An Entrepreneur's Problem Under Perfect Foresight

Consider a firm characterized by the following:

k_t	-	Firm's capital stock at the beginning of period t
$\mathrm{f}(k)$	-	The firm's total output depends only on k
i_t	-	Investment in period t
$\mathrm{j}(i,k)$	-	AdJustment costs associated with investment i given capital k
$\xi_t = i_t + \mathrm{j}_t$	-	eXpenditures (purchases plus adjustment costs) on investment
$\beta = 1/\mathsf{R}$	-	Discount factor for future profits (inverse of interest factor)

Suppose that the firm's goal is to pick the sequence i_t that solves:

$$\mathrm{e}(k_t) = \max_{\{\mathrm{i}\}_t^\infty} \sum_{n=0}^{\infty} \beta^n \left(f_{t+n} - i_{t+n} - \mathrm{j}_{t+n}\right) \tag{1}$$

subject to the transition equation for capital,

$$k_{t+1} = (k_t + i_t)\daleth \tag{2}$$

where $\daleth = (1-\delta)$ is the amount of capital left after one period of depreciation at rate δ.[1] e_t is the value of the profit-maximizing firm: If capital markets are efficient this is the equity value that the firm would command if somebody wanted to buy it.

The firm's Bellman equation can be written:

$$\begin{aligned}
\mathrm{e}_t(k_t) &= \max_{\{\mathrm{i}\}_t^\infty} \sum_{n=0}^{\infty} \beta^n \left(f_{t+n} - i_{t+n} - \mathrm{j}_{t+n}\right) \\
&= \max_{\{i_t\}} \ f_t - i_t - \mathrm{j}(i_t, k_t) + \beta \left[\max_{\{i\}_{t+1}^\infty} \sum_{n=0}^{\infty} \beta^n \left(f_{t+1+n} - i_{t+1+n} - \mathrm{j}_{t+1+n}\right) \right] \\
&= \max_{\{i_t\}} \ f_t - i_t - \mathrm{j}(i_t, k_t) + \beta \mathrm{e}_{t+1}\left((k_t + i_t)\daleth\right)
\end{aligned}$$

Define j_t^i as the derivative of adjustment costs with respect to the level of investment.

The first order condition for optimal investment implies:

$$\begin{aligned}
0 &= -1 - j_t^i + \daleth\beta \mathrm{e}_{t+1}^k(k_{t+1}) \\
1 + j_t^i &= \daleth\beta \mathrm{e}_{t+1}^k(k_{t+1})
\end{aligned} \tag{3}$$

[1] There are some small differences between the formulation of the model here and in `qModel`. Here, investment costs are paid at the time of investment and the depreciation factor applies to $(k_t + i_t)$ rather than just k_t. These changes simplify the computational solution without changing any key results.

In words: The marginal cost of an additional unit of investment (the LHS) should be equal to the discounted marginal value of the resulting extra capital (the RHS).

The Envelope theorem says

$$
\begin{aligned}
\mathrm{e}^k_t(k_t) &= \mathrm{f}^k(k_t) - \mathrm{j}^k_t + \beta \mathrm{e}^k_{t+1}(k_{t+1}) \overbrace{\left(\frac{\partial k_{t+1}}{\partial k_t}\right)}^{\daleth} \\
&= \mathrm{f}^k(k_t) - \mathrm{j}^k_t + \underbrace{\beta \daleth \mathrm{e}^k_{t+1}(k_{t+1})}_{=(1+\mathrm{j}^i_t) \text{ from } (3)}
\end{aligned}
$$

So the corresponding $t+1$ equation can be substituted into (3) to obtain

$$
(1+\mathrm{j}^i_t) = \left(\mathrm{f}^k(k_{t+1}) + (1 + \mathrm{j}^i_{t+1} - \mathrm{j}^k_{t+1})\right) \daleth \beta \tag{4}
$$

which is the Euler equation for investment.

Now suppose that a steady state exists in which the capital stock is at its optimal level and is not adjusting, so costs of adjustment are zero: $\mathrm{j}_t = \mathrm{j}_{t+1} = j^i_t = j^i_{t+1} = j^k_t = j^k_{t+1} = 0$.

If $j^i_t = j^i_{t+1} = j^k_{t+1}$ then (4) reduces to

$$
\begin{aligned}
1 &= \overbrace{\beta}^{=\mathsf{R}^{-1}} \daleth \left[\mathrm{f}^k(\check{k}) + 1\right] \\
\mathsf{R} &= \daleth(\mathrm{f}^k(\check{k}))
\end{aligned} \tag{5}
$$

so that the capital stock is equal to the value that causes its marginal product to match the interest factor, after compensating for depreciation.

Another way to analyze this problem is in terms of the marginal value of capital, $\lambda_t \equiv \mathrm{e}^k_t(k_t)$.

