$b_{1t} \perp b_{2t} \perp b_{3t} \mid s_t^*$$

- Li Perrigne & Vuong (2000), Krasnokutskaya (2011), Hu McAdams & Shum (2013 JE)

Auctions with heterogeneous beliefs

- An (2016): empirical analysis on Level-k belief in auctions
- Bidders have different levels of sophistication $\Rightarrow$ Heterogenous (possibly incorrect) beliefs about others' behavior
- Beliefs (types) have a hierarchical structure

Type	Belief about other bidders' behavior
1	all other bidders are type-$L0$ (bid naïvely)
2	all other bidders are type-1
$\vdots$	$\vdots$
k	all other bidders are type-$(k-1)$

  - Specification of type-$L0$ is crucial, assumed by the researchers
- Help explain overbidding and non-equilibrium behavior
- Observe joint distribution of a bidder's bids in three auctions, assuming bidder's belief level doesn't change across auctions
- three bids are independent conditional on belief level

Multiple equilibria in incomplete information games

- Xiao (2014): a static simultaneous move game
- utility function

$$u_i(a_i, a_{-i}, \epsilon_i) = \pi_i(a_i, a_{-i}) + \epsilon_i(a_i)$$

- expected payoff of player i from choosing action a_i

$$\sum_{a_{-i}} \pi_i(a_i, a_{-i}) \Pr(a_{-i}) + \epsilon_i(a_i) \equiv \Pi_i(a_i) + \epsilon_i(a_i)$$

- Bayesian Nash Equilibrium is defined as a set of choice probabilities $\Pr(a_i)$ s.t.

$$\Pr(a_i = k) = \Pr\left(\left\{\Pi_i(k) + \epsilon_i(k) > \max_{j \neq k} \Pi_i(j) + \epsilon_i(j)\right\}\right)$$

- let e^* denote the index of equilibria

$$a_1 \perp a_2 \perp ... \perp a_N \mid e^*$$

Dynamic learning models

- Hu Kayaba & Shum (2013 GEB): observe choices Y_t, rewards R_t, proxy Z_t for the agent's belief X_t^*
- Z_t: eye movement

$$\begin{array}{ccccccc} & Y_{t-1} & & Y_t & & Y_{t+1} & \\ & \uparrow & & \uparrow & & \uparrow & \\ \longrightarrow & X_{t-1}^* & \longrightarrow & X_t^* & \longrightarrow & X_{t+1}^* & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ & Z_{t-1} & & Z_t & & Z_{t+1} & \end{array}$$

- a 3-measurement model

$$Z_t \perp Y_t \perp Z_{t-1} \mid X_t^*$$

- learning rule $\Pr\left(X_{t+1}^* | X_t^*, Y_t, R_t\right)$ can be identified from

$$\begin{aligned} & \Pr\left(Z_{t+1}, Y_t, R_t, Z_t\right) \\ = & \sum_{X_{t+1}^*} \sum_{X_t^*} \Pr\left(Z_{t+1} | X_{t+1}^*\right) \Pr\left(Z_t | X_t^*\right) \Pr\left(X_{t+1}^*, X_t^*, Y_t, R_t\right). \end{aligned}$$

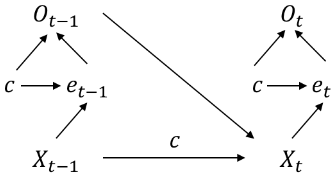
Effort and type in contract models: Xin (2018)

- Online credit markets for peer-to-peer lending attract dispersed and anonymous borrowers and lenders, and often require no collateral.
- The problems of asymmetric information are two-fold:
  1. Borrowers differ in their **inherent risks** $\Longrightarrow$ Adverse Selection;
  2. Additional **incentives** are necessary to motivate borrowers to exert effort $\Longrightarrow$ Moral Hazard.
- Xin (2018, Job market paper) sets up a dynamic structural model to formalize
  1. borrowers' repayment decisions,
  2. lenders' investment strategies,
  3. websites' pricing schemes,

  when both **hidden information** (adverse selection) and **hidden actions** (moral hazard) are present.
- identification strategies to recover the dist. of borrowers' private types and costs of effort, and utility primitives, and estimate the model using a large dataset from Prosper.com.

