

The Lucas Asset Pricing Model

0.1 Introduction/Setup

Lucas (1978) considers an economy populated by infinitely many¹ identical individual consumers, in which the only assets are some identical infinitely-lived trees. Aggregate output is the fruit that falls from the trees, and cannot be stored (it would rot!); because $u'(c) > 0 \forall c$, it is all eaten:

$$c_t L_t = d_t K_t \quad (1)$$

where c_t is consumption of fruit per person, L_t is the population, d_t is the exogenous output of fruit that drops from each tree and K_t is the stock of trees. (In a given year, each tree produces exactly the same amount of fruit as every other tree, but d_t varies from year to year depending on the weather.) An economy like this, in which output arrives without any deliberate actions on the part of residents, is called an ‘endowment’ economy (or, sometimes, an ‘exchange’ economy).

0.2 The Market for Trees

In equilibrium, the price of trees must be such that, each period, each (identical) consumer does not want either to increase or to decrease his holding of trees (if, at a hypothesized equilibrium price, every identical consumer wanted (say) to increase their holdings, that price could not be an equilibrium price!).

Let P_t denote the equilibrium price, and assume that if the tree is sold, the sale occurs after the existing owner receives that period’s fruit (P_t is the ‘ex-dividend’ price). The total resources available to consumer i in period t are the sum of the fruit received from the trees owned, $d_t k_t^i$, plus the potential proceeds if the consumer were to sell all his stock of trees, $P_t k_t^i$. Total resources are divided into two uses: Current consumption c_t^i and the purchase of trees for next period k_{t+1}^i at price P_t ,

$$\overbrace{k_{t+1}^i P_t + c_t^i}^{\text{Uses of resources}} = \overbrace{d_t k_t^i + P_t k_t^i}^{\text{Total resources}} \quad (2)$$

$$k_{t+1}^i = (1 + d_t/P_t)k_t^i - c_t^i/P_t. \quad (3)$$

¹As in **Aggregation**.

0.3 The Problem of an Individual Consumer

Consumer i maximizes

$$\begin{aligned} v(m_t^i) &= \max \mathbb{E}_t^i \left[\sum_{n=0}^{\infty} \beta^n u(c_{t+n}^i) \right] \\ &\text{s.t.} \\ k_{t+1}^i &= (1 + d_t/P_t)k_t^i - c_t^i/P_t \\ m_{t+1}^i &= (P_{t+1} + d_{t+1})k_{t+1}^i. \end{aligned} \quad (4)$$

Rewriting in the form of Bellman's equation,

$$v(m_t^i) = \max_{\{c_t^i\}} u(c_t^i) + \beta \mathbb{E}_t^i [v(m_{t+1}^i)],$$

the first order condition tells us that

$$0 = u'(c_t^i) + \beta \mathbb{E}_t^i \left[v'(m_{t+1}^i) \frac{d}{dc_t^i} \left(\underbrace{(P_{t+1} + d_{t+1}) \left((1 + d_t/P_t) k_t^i - c_t^i/P_t \right)}_{m_{t+1}^i} \right) \right]$$

so

$$\begin{aligned} u'(c_t^i) &= \beta \mathbb{E}_t^i \left[v'(m_{t+1}^i) \left(\underbrace{\frac{P_{t+1} + d_{t+1}}{P_t}}_{\equiv \mathbf{R}_{t+1}} \right) \right] \\ &= \beta \mathbb{E}_t^i [\mathbf{R}_{t+1} v'(m_{t+1}^i)] \end{aligned} \quad (5)$$

where \mathbf{R}_{t+1} is the return factor that measures the resources in period $t + 1$ that are the reward for owning a unit of trees at the end of t .

