The Equity Premium Puzzle and the Riskfree Rate

This handout derives the equity premium puzzle (Mehra and Prescott (1985)). Consider a representative agent solving the joint consumption and portfolio allocation problem:

\[
v(m_t) = \max_{\{c_t, \xi_t\}} u(c_t) + \mathbb{E}_t \left[ \sum_{n=1}^{\infty} \beta^n u(c_{t+n}) \right] \tag{1}\]

s.t.

\[
m_{t+1} = (m_t - c_t)R_{t+1} + y_{t+1} \tag{2}\]

\[
R_{t+1} = \xi_t R_{t+1} + (1 - \xi_t) \mathbb{R} \tag{3}\]

where \(R\) denotes the return on a perfectly riskless asset and \(R_{t+1}\) denotes the return on equities (the risky asset) held between periods \(t\) and \(t+1\), \(\xi_t\) is the share of end-of-period savings invested in the risky asset, \(R_{t+1}\) is the portfolio-weighted rate of return, and \(y_{t+1}\) is noncapital income in period \(t+1\).

As usual, the objective can be rewritten in recursive form:

\[
v(m_t) = \max_{\{c_t, \xi_t\}} u(c_t) + \mathbb{E}_t \left[ v\left( \frac{\xi_t R_{t+1} + (1 - \xi_t) \mathbb{R}}{R_{t+1}} (m_t - c_t) + y_{t+1} \right) \right] \tag{4}\]

The first order condition with respect to \(c_t\) is

\[
u'(c_t) = \beta \mathbb{E}_t \left[ R_{t+1} v'(m_{t+1}) \right]. \tag{5}\]

and the FOC with respect to \(\xi_t\) is

\[
\mathbb{E}_t[(R_{t+1} - \mathbb{R}) v'(m_{t+1})(m_t - c_t)] = 0 \tag{6}\]

\[
\mathbb{E}_t[(R_{t+1} - \mathbb{R}) v'(m_{t+1})] = 0. \tag{7}\]

But the usual logic of the Envelope theorem tells us that

\[
u'(c_{t+1}) = v'(m_{t+1}), \tag{8}\]

so, substituting (8) into (5) and (7) we have

\[
u'(c_t) = \mathbb{E}_t \left[ \beta R_{t+1} u'(c_{t+1}) \right] \tag{9}\]

\[
\mathbb{E}_t[(R_{t+1} - \mathbb{R}) u'(c_{t+1})] = 0 \tag{10}\]

Now assume CRRA utility, \(u(c) = c^{1-\rho}/(1-\rho)\) and divide both sides of (10) by \(c_t^{1-\rho}\) to get

\[
\mathbb{E}_t[(c_{t+1}/c_t)^{-\rho}(R_{t+1} - \mathbb{R})] = 0. \tag{11}\]
Now recall the following two facts:

Fact 1: If $\Delta c_{t+1}/c_t$ is small, $c_{t+1}/c_t \approx 1 + \Delta \log c_{t+1}$.
Fact 2: If $z$ is small, $(1 + z)^\lambda \approx 1 + \lambda z$.

Using these two facts, equation (11) can be approximated by

$$\mathbb{E}_t[(1 - \rho \Delta \log c_{t+1})(R_{t+1} - R)] \approx 0.$$ (12)  

Using one more fact,

Fact 3: $\mathbb{E}[xy] = \mathbb{E}[x] \mathbb{E}[y] + \text{cov}(x, y)$,

we get

$$(1 - \rho \mathbb{E}_t[\Delta \log c_{t+1}]) (R - \mathbb{E}_t[R_{t+1}]) + \text{cov}_t(-\rho \Delta \log c_{t+1}, -R_{t+1}) \approx 0$$ (13)

or

$$\mathbb{E}_t[R_{t+1}] - R \approx \frac{\rho \text{cov}_t(\Delta \log c_{t+1}, R_{t+1})}{1 - \rho \mathbb{E}_t[\Delta \log c_{t+1}]}$$ (14)

$$\approx \rho \text{cov}_t(\Delta \log c_{t+1}, R_{t+1})$$ (15)

where the last approximation holds because $\mathbb{E}_t[\Delta \log c_{t+1}]$ is small.

**The Equity Premium Puzzle**

Because this expression must hold at all $t$, we can check it empirically by calculating

empirical estimates of the two components and assuming that the sample averages

are equal to the representative agent’s expectations. That is, if we have data

for periods $1 \ldots n$, we assume that the unconditional expectations correspond to

the sample means, $\mathbb{E}[R] = (1/n) \sum_{s=1}^n R_s$; $\mathbb{E}[\Delta \log c] = (1/n) \sum_{s=1}^n \Delta \log c_s$; and $\text{cov}(\Delta \log c, R) = (1/n) \sum_{s=1}^n (\Delta \log c_s - \mathbb{E}[\Delta \log c])(R_s - \mathbb{E}[R])$.

The equity premium puzzle is essentially that $\text{cov}(\Delta \log c, R)$ is very small (about 0.004) but $\mathbb{E}[R] - R$ is about 0.08 (stocks have earned real returns of about 8 percent more than riskless assets over the historical period), which means that the only way equation (15) can hold is if $\rho$ is implausibly large (these values imply a value of $\rho = 20$).

How do we know what plausible values of $\rho$ are? Consider the following. You must

choose between a gamble in which you consume $50,000 for the rest of your life with

probability 0.5 and $100,000 with probability 0.5, or consuming some amount $X$ with

certainty. The coefficient of relative risk aversion determines the $X$ which would make

you indifferent between consuming $X$ or being exposed to the gamble. For example,

if $\rho = 0$, then you have no risk aversion at all and you will be indifferent between

$\$75,000 with certainty and the 50/50 gamble with expected value of $\$75,000. Here
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</table>

are the values of $X$ associated with different values of $\rho$ (table taken from Mankiw and Zeldes Mankiw and Zeldes (1989).)

The Riskfree Rate Puzzle

Rewrite the consumption Euler equation (9) as

$$u'(c_t) = \mathbb{E}_t[\beta (R + \varsigma_t[R_{t+1} - R])u'(c_{t+1})]$$

and note that from (10) we know that $\mathbb{E}_t[\beta \varsigma_t(R_{t+1} - R)u'(c_{t+1})] = 0$ so that (16) reduces to the ordinary Euler equation

$$u'(c_t) = \mathbb{E}_t[\beta Ru'(c_{t+1})]$$

$$1 = \beta R \mathbb{E}_t[(c_{t+1}/c_t)^{-\rho}].$$

Using the same ‘facts’ and approximations as above, we get the standard approximation to the Euler equation,

$$\Delta \log c_{t+1} \approx (1/\rho)(r - \vartheta).$$

The ‘riskfree rate puzzle’ is that average consumption growth per capita has been about 1.5 percent (in the US in the postwar period) while real riskfree interest rates have been at most 1 percent. Even if we assume a time preference rate of $\vartheta = 0$ (no impatience at all, e.g. $\beta = 1$), the only way this equation can hold is if $\rho$ is a very small number (maybe even less than one). Of course, this is precisely the opposite of the conclusion of the equity premium puzzle, which implies the $\rho$ must be very large.
References