Rewrite (4) as

$$
\begin{aligned}
\lambda_t &= \mathrm{f}^k(k_t) - \mathrm{j}^k_t + \beta\daleth(\lambda_t + \lambda_{t+1} - \lambda_t) \\
&= \mathrm{f}^k(k_t) - \mathrm{j}^k_t + \beta\daleth(\lambda_t + \Delta\lambda_{t+1}) \\
(1-\beta\daleth)\lambda_t &= \mathrm{f}^k(k_t) - \mathrm{j}^k_t + \Delta\lambda_{t+1} \\
\lambda_t &= \frac{\mathrm{f}^k(k_t) - \mathrm{j}^k_t + \Delta\lambda_{t+1}}{(1-\beta\daleth)}
\end{aligned} \tag{6}
$$

and the phase diagram is constructed using the $\Delta\lambda_{t+1} = 0$ locus. In the vicinity of the steady state, we can assume $\mathrm{j}^k_t \approx 0$ in which case the $\Delta\lambda_{t+1} = 0$ locus becomes

$$
\lambda_t = \frac{\mathrm{f}^k(k_t)}{(1-\beta\daleth)} \tag{7}
$$

which implies (since $\mathrm{f}^k(k_t)$ is downward sloping in k_t) that the $\Delta\lambda_t = 0$ locus (that is, the $\lambda_t(k_t)$ function that corresponds to $\Delta\lambda_t = 0$) is downward sloping.

The phase diagram is depicted below.

The steady state of the model will be the point at which $k_{t+1} = k_t = \check{k}$, implying from (2) a steady-state investment rate of

$$\begin{aligned} \check{k} &= (\check{k} + \check{i})\daleth \\ \check{i} &= (1 - \daleth)\check{k}/\daleth = (\delta/\daleth)\check{k} \end{aligned} \tag{8}$$

and solving (5) for $\mathrm{f}^k(\check{k})$

$$\left(\frac{(1 - \beta\daleth)}{\beta\daleth}\right) = \mathrm{f}^k(\check{k}) \tag{9}$$

which can be substituted into (7) to obtain the steady-state value of λ:

$$\check{\lambda} = \left(\frac{\mathsf{R}}{\daleth}\right). \tag{10}$$

We now wish to modify the problem in two ways. First, we have been assuming that the firm has only physical capital, and no financial assets. Second, we have been assuming that the manager running the firm only cares about the PDV of profits; suppose instead we want to assume that the firm is a small business run by an entrepreneur who must live off the dividends of the firm, and thus they are maximizing the discounted sum of utility from dividends $\mathrm{u}(c_t)$ rather than just the level of discounted profits. (Note that we designate dividends by c_t; dividends were not explicitly chosen in the φ-model version of the problem, because the Modigliani-Miller theorem says that the firm's value is unaffected by its dividend policy).

We call the maximizer running this firm the 'entrepreneur.' The entrepreneur's level of monetary assets m_t evolves according to

$$m_{t+1} = f_{t+1} + (m_t - i_t - \mathrm{j}_t - c_t)\,\mathsf{R}. \tag{11}$$

That is, next period the firm's money is next period's profits plus the return factor on the money at the beginning of this period, minus this period's investment and associated adjustment costs, minus dividends paid out (which, having been paid out, are no longer part of the firm's money).

The entrepreneur's Bellman equation can now be written

$$\mathrm{v}_t(k_t, m_t) = \max_{\{i_t, c_t\}} \ \mathrm{u}(c_t) + \beta \mathrm{v}_{t+1}(k_{t+1}, m_{t+1})$$

Value is simply the discounted sum of utility from future dividends:

$$
\begin{aligned}
\mathrm{v}_t(k_t, m_t) &= \max_{\{i,c\}_t^\infty} \sum_{n=0}^{\infty} \beta^n \mathrm{u}(c_{t+n}) \\
&= \max_{\{i,c\}_t^\infty} \left(\mathrm{u}(c_t) + \beta \sum_{n=0}^{\infty} \beta^n \mathrm{u}(c_{t+1+n}) \right) \\
&= \max_{\{i_t,c_t\}} \quad \mathrm{u}(c_t) + \beta \mathrm{v}_{t+1}(k_{t+1}, m_{t+1}).
\end{aligned}
$$

Assume that f and j do not depend directly on m_t. That is, their partial derivatives with respect to m_t are zero.

Then we will have

FOC wrt c_t:

$$\mathrm{u}'(c_t) = \mathsf{R}\beta v_{t+1}^m \tag{12}$$

Envelope wrt m_t:

$$v_t^m = \mathsf{R}\beta v_{t+1}^m \tag{13}$$

and combining the FOC with the Envelope theorem we get the usual

$$
\begin{aligned}
v_t^m &= \mathsf{R}\beta v_{t+1}^m \\
&= \mathrm{u}'(c_t) \\
&= \mathsf{R}\beta \mathrm{u}'(c_{t+1}) \\
&= \mathrm{u}'(c_{t+1})
\end{aligned}
$$

where the last line follows because we have assumed $\mathsf{R}\beta = 1$.

