

# The Envelope Theorem and the Euler Equation

This handout shows how the Envelope theorem is used to derive the consumption Euler equation in a multiperiod optimization problem with geometric discounting and intertemporally separable utility.

The consumer's goal is to

$$\max \sum_{s=t}^T \beta^{s-t} u(c_s) \quad (1)$$

subject to the dynamic budget constraint

$$m_{t+1} = (m_t - c_t)R + y_{t+1}. \quad (2)$$

The problem can be written in Bellman equation form as

$$v_t(m_t) = \max_{\{c_t\}} \{u(c_t) + \beta v_{t+1}((m_t - c_t)R + y_{t+1})\}. \quad (3)$$

The first order condition for (3) can be written as

$$0 = u'(c_t) + \overbrace{\left(\frac{dm_{t+1}}{dc_t}\right)}{=-R \text{ from (2)}} \beta v'_{t+1}(m_{t+1}) \quad (4)$$

$$u'(c_t) = R\beta v'_{t+1}(m_{t+1}), \quad (5)$$

and we can define a function  $c_t(m_t)$  that returns the  $c_t$  that solves the max problem for any given  $m_t$ . That is, for  $c_t = c_t(m_t)$  the first order condition (5) will hold so that

$$u'(c_t(m_t)) - R\beta v'_{t+1}((m_t - c_t(m_t))R + y_{t+1}) = 0. \quad (6)$$

Now define a function

$$\underline{v}_t(m_t, c_t) = u(c_t) + \beta v_{t+1}((m_t - c_t)R + y_{t+1}) \quad (7)$$

with partial derivatives

$$\left(\frac{\partial \underline{v}_t}{\partial c_t}\right) \equiv \underline{v}_t^c(m_t, c_t) = u'(c_t) - R\beta v_{t+1}^m((m_t - c_t)R + y_{t+1}) \quad (8)$$

$$\left(\frac{\partial \underline{v}_t}{\partial m_t}\right) \equiv \underline{v}_t^m(m_t, c_t) = R\beta v_{t+1}^m(m_{t+1}) \quad (9)$$

and note that by definition

$$v_t(m_t) = \underline{v}_t(m_t, c_t(m_t)). \quad (10)$$

The Chain Rule of differentiation tells us that

$$v'_t(m) \equiv v_t^m(m_t) \equiv \left( \frac{dv_t}{dm_t} \right) = \underline{v}_t^m(m_t, c_t(m_t)) + \left( \frac{\partial c_t(m_t)}{\partial m_t} \right) \underline{v}_t^c(m_t, c_t(m_t)). \quad (11)$$

Now here's the key insight: The assumption that consumers are optimizing means that we will always be evaluating the value function and its derivatives at a  $c_t$  that satisfies the first-order optimality condition (6).<sup>1</sup> Thus we have from (8) that

$$\underline{v}_t^c(m_t, c_t(m_t)) = u'(c_t(m_t)) - R\beta v'_{t+1}((m_t - c_t(m_t))R + y_{t+1}) \quad (12)$$

$$= 0. \quad (13)$$

This means that the second term in (11) is always equal to zero, so from (9) we obtain

$$v'_t(m_t) = R\beta v'_{t+1}(m_{t+1}). \quad (14)$$

Now notice that the RHS's of (5) and (14) are identical, so we can equate the left hand sides,

$$v'_t(m_t) = u'(c_t) \quad (15)$$

and since a corresponding equation will hold in period  $t + 1$  we can rewrite (14) as

$$u'(c_t) = R\beta u'(c_{t+1}). \quad (16)$$

The general principle can be condensed by realizing that the Envelope theorem will always imply that the derivative of a value function with respect to any choice variable must be equal to zero for optimizing consumers. Thus we could have obtained the result immediately by treating  $c_t$  as though it were a constant (that is, treating the problem as though  $c'_t(m_t) = 0$ ) and taking the derivative of Bellman's equation with respect to  $m_t$  directly. This leads immediately to the key result:

$$v_t(m_t) = u(c(m_t)) + \beta v_{t+1}((m_t - c_t(m_t))R + y_{t+1}) \quad (17)$$

$$v'_t(m_t) = \beta R v'_{t+1}(m_{t+1}). \quad (18)$$

---

<sup>1</sup>Unless there is some constraint that prevents them from doing so - like a liquidity constraint; effects of liquidity constraints are examined in a later handout.