CRRA Portfolio Choice with Two Risky Assets

Merton (1969) and Samuelson (1969) study optimal portfolio allocation for a consumer with Constant Relative Risk Aversion utility $u(c) = (1 - \rho)^{-1} c^{1-\rho}$ who can choose among many risky investment options.

Using their framework, here we study a consumer who has wealth $a_t$ at the end of period $t$, and is deciding how much to invest in two risky assets with lognormally distributed return factors $R_{t+1} = (R_{1,t+1}, R_{2,t+1})'$, log $R_{t+1} = r_{t+1} = (r_{1,t+1}, r_{2,t+1})' \sim (\mathcal{N}(r_1, \sigma_1^2), (\mathcal{N}(r_2, \sigma_2^2))'$, with covariance matrix

$$
\begin{pmatrix}
\sigma_1^2 & \sigma_{12} \\
\sigma_{12} & \sigma_2^2
\end{pmatrix}.
$$

If the period-$t$ consumer invests proportion $\varsigma_i$ of $a_t$ in risky asset $i$, $i = 1, 2$ (so that $\varsigma_1 = (1 - \varsigma_2)$ and vice-versa), spending all available resources in the last period of life $t + 1$ will yield:

$$c_{t+1} = (\varsigma \cdot R_{t+1}) a_t$$

where $R_{t+1}$ is the portfolio-weighted return factor.

Campbell and Viceira (2002) point out that a good approximation to the portfolio rate of return is obtained by

$$r_{t+1} = r_{1,t+1} + \varsigma_2 (r_{2,t+1} - r_{1,t+1}) + \varsigma_2 (1 - \varsigma_2) \eta/2$$

where

$$\eta = (\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}).$$

Using this approximation, the expectation as of date $t$ of utility at date $t + 1$ is:

$$E_t[u(c_{t+1})] \approx (1 - \rho)^{-1} E_t \left[ a_t e^{r_{1,t+1} + \varsigma_2 (r_{2,t+1} - r_{1,t+1}) + \varsigma_2 (1 - \varsigma_2) \eta/2} \right]^{1-\rho}$$

$$\approx (1 - \rho)^{-1} a_t^{1-\rho} e^{(1-\rho)\varsigma_2 (1-\varsigma_2) \eta/2} E_t \left[ e^{(r_{1,t+1} + \varsigma_2 (r_{2,t+1} - r_{1,t+1})) (1-\rho)} \right]$$

where the first term is a negative constant under the usual assumption that relative risk aversion $\rho > 1$.

Our foregoing assumptions imply that

$$(1 - \rho)(\varsigma_1 R_{1,t+1} + \varsigma_2 R_{2,t+1}) \sim \mathcal{N}((1 - \rho)(\varsigma_1 R_1 + \varsigma_2 R_2), (1 - \rho)^2 (\varsigma_1^2 \sigma_1^2 + \varsigma_2^2 \sigma_2^2 + 2\varsigma_1 \varsigma_2 \sigma_{12}))$$

\footnote{The portfolio allocation solution obtained below induces back to earlier periods of life, as Samuelson (1963, 1989) famously emphasized.}
(using \([\text{LogELogNormTimes}]\)). With a couple of extra lines of derivation we can show that the log of the expectation in (2) is

\[
\log \mathbb{E}_t \left[ e^{(r_{1,t+1} + \varsigma_2(r_{2,t+1} - r_{1,t+1}))} \right] = (1 - \rho)(\varsigma_1 r_1 + \varsigma_2 r_2) + (1 - \rho)^2(\varsigma_1^2 \sigma_1^2 + \varsigma_2^2 \sigma_2^2 + 2 \varsigma_1 \varsigma_2 \sigma_{12})/2
\]

Substituting from (3) for the log of the expectation in (2), the log of the ‘excess return utility factor’ in (2) is

\[
(1 - \rho)\varsigma_2(1 - \varsigma_2)\eta/2 + (1 - \rho)(\varsigma_1 r_1 + \varsigma_2(r_2 - r_1)) + (\rho - 1)^2(\sigma_1^2 + \varsigma_2^2 \eta + 2 \varsigma_2 (\sigma_{12} - \sigma_2^2))/2.
\]

The \(\varsigma\) that minimizes this log will also minimize the level; the FOC for minimizing this expression is

\[
(1 - 2 \varsigma_2)\eta/2 + r_2 - r_1 + (1 - \rho)(\varsigma_2 \eta + (\sigma_{12} - \sigma_1^2)) = 0
\]

\[
(r_2 - r_1 + \eta/2) + (1 - \rho)(\sigma_{12} - \sigma_1^2) = \rho \eta \varsigma_2.
\]

So

\[
\varsigma_2 = \left( \frac{r_2 - r_1 + \eta/2 + (1 - \rho)(\sigma_{12} - \sigma_1^2)}{\rho \eta} \right) (5)
\]

and note that if the first asset is riskfree so that \(\sigma_1 = \sigma_{12} = 0\) then this reduces to

\[
\varsigma_2 = \left( \frac{r_2 - r_1 + \sigma_2^2/2}{\rho \sigma_2^2} \right) (6)
\]

but the log of the expected return premium (in levels) on the risky over the safe asset in this case is \(\varphi \equiv \log \left( R_2/R_1 \right) = r_2 - r_1 + \sigma_2^2/2\) (recalling that we have assumed \(\sigma_{12} = \sigma_1^2 = 0\)), so (6) becomes

\[
\varsigma_2 = \left( \frac{\varphi}{\rho \sigma_2^2} \right) (7)
\]

which corresponds to the solution obtained for the case of a single risky asset in Portfolio-CRRA.
References