The **Envelope** theorem tells us that $v'(m_{t+1}^i) = u'(c_{t+1}^i)$, so (5) becomes

$$\begin{aligned} u'(c_t^i) &= \beta \mathbb{E}_t^i \left[u'(c_{t+1}^i) \left(\frac{P_{t+1} + d_{t+1}}{P_t} \right) \right] \\ P_t &= \beta \mathbb{E}_t^i \left[\left(\frac{u'(c_{t+1}^i)}{u'(c_t^i)} \right) (P_{t+1} + d_{t+1}) \right]. \end{aligned} \quad (6)$$

0.4 Aggregation

The assumption that all consumers are identical says that $c_t^i = c_t^j \forall i, j$, so henceforth we just call consumption per capita c_t . Since aggregate consumption must equal aggregate production because fruit cannot be stored, normalizing the population to

$L_t = 1 \forall t$ and stock of trees to $K_t = 1 \forall t$, equation (1) becomes:

$$c_t = d_t. \quad (7)$$

Substituting c_t and c_{t+1} for c_t^i and c_{t+1}^i in (6) and then substituting d_t for c_t we get

$$P_t = \beta \mathbb{E}_t \left[\left(\frac{u'(d_{t+1})}{u'(d_t)} \right) (P_{t+1} + d_{t+1}) \right]. \quad (8)$$

We can rewrite this more simply if we define an object

$$M_{t,t+n} = \beta^n \left(\frac{u'(d_{t+n})}{u'(d_t)} \right) \quad (9)$$

which is called the ‘stochastic discount factor’ because (a) it is stochastic (thanks to the shocks between t and $t+n$ that determine the value of d_{t+n}); and (b) it measures the rate at which all agents in this economy in period t will discount a unit of value received in a future period, e.g. $t+1$:

$$P_t = \mathbb{E}_t [M_{t,t+1} (P_{t+1} + d_{t+1})]. \quad (10)$$

A corresponding equation will hold in period $t+1$, and in period $t+2$, and beyond:

$$P_{t+1} = \mathbb{E}_{t+1} [M_{t+1,t+2} (P_{t+2} + d_{t+2})] \quad (11)$$

$$P_{t+2} = \mathbb{E}_{t+2} [M_{t+2,t+3} (P_{t+3} + d_{t+3})], \quad (12)$$

so we can use repeated substitution, e.g. of (11) into (10), and of (12) into the result, etc to get

$$P_t = \mathbb{E}_t [M_{t,t+1}d_{t+1}] + \mathbb{E}_t [M_{t,t+1} \mathbb{E}_{t+1} [M_{t+1,t+2}d_{t+2}]] + \dots \quad (13)$$

The ‘law of iterated expectations’ says that $\mathbb{E}_t [\mathbb{E}_{t+1} [P_{t+2}]] = \mathbb{E}_t [P_{t+2}]$; given this, and noting that $M_{t,t+2} = M_{t,t+1}M_{t+1,t+2}$, (13) becomes:

$$P_t = \mathbb{E}_t [M_{t,t+1}d_{t+1} + M_{t,t+2}d_{t+2} + M_{t,t+3}d_{t+3} + \dots]. \quad (14)$$

So, the price of the asset is the present discounted value of the stream of future ‘dividends,’ where the stochastic factor by which (potentially stochastic) dividends received in $t+n$ are discounted back to t is $M_{t,t+n}$.

0.5 Specializing the Model

This is as far as we can go without making explicit assumptions about the structure of utility. If utility is CRRA, $u(c) = (1-\rho)^{-1}c^{1-\rho}$, substituting $u'(d) = d^{-\rho}$ into (8) yields

$$P_t = \beta d_t^\rho \mathbb{E}_t [d_{t+1}^{-\rho} (P_{t+1} + d_{t+1})] \quad (15)$$

and the particularly special case of logarithmic utility (which Lucas emphasizes) corresponds to $\rho = 1$, which (again using the law of iterated expectations) allows