Effort and type in contract models: Xin (2018)

- Let the index for two loans be $t-1$ and t.
- Key elements in the model:
  1. Outcomes of the loan (default, late payment): O_t, O_{t-1};
  2. Observed characteristics (debt-to-income ratio, credit grade): X_t, X_{t-1};
  3. Effort choices: e_t, e_{t-1};
  4. Borrower's type: c.
- Dynamic structure motivated by the model:



Effort and type in contract models: Xin (2018)

- Step 1: Identify Type Distribution
- Observables, $X_t = \{\text{Financial Status}(Z_t), \text{Credit Grade}(K_t)\}$.
- Three pieces of information, independent conditional on type.

$$f(O_t, X_t, O_{t-1}, X_{t-1}) = \sum_c \underbrace{f(c, X_{t-1}, O_{t-1})}_{\text{Init. Char.}} \underbrace{f(X_t|X_{t-1}, O_{t-1}, c)}_{\text{Transition of States}} \underbrace{f(O_t|c, X_t)}_{\text{Outcome Realized}}$$

- Type distribution $f(c|X_{t-1}, O_{t-1})$ is identified for borrowers with multiple loans. (Hu and Shum, 2012)

Effort and type in contract models: Xin (2018)

- Step 2: Identify Effort Choice Probabilities
- Loan outcomes include borrowers' default and late payment performances, $O_t = \{D_t, L_t\}$.

$$\underbrace{f(O_t|c, X_t)}_{\text{identified}} = \sum_{e_t} f(D_t|e_t) f(L_t|e_t) f(e_t|c, X_t)$$

1. Conditional on effort, default and late payment are independent.
2. Effort choice is related to borrower's type.

- Following Hu (2008), effort choice probabilities and outcome realization process are identified.

Unemployment and labor market participation

- Feng & Hu (2013 AER): Let X_t^* and X_t denote the true and self-reported labor force status.
- monthly CPS $\{X_{t+1}, X_t, X_{t-9}\}_i$
- local independence

$$\Pr(X_{t+1}, X_t, X_{t-9}) = \sum_{X_{t+1}^*} \sum_{X_t^*} \sum_{X_{t-9}^*} \Pr(X_{t+1}|X_{t+1}^*) \times$$
$$\times \Pr(X_t|X_t^*) \Pr(X_{t-9}|X_{t-9}^*) \Pr(X_{t+1}^*, X_t^*, X_{t-9}^*).$$

- assume

$$\Pr(X_{t+1}^*|X_t^*, X_{t-9}^*) = \Pr(X_{t+1}^*|X_t^*)$$

- a 3-measurement model

$$\begin{aligned} & \Pr(X_{t+1}, X_t, X_{t-9}) \\ = & \sum_{X_t^*} \Pr(X_{t+1}|X_t^*) \Pr(X_t|X_t^*) \Pr(X_t^*, X_{t-9}), \end{aligned}$$

Cognitive and noncognitive skill formation

- Cunha Heckman & Schennach (2010 ECMA)
  $X_t^* = \left(X_{C,t}^*, X_{N,t}^*\right)$ cognitive and noncognitive skill
  $I_t = \left(I_{C,t}, I_{N,t}\right)$ parental investments
- for $k \in \{C, N\}$, skills evolve as

$$X_{k,t+1}^* = f_{k,s}\left(X_t^*, I_t, X_P^*, \eta_{k,t}\right),$$

  where $X_P^* = \left(X_{C,P}^*, X_{N,P}^*\right)$ are parental skills
- latent factors

$$X^* = \left(\left\{X_{C,t}^*\right\}_{t=1}^T, \left\{X_{N,t}^*\right\}_{t=1}^T, \left\{I_{C,t}\right\}_{t=1}^T, \left\{I_{N,t}\right\}_{t=1}^T, X_{C,P}^*, X_{N,P}^*\right)$$

- measurements of these factors

$$X_j = g_j\left(X^*, \varepsilon_j\right)$$

- key identification assumption

$$X_1 \perp X_2 \perp X_3 \mid X^*$$

- a 3-measurement model

Dynamic discrete choice with unobserved state variables

- Hu & Shum (2012 JE)
- $W_t = (Y_t, M_t)$
  Y_t agent's choice in period t
  M_t observed state variable
  X_t^* unobserved state variable
- for Markovian dynamic optimization models

$$f_{W_t, X_t^* | W_{t-1}, X_{t-1}^*} = f_{Y_t | M_t, X_t^*} f_{M_t, X_t^* | Y_{t-1}, M_{t-1}, X_{t-1}^*}$$

  $f_{Y_t | M_t, X_t^*}$ conditional choice probability for the agent's optimal
  $f_{M_t, X_t^* | Y_{t-1}, M_{t-1}, X_{t-1}^*}$ joint law of motion of state variables
- $f_{W_{t+1}, W_t, W_{t-1}, W_{t-2}}$ uniquly determines $f_{W_t, X_t^* | W_{t-1}, X_{t-1}^*}$

