CRRA Portfolio Choice with Two Risky Assets

Merton (1969) and Samuelson (1969) study optimal portfolio allocation for a consumer with Constant Relative Risk Aversion utility $u(c) = (1 - \rho)^{-1}e^{1 - \rho}$ who can choose among many risky investment options.

Using their framework, here we study a consumer who has wealth $a_t$ at the end of period $t$, and is deciding how much to invest in two risky assets with lognormally distributed return factors $R_{t+1} = (R_{1,t+1}, R_{2,t+1})'$, $\log R_{t+1} = r_{t+1} = (r_{1,t+1}, r_{2,t+1})' \sim (\mathcal{N}(\mu_1, \sigma_1^2), (\mathcal{N}(\mu_2, \sigma_2^2))'$, with covariance matrix

$$
\begin{pmatrix}
\sigma_1^2 & \sigma_{12} \\
\sigma_{12} & \sigma_2^2
\end{pmatrix}.
$$

If the period-$t$ consumer invests proportion $\varsigma_i$ of $a_t$ in risky asset $i$, $i = 1, 2$ (so that $\varsigma_1 = (1 - \varsigma_2)$ and vice-versa), spending all available resources in the last period of life$^1$ $t + 1$ will yield:

$$
c_{t+1} = (\varsigma \cdot R_{t+1})a_t \\
= \sum_{R_{t+1}}
$$

where $R_{t+1}$ is the portfolio-weighted return factor.

Campbell and Viceira (2002) point out that a good approximation to the portfolio rate of return is obtained by

$$
r_{t+1} = r_{1,t+1} + \varsigma_2(r_{2,t+1} - r_{1,t+1}) + \varsigma_2(1 - \varsigma_2)\eta/2
$$

where

$$
\eta = (\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}).
$$

Using this approximation, the expectation as of date $t$ of utility at date $t + 1$ is:

$$
\mathbb{E}_t[u(c_{t+1})] \approx (1 - \rho)^{-1}\mathbb{E}_t\left[\left(a_te^{r_{1,t+1}}e^{\varsigma_2(r_{2,t+1} - r_{1,t+1}) + \varsigma_2(1 - \varsigma_2)\eta/2}\right)^{1 - \rho}\right]
$$

$$
\approx (1 - \rho)^{-1}a_t^{1 - \rho}e^{(1 - \rho)(\varsigma_2(1 - \varsigma_2)\eta/2)}\mathbb{E}_t\left[e^{[(r_{1,t+1} + \varsigma_2(r_{2,t+1} - r_{1,t+1}))(1 - \rho)]}\right]
$$

where the first term is a negative constant under the usual assumption that relative risk aversion $\rho > 1$.

Our foregoing assumptions imply that

$$
(1 - \rho)(\varsigma_1 r_{1,t+1} + \varsigma_2 r_{2,t+1}) \sim \mathcal{N}((1 - \rho)(\varsigma_1 r_1 + \varsigma_2 r_2), (1 - \rho)^2(\varsigma_1^2 \sigma_1^2 + \varsigma_2^2 \sigma_2^2 + 2\varsigma_1\varsigma_2\sigma_{12}))
$$

(-using $[\text{LogELogNormTimes}]$). With a couple of extra lines of derivation we can show that the log of the expectation in (2) is

$$
\log \mathbb{E}_t\left[e^{[(r_{1,t+1} + \varsigma_2(r_{2,t+1} - r_{1,t+1}))(1 - \rho)]}\right] = (1 - \rho)(\varsigma_1 r_1 + \varsigma_2 r_2) + (1 - \rho)^2(\varsigma_1^2 \sigma_1^2 + \varsigma_2^2 \sigma_2^2 + 2\varsigma_1\varsigma_2\sigma_{12})/2
$$

$^1$The portfolio allocation solution obtained below induces back to earlier periods of life, as Samuelson (1963, 1989) famously emphasized.
Substituting from (2) for the log of the expectation in (2), the log of the ‘excess return utility factor’ in (2) is

$$(1 - \rho)\varsigma_2(1 - \varsigma_2)\eta/2 + (1 - \rho)(r_1 + \varsigma_2(r_2 - r_1)) + (\rho - 1)^2(\sigma_1^2 + \varsigma_2^2\eta + 2\varsigma_2(\sigma_{12} - \sigma_1^2))/2.$$  

The $\varsigma$ that minimizes this log will also minimize the level; the FOC for minimizing this expression is

$$(1 - 2\varsigma_2)\eta/2 + r_2 - r_1 + (1 - \rho)(\varsigma_2\eta + (\sigma_{12} - \sigma_1^2)) = 0$$

$$(r_2 - r_1 + \eta/2) + (1 - \rho)(\sigma_{12} - \sigma_1^2) = \rho\eta\varsigma_2.$$

So

$$\varsigma_2 = \left(\frac{r_2 - r_1 + \eta/2 + (1 - \rho)(\sigma_{12} - \sigma_1^2)}{\rho\eta}\right)$$  

(3)

and note that if the first asset is riskfree so that $\sigma_1 = \sigma_{12} = 0$ then this reduces to

$$\varsigma_2 = \left(\frac{r_2 - r_1 + \sigma_2^2/2}{\rho\sigma_2^2}\right)$$  

(4)

but the log of the expected return premium (in levels) on the risky over the safe asset in this case is $\varphi \equiv \log R_2/R_1 = r_2 - r_1 + \sigma_2^2/2$ (recalling that we have assumed $\sigma_{12} = \sigma_1^2 = 0$), so (4) becomes

$$\varsigma_2 = \left(\frac{\varphi}{\rho\sigma_2^2}\right)$$  

(5)

which corresponds to the solution obtained for the case of a single risky asset in Portfolio-CRRA.

References