us to simplify (15) to

$$\begin{aligned}
\frac{P_t}{d_t} &= \beta \mathbb{E}_t [d_{t+1}^{-1}(P_{t+1} + d_{t+1})] \\
&= \beta \left(1 + \mathbb{E}_t \left[\frac{P_{t+1}}{d_{t+1}} \right] \right) \\
&= \beta \left(1 + \beta \left(1 + \mathbb{E}_t \left[\frac{P_{t+2}}{d_{t+2}} \right] \right) \right) \\
&= \beta \left(1 + \beta + \beta^2 + \dots + \mathbb{E}_t \left[\lim_{n \rightarrow \infty} \beta^{n-1} \left[\frac{P_{t+n}}{d_{t+n}} \right] \right] \right) \\
&= \frac{\beta}{1 - \beta} + \beta \mathbb{E}_t \left\{ \lim_{n \rightarrow \infty} \beta^{n-1} \left[\frac{P_{t+n}}{d_{t+n}} \right] \right\}.
\end{aligned}$$

If the price is bounded (it cannot ever go, for example, to a value such that it would cost more than the economy's entire output to buy a single tree), it is possible to show that the limit term in this equation goes to zero. Using the usual definition of the time preference factor as $\beta = 1/(1 + \vartheta)$ where ϑ is the time preference rate, the equilibrium price is:

$$\begin{aligned}
P_t &= d_t \left(\frac{\beta}{1 - \beta} \right) \\
&= d_t \left(\frac{1}{1/\beta - 1} \right) \\
&= d_t \left(\frac{1}{1 + \vartheta - 1} \right) \\
&= \frac{d_t}{\vartheta}
\end{aligned} \tag{16}$$

or, equivalently, the 'dividend-price ratio' is always $d_t/P_t = \vartheta$.²

It may surprise you that the equilibrium price of trees today does not depend on the expected level of fruit output in the future. You might reason that higher expected future fruit production increases the attractiveness today of buying the trees that will produce that abundant future fruit, so demand for (and thus price of) trees should be higher. This logic is not wrong; but it is (exactly) counterbalanced by another, and subtler, fact: Since future consumption will equal future fruit output, higher expected fruit output means lower marginal utility of consumption in that future period of abundant fruit (basically, people get weary of eating fruit), which reduces the attractiveness of buying the trees. These two forces are the manifestation of the (pure) income effect and substitution effect in this model (there is no human wealth,

²A derivation parallel to the one above shows that in the CRRA utility case the solution is $d_t^p/P_t = \vartheta$.

and therefore no human wealth effect). In the special case of logarithmic utility considered here, income and substitution effects are of the same size and opposite sign so the two forces exactly offset each other.

0.6 The Interest Rate and the ‘Rate of Return’ in a Lucas Model

We can decompose the return factor attributable to ownership of a share of capital (cf. (5)) by adding and subtracting P_t in the numerator:

$$\begin{aligned} \mathbf{R}_{t+1} &= \left(\frac{P_{t+1} + P_t - P_t + d_{t+1}}{P_t} \right) \\ &= \left(1 + \frac{\Delta P_{t+1}}{P_t} + \frac{d_{t+1}}{P_t} \right) \end{aligned} \tag{17}$$

so the ‘rate of return’ is

$$\mathbf{r}_{t+1} = \frac{\Delta P_{t+1}}{P_t} + \frac{d_{t+1}}{P_t} \tag{18}$$

which is a useful decomposition because the two components have natural interpretations: The first is a ‘capital gain’ (or loss), and the second can plausibly be identified as ‘the interest rate’ paid by the asset (because it corresponds to income received regardless of whether the asset is liquidated).

In models that do not explicitly discuss asset pricing, the implicit assumption is usually that the price of capital is constant (which might be plausible if capital consists mostly of reproducible items like machines,³ rather than Lucas trees). In this case

$$\mathbf{R}_{t+1} = \left(1 + \frac{d_{t+1}}{P_t} \right)$$

says that the only risk in the rate of return is attributable to unpredictable variation in the size of dividend/interest payments. Indeed, if additional assumptions are made (e.g., perfect capital markets) that yield the conclusion that the interest rate matches the marginal product of capital, then such models generally imply that variation in returns (at least at high frequencies) is very small, because aggregate capital typically is very stable from one period to the next in such models, and, if the aggregate production function is stable, this implies great stability in the marginal product of capital.