Latent indices in matching models

- Diamond & Agarwal (2017): an economy containing n workers with characteristics (X_i, ε_i) and n firms described by (Z_j, η_j)
- researchers observe X_i and Z_j
- a firm ranks workers by a human capital index as

$$v(X_i, \varepsilon_i) = h(X_i) + \varepsilon_i. \tag{1}$$

- the workers' preference for firm j is described by

$$u(Z_j, \eta_j) = g(Z_j) + \eta_j. \tag{2}$$

- the preferences on both sides are public information in the market. Researchers are interested in the preferences, including functions h, g, and distributions of ε_i and η_j.
- a pairwise stable equilibrium, where no two agents on opposite sides of the market prefer each other over their matched partners.

Matching models with latent indices

- when the numbers of firms and workers are both large, The joint distribution of (X, Z) from observed pairs then satisfies

$$f(X,Z) = \int_0^1 f(X|q)\, f(Z|q)\, dq$$

$$\begin{aligned} f(X|q) &= f_\varepsilon\left(F_V^{-1}(q) - h(X)\right) \\ f(Z|q) &= f_\eta\left(F_U^{-1}(q) - g(Z)\right) \end{aligned}$$

  a 2-measurement model
- h and g may be identified up to a monotone transformation. intuition: $f_{Z|X}(z|x_1) = f_{Z|X}(z|x_2)$ for all z implies $h(x_1) = h(x_2)$
- in many-to-one matching

$$f(X_1, X_2, Z) = \int_0^1 f(X_1|q)\, f(X_2|q)\, f(Z|q)\, dq$$

  a 3-measurement model

Income dynamics

- Arellano Blundell & Bonhomme (2017): nonlinear aspect of income dynamics
- pre-tax labor income y_{it} of household i at age t

$$y_{it} = \eta_{it} + \varepsilon_{it}$$

- persistent component η_{it} follows a first-order Markov process

$$\eta_{it} = Q_t\left(\eta_{i,t-1}, u_{it}\right)$$

- transitory component ε_{it} is independent over time
- $\{y_{it}, \eta_{it}\}$ is a hidden Markov process with

$$y_{i,t-1} \perp y_{it} \perp y_{i,t+1} \mid \eta_{it}$$

- a 3-measurement model

A canonical model of income dynamics: a revisit

- Permanent income: a random walk process
- Transitory income: an ARMA process

$$\begin{aligned} X_t &= X_t^* + v_t \\ X_t^* &= X_{t-1}^* + \eta_t \\ v_t &= \rho_t v_{t-1} + \lambda_t \epsilon_{t-1} + \epsilon_t \end{aligned}$$

$$\begin{cases} \eta_t : & \text{permanent income shock in period } t \\ \epsilon_t : & \text{transitory income shock} \\ X_t^* : & \text{latent permanent income} \\ v_t : & \text{latent transitory income} \end{cases}$$

- Can a sample of $\{X_t\}_{t=1,...,T}$ uniquely determine distributions of latent variables η_t, ϵ_t, X_t^*, and v_t?

A canonical model of income dynamics: a revisit

- Define

$$\Delta X_{t+1} = X_{t+1} - X_t$$

- Estimate AR coefficient

$$\rho_{t+1}\frac{1-\rho_{t+2}}{1-\rho_{t+1}} = \frac{cov\left(\Delta X_{t+2}, X_{t-1}\right)}{cov\left(\Delta X_{t+1}, X_{t-1}\right)}$$

- Use Kotlarski's identity

$$\begin{aligned} X_t &= v_t + X_t^* \\ \frac{\Delta X_{t+2}}{\rho_{t+2}-1} - \Delta X_{t+1} &= v_t + \frac{\lambda_{t+2}\epsilon_{t+1} + \epsilon_{t+2} + \eta_{t+2}}{\rho_{t+2}-1} - \eta_{t+1} \end{aligned}$$

- Joint distribution of $\{X_t\}_{t=1,\ldots,T\geqslant 3}$ uniquely determines distributions of latent variables η_t, ϵ_t, X_t^*, and v_t. (Hu, Moffitt, and Sasaki, 2016)

Summary

The Econometrics of Unobservables

- a solution to the endogeneity problem
- integration of microeconomic theory and econometric methodology
- economic theory motivates our intuitive assumptions
- global nonparametric point identification and estimation
- flexible nonparametrics applies to large range of economic models
- latent variable approach allows researchers to go beyond observables

See the online book for details

The Econometrics of Unobservables

– Latent Variable and Measurement Error Models and Their Applications

at ▸ Yingyao Hu's webpage

Comments are welcome. Thank you for your interest.