CRRA Portfolio Choice with Two Risky Assets

Merton (1969) and Samuelson (1969) study optimal portfolio allocation for a consumer with Constant Relative Risk Aversion utility \( u(c) = (1 - \rho)^{-1}c^{1-\rho} \) who can choose among many risky investment options.

Using their framework, here we study a consumer who has wealth \( a_t \) at the end of period \( t \), and is deciding how much to invest in two risky assets with lognormally distributed return factors \( R_{t+1} = (R_{1,t+1}, R_{2,t+1})' \), \( \log R_{t+1} = r_{t+1} = (r_{1,t+1}, r_{2,t+1})' \sim (N(r_1, \sigma_1^2), (N(r_2, \sigma_2^2))' \), with covariance matrix

\[
\begin{pmatrix}
\sigma_1^2 & \sigma_{12} \\
\sigma_{12} & \sigma_2^2 
\end{pmatrix}
\]

If the period-\( t \) consumer invests proportion \( \varsigma_i \) of \( a_t \) in risky asset \( i, i = 1, 2 \) (so that \( \varsigma_1 = (1 - \varsigma_2) \) and vice-versa), spending all available resources in the last period of life \( t + 1 \) will yield:

\[
c_{t+1} = (\varsigma \cdot R_{t+1}) a_t
\]

where \( R_{t+1} \) is the portfolio-weighted return factor.

Campbell and Viceira (2002) point out that a good approximation to the portfolio rate of return is obtained by

\[
r_{t+1} = r_{1,t+1} + \varsigma_2(r_{2,t+1} - r_{1,t+1}) + \varsigma_2(1 - \varsigma_2)\eta/2
\]

where

\[
\eta = (\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}).
\]

Using this approximation, the expectation as of date \( t \) of utility at date \( t + 1 \) is:

\[
\mathbb{E}_t[u(c_{t+1})] \approx (1 - \rho)^{-1} \mathbb{E}_t \left[ (a_t e^{r_{1,t+1} + \varsigma_2(r_{2,t+1} - r_{1,t+1}) + \varsigma_2(1 - \varsigma_2)\eta/2})^{1-\rho} \right]
\]

\[
\approx (1 - \rho)^{-1} a_t^{1-\rho} e^{(1-\rho)\varsigma_2(1 - \varsigma_2)\eta/2} \mathbb{E}_t \left[ e^{(r_{1,t+1} + \varsigma_2(r_{2,t+1} - r_{1,t+1}))}(1-\rho)} \right]
\]

where the first term is a negative constant under the usual assumption that relative risk aversion \( \rho > 1 \).

Our foregoing assumptions imply that

\[
(1 - \rho)(\varsigma_1 r_{1,t+1} + \varsigma_2 r_{2,t+1}) \sim N((1 - \rho)(\varsigma_1 r_1 + \varsigma_2 r_2), (1 - \rho)^2(\varsigma_1^2 \sigma_1^2 + \varsigma_2^2 \sigma_2^2 + 2\varsigma_1\varsigma_2\sigma_{12}))
\]

\[1\]The portfolio allocation solution obtained below induces back to earlier periods of life, as Samuelson (1963, 1989) famously emphasized.
(using \([\text{LogELogNormTimes}]\)). With a couple of extra lines of derivation we can show that the log of the expectation in (2) is
\[
\log \mathbb{E}_t \left[ e^{(r_{1,t+1} + \varsigma_2(r_{2,t+1} - r_{1,t+1}))(1-\rho)} \right] = (1-\rho)(\varsigma_1 r_1 + \varsigma_2 r_2) + (1-\rho)^2(\varsigma_1^2 \sigma_1^2 + \varsigma_2^2 \sigma_2^2 + 2\varsigma_1 \varsigma_2 \sigma_{12})/2
\]
Substituting from (3) for the log of the expectation in (2), the log of the ‘excess return utility factor’ in (2) is
\[
(1-\rho)\varsigma_2(1-\varsigma_2)\eta/2 + (1-\rho)(r_1 + \varsigma_2(r_2 - r_1)) + (\rho - 1)^2(\sigma_1^2 + \varsigma_2^2 \eta + 2\varsigma_2(\sigma_{12} - \sigma_2^2))/2.
\]
The \(\varsigma\) that minimizes this log will also minimize the level; the FOC for minimizing this expression is
\[
(1-2\varsigma_2)\eta/2 + r_2 - r_1 + (1-\rho)(\varsigma_2 \eta + (\sigma_{12} - \sigma_1^2)) = 0 \quad (3)
\]
\[
(r_2 - r_1 + \eta/2) + (1-\rho)(\sigma_{12} - \sigma_1^2) = \rho \eta \varsigma_2. \quad (4)
\]
So
\[
\varsigma_2 = \left( \frac{r_2 - r_1 + \eta/2 + (1-\rho)(\sigma_{12} - \sigma_1^2)}{\rho \eta} \right) \quad (5)
\]
and note that if the first asset is riskfree so that \(\sigma_1 = \sigma_{12} = 0\) then this reduces to
\[
\varsigma_2 = \left( \frac{r_2 - r_1 + \sigma_2^2/2}{\rho \sigma_2^2} \right) \quad (6)
\]
but the log of the expected return premium (in levels) on the risky over the safe asset in this case is \(\varphi \equiv \log R_2/R_1 = r_2 - r_1 + \sigma_2^2/2\) (recalling that we have assumed \(\sigma_{12} = \sigma_1^2 = 0\)), so (6) becomes
\[
\varsigma_2 = \left( \frac{\varphi}{\rho \sigma_2^2} \right) \quad (7)
\]
which corresponds to the solution obtained for the case of a single risky asset in \text{Portfolio-CRRA}. 

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References