³The key insights below remain true even if there is a gradual trend in the real price of capital goods, as has in fact been true.

0.7 Aggregate Returns Versus Individual Returns

One of the subtler entries in [Aristotle \(350 BC\)](#)'s catalog of common human reasoning errors was the 'fallacy of composition,' in which the reasoner supposes that if a proposition is true of each element of a whole, then it must be true of the whole.

The Lucas model provides a vivid counterexample. From the standpoint of any individual (atomistic) agent, it is quite true that a decision to save one more unit will yield greater future resources, in the amount \mathbf{R}_{t+1} . But from the standpoint of the society as a whole, if everyone decided to do the same thing (save one more unit), there would be no effect on aggregate resources in period $t + 1$. Put another way, for any individual agent, it appears that the 'marginal product of capital' is \mathbf{r}_{t+1} , but for the society as a whole the marginal product of capital is zero.

The proposition that the return for society as a whole must be the same as the return that is available to individuals is an error because it implicitly assumes that there are no general equilibrium effects of a generalized desire to save more (or, more broadly, that there is no interaction between the decisions one person makes and the outcomes for another person). The Lucas model provides a counterexample in which, if everyone's preferences change (e.g., ϑ goes down for everyone), the price of the future asset is affected – indeed, it is affected in a way that is sufficient to exactly counteract the increased desire for ownership of future dividends (since there is a fixed supply of assets to be owned, the demand must be reconciled with that preexisting supply).

Appendix: Analytical Solutions in CRRA Utility Case

0.1 When Dividends are IID

Suppose d_{t+n} is identically individually distributed in every future period, so that its expectation as of t is the same for any date $n > 0$:

$$\dot{d} \equiv \mathbb{E}_t[d_{t+n}^{1-\rho}]. \quad (19)$$

Now note that (15) can be rewritten as

$$\begin{aligned} \frac{P_t}{d_t^\rho} &= \beta \left(\dot{d} + \mathbb{E}_t \left[\frac{P_{t+1}}{d_{t+1}^\rho} \right] \right) \\ &= \beta \dot{d} \left(1 + \beta \dot{d} + \beta \mathbb{E}_t \left[\frac{P_{t+2}}{d_{t+2}^\rho} \right] \right) \\ &= \beta \dot{d} \left(1 + (\beta \dot{d}) + (\beta \dot{d})^2 + \dots + \underbrace{\mathbb{E}_t \left[\lim_{n \rightarrow \infty} \beta^{n-1} \left[\frac{P_{t+n}}{d_{t+n}^\rho} \right] \right]}_{\text{assume goes to zero}} \right) \end{aligned} \quad (20)$$

$$= \left(\frac{\beta \dot{d}}{1 - \beta \dot{d}} \right) \quad (21)$$

$$= \left(\frac{1}{(\beta \dot{d})^{-1} - 1} \right) \quad (22)$$

To make further progress, suppose that the iid process for dividends is a mean-one lognormal: $\log d_{t+n} \sim \mathcal{N}(-\sigma^2/2, \sigma^2) \forall n$ so that $\mathbb{E}_t[d_{t+n}] = 1 \forall n$ (see [\[ELogNormMeanOne\]](#)), in which case [\[ELogNormTimes\]](#) can be used to show that

$$\dot{d} = e^{\rho(\rho-1)(1/2)\sigma^2} \quad (23)$$

and if we approximate $\beta \approx e^{-\vartheta}$ then $(\beta \dot{d})^{-1} \approx e^{\vartheta - \rho(\rho-1)(1/2)\sigma^2}$ and so (22) becomes

$$\begin{aligned} \frac{P_t}{d_t^\rho} &\approx \left(\frac{1}{e^{\vartheta - (\rho(\rho-1)\sigma^2/2)} - 1} \right) \\ P_t &\approx \left(\frac{d_t^\rho}{\vartheta - \rho(\rho-1)\sigma^2/2} \right) \end{aligned}$$

(where we used [\[ExpEps\]](#) to get from the first to the second equation). So the log of (22) is

$$\log P_t \approx \rho \log d_t - \log(\vartheta - (1/2)\rho(\rho-1)\sigma^2) \quad (24)$$

and thus the variances obey

$$\text{var}(\log P) = \rho^2 \text{var}(\log d). \quad (25)$$

Given that $\rho > 1$, this derivation yields some interesting insights:

1. (the log of) asset prices will be more volatile than (the log of) dividends
2. An increase in risk aversion ρ increases P_t (because $\rho(\rho - 1)\sigma^2/2 > 0$ and an increase in ρ increases its size)

The second point is surprising, so let me say it again: an *increase* in risk aversion *increases* asset prices. In a sense, this is an implication of the proposition that risk aversion increases the volatility of asset prices (when they are high, they must be *very* high; when low, *very* low). But, it does not correspond very well to the common narrative in which market analysts often attribute a decline in asset prices to ‘increased risk aversion.’

0.2 When Dividends Follow a Random Walk

The polar alternative to IID shocks would be for dividends to follow a random walk: $\log(d_{t+1}/d_t) \sim \mathcal{N}(-\sigma^2/2, \sigma^2)$.

Now divide both sides of (15) by d_t , and rewrite the object inside the expectations operator by multiplying the first term by d_{t+1} and dividing the second term by d_{t+1} , yielding

$$\left(\frac{P_t}{d_t}\right) = \beta d_t^{-(1-\rho)} \mathbb{E}_t \left[d_{t+1}^{1-\rho} \left(\frac{P_{t+1}}{d_{t+1}} + 1\right) \right] \quad (26)$$

$$= \beta \mathbb{E}_t \left[\left(\frac{d_{t+1}}{d_t}\right)^{1-\rho} \left(\frac{P_{t+1}}{d_{t+1}} + 1\right) \right]. \quad (27)$$

Now (1) note that our assumption here about the distribution of d_{t+1}/d_t is identical to the assumption about d_{t+1} above, so the expectation will be the same \dot{d} ; and (2) hypothesize that there will be a solution under which the price-dividend ratio is a constant; call it \mathbf{r}^{-1} :

$$\mathbf{r}^{-1} = \beta \left[\dot{d}(\mathbf{r}^{-1} + 1) \right] \quad (28)$$

$$1 = \beta \dot{d}(1 + \mathbf{r}) \quad (29)$$

$$\left(\frac{1 - \beta \dot{d}}{\beta \dot{d}}\right) = \mathbf{r} \quad (30)$$

$$\left(\frac{1}{(\beta \dot{d})^{-1} - 1}\right) = \mathbf{r}^{-1} \quad (31)$$

so that (remarkably) we obtain a formula for $\mathbf{r}^{-1} = \mathbf{P}_t/d_t$ that is identical to the formula for \mathbf{P}_t/d_t^ρ in (22); corresponding derivations lead to

$$\log \mathbf{P}_t \approx \log d_t - \log(\vartheta - (1/2)\rho(\rho - 1)\sigma^2) \quad (32)$$

The difference with (44) is only the absence of the ρ multiplying $\log d_t$. The main substantive difference is therefore that the variance of (log) prices and the variance of (log) dividends is now the same. The surprising result that the price-dividend ratio increases when risk aversion increases continues to hold.

0.3 When Dividends Follow an AR(1) Process

Start with (34):

$$\left(\frac{\mathbf{P}_t}{d_t}\right) = \beta d_t^{-(1-\rho)} \mathbb{E}_t \left[d_{t+1}^{1-\rho} \left(\frac{\mathbf{P}_{t+1}}{d_{t+1}} + 1\right) \right] \quad (33)$$

$$= \beta \mathbb{E}_t \left[\left(\frac{d_{t+1}}{d_t}\right)^{1-\rho} \left(\frac{\mathbf{P}_{t+1}}{d_{t+1}} + 1\right) \right]. \quad (34)$$

and substitute for $d_{t+1} = \alpha d_t + z_{t+1}$:

$$\left(\frac{\mathbf{P}_t}{d_t}\right) = \beta \mathbb{E}_t \left[\left(\frac{\alpha d_t + z_{t+1}}{d_t}\right)^{1-\rho} \left(\frac{\mathbf{P}_{t+1}}{d_{t+1}} + 1\right) \right] \quad (35)$$

We cannot make further analytical progress so long as the z_{t+1} term is present.

Numerical solutions tend to work best when it is possible to define the limits as the state variables approach their maximum possible values, so the next step is to try to compute such limits.

0.3.1 As $d \uparrow \infty$

In the limit as d_t approaches ∞ , the z_{t+1} term becomes arbitrarily small (relative to d_t). Thus,

$$\lim_{d_t \uparrow \infty} \left(\frac{\mathbf{P}_t}{d_t}\right) = \beta \left[\alpha^{1-\rho} \left(\frac{\mathbf{P}_{t+1}}{d_{t+1}} + 1\right) \right] \quad (36)$$

$$= \left(\frac{\beta \alpha^{1-\rho}}{1 - (\beta \alpha^{1-\rho})}\right) \quad (37)$$

$$= \left(\frac{1}{\beta^{-1} \alpha^{\rho-1} - 1}\right) \quad (38)$$

0.3.2 As $d \downarrow 0$

Suppose that $\log z_{t+1} \sim \mathcal{N}(-\sigma^2/2, \sigma^2)$. Then [ELogNormTimes] says:

$$\log \mathbb{E}_t[z_{t+1}d_t^{-1}] = -(1-\rho)d_t^{-1}\sigma^2/2 + \left(\frac{(1-\rho)}{d_t^{-1}}\right)^2 \sigma^2/2 \quad (39)$$

$$= d_t^{-1} \left(-(1-\rho)\sigma^2/2 + \left(\frac{(1-\rho)^2}{d_t}\right) \sigma^2/2 \right) \quad (40)$$

whose limit is

$$\lim_{d_t \downarrow 0} \log \mathbb{E}_t[z_{t+1}d_t^{-1}] = \left(\left(\frac{(1-\rho)}{d_t}\right)^2 \sigma^2/2 \right) \quad (41)$$

so

$$\lim_{d_t \downarrow 0} \left(\frac{P_t}{d_t}\right) = \beta \left[\left(\left(\frac{(1-\rho)}{d_t}\right)^2 \sigma^2/2 \right) \left(\frac{P_{t+1}}{d_{t+1}} + 1\right) \right] \quad (42)$$

so since P_{t+1}/d_{t+1} is a finite number we should have that

$$d_t^2 \lim_{d_t \downarrow 0} \left(\frac{P_t}{d_t}\right) = \beta \left[\left(\left(\frac{(1-\rho)}{1}\right)^2 \sigma^2/2 \right) \left(\frac{P_{t+1}}{d_{t+1}} + 1\right) \right] \quad (43)$$

which should imply that $P_t d_t$ is a finite number even as $d_t \downarrow 0$. To have both limits be finite, we might be able to use a trick like the ones proposed by Boyd (1990). This would involve multiplying by some $f(d)$ that approaches d_t^2 as d_t approaches zero but approaches 1 as d_t approaches infinity. Like, $f(d) = d^2 \left(\frac{1}{1+d^2}\right)$? (The idea is that $f(d)P_t/d_t$ might be finite in both limits (and everywhere in between) even if P_t/d_t is not). [Think more about this later].

Alternative. The solution to the AR(1) case is surely somewhere between the solutions to the IID and RW cases. That means that it is between

$$\log P_t \approx \rho \log d_t - \log(\vartheta - (1/2)\rho(\rho-1)\sigma^2) \quad (44)$$

and

$$\log P_t \approx \log d_t - \log(\vartheta - (1/2)\rho(\rho-1)\sigma^2) \quad (45)$$

which can surely somehow be used to produce a reasonable limit. Actually, it seems pretty clear that the relevant comparison is to the IID case.

References

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