Now note that the value function can be rewritten as

$$\mathrm{v}_t(k_t, m_t) = \max_{\{i_t, m_{t+1}\}} \quad \mathrm{u}((f_{t+1} - m_{t+1})/\mathsf{R} + m_t - i_t - \mathrm{j}_t) + \beta \mathrm{v}_{t+1}(k_{t+1}, m_{t+1})$$

This holds because maximizing with respect to m_{t+1} (subject to the accumulation equation) is equivalent to maximizing with respect to the components of m_{t+1}.

For the version in (14) the FOC with respect to i_t is

$$\mathrm{u}'(c_t)((1 + \mathrm{j}_t^i) - f_{t+1}^k \daleth/\mathsf{R}) = \daleth\beta v_{t+1}^k \tag{14}$$

This holds because the derivative of the RHS of (14) with respect to i_t is

$$\mathrm{u}'(c_t) \left(\left(\frac{\partial f_{t+1}}{\partial k_{t+1}} \frac{\partial k_{t+1}}{\partial i_t} \right) /\mathsf{R} - \frac{\partial i_t}{\partial i_t} - \frac{\partial \mathrm{j}_t}{\partial i_t} \right) + \beta \left(\frac{\partial k_{t+1}}{\partial i_t} \right) \mathrm{v}_{t+1}^k(k_{t+1}, m_{t+1}) \tag{15}$$

(remember that m_{t+1} is a control variable and thus its derivative with respect to investment is zero) so the FOC translates to

$$\mathrm{u}'(c_t)(f^k_{t+1}\daleth/\mathsf{R} - 1 - \mathrm{j}^i_t) + \beta\daleth v^k_{t+1} = 0 \tag{16}$$

which reduces to (14).

Now we can use the envelope theorem with respect to k_t to show that

$$v^k_t = \mathrm{u}'(c_t)(f^k_{t+1}\daleth/\mathsf{R} - \mathrm{j}^k_t) + \beta\daleth v^k_{t+1} \tag{17}$$

This can be seen by directly taking the derivative of the RHS of (14) with respect to k_t:

$$\mathrm{u}'(c_t)\left(\left(\frac{\partial f_{t+1}}{\partial k_{t+1}}\frac{\partial k_{t+1}}{\partial k_t}\right)/\mathsf{R} - \frac{\partial \mathrm{j}_t}{\partial k_t}\right) + \beta\left(\frac{\partial k_{t+1}}{\partial k_t}\right)v^k_{t+1} \tag{18}$$

and noting that the Envelope theorem tells us the derivatives with respect to the controls m_{t+1} and i_t are zero while $\partial k_{t+1}/\partial k_t = \daleth$.

Now we can combine (14) and (17) to derive the Euler equation for investment

$$(1 + \mathrm{j}^i_t) = \daleth\beta\left[\mathrm{f}^k(k_{t+1}) + (1 + \mathrm{j}^i_{t+1} - \mathrm{j}^k_{t+1})\right]. \tag{19}$$

To see this, start with the Envelope theorem,

$$\begin{aligned} v^k_t &= \mathrm{u}'(c_t)(f^k_{t+1}\daleth/\mathsf{R} - \mathrm{j}^k_t) + \overbrace{\daleth\beta v^k_{t+1}}^{=\mathrm{u}'(c_t)((1+\mathrm{j}^i_t)-f^k_{t+1}\daleth/\mathsf{R}) \text{ from } (14)} \\ &= \mathrm{u}'(c_t)(f^k_{t+1}\daleth/\mathsf{R} - \mathrm{j}^k_t) + \mathrm{u}'(c_t)((1 + \mathrm{j}^i_t) - f^k_{t+1}\daleth/\mathsf{R}) \\ &= \mathrm{u}'(c_t)\left(1 + \mathrm{j}^i_t - \mathrm{j}^k_t\right) \end{aligned} \tag{20}$$

which means that we can rewrite (14) substituting the rolled-forward version of (20)

$$\begin{aligned} \mathrm{u}'(c_t)((1 + \mathrm{j}^i_t) - f^k_{t+1}\daleth/\mathsf{R}) &= \daleth\beta v^k_{t+1} \\ &= \daleth\beta\mathrm{u}'(c_{t+1})\left(1 + \mathrm{j}^i_{t+1} - \mathrm{j}^k_{t+1}\right) \\ (1 + \mathrm{j}^i_t) &= \daleth\beta\left[\mathrm{f}^k(k_{t+1}) + (1 + \mathrm{j}^i_{t+1} - \mathrm{j}^k_{t+1})\right] \end{aligned}$$

where the last line follows because with $\mathsf{R}\beta = 1$ we know that $c_{t+1} = c_t$ implying $\mathrm{u}'(c_{t+1}) = \mathrm{u}'(c_t)$.

Since behavior (for either a firm manager or a consumer) is determined by Euler equations, and the Euler equations for both consumption and investment are identical in this model to the Euler equations for the standard models, there is no observable consequence for investment of the fact that the firm is being run by a utility maximizer, and there is no observable consequence for consumption of the fact that the consumer owns a business enterprise with costly capital adjustment.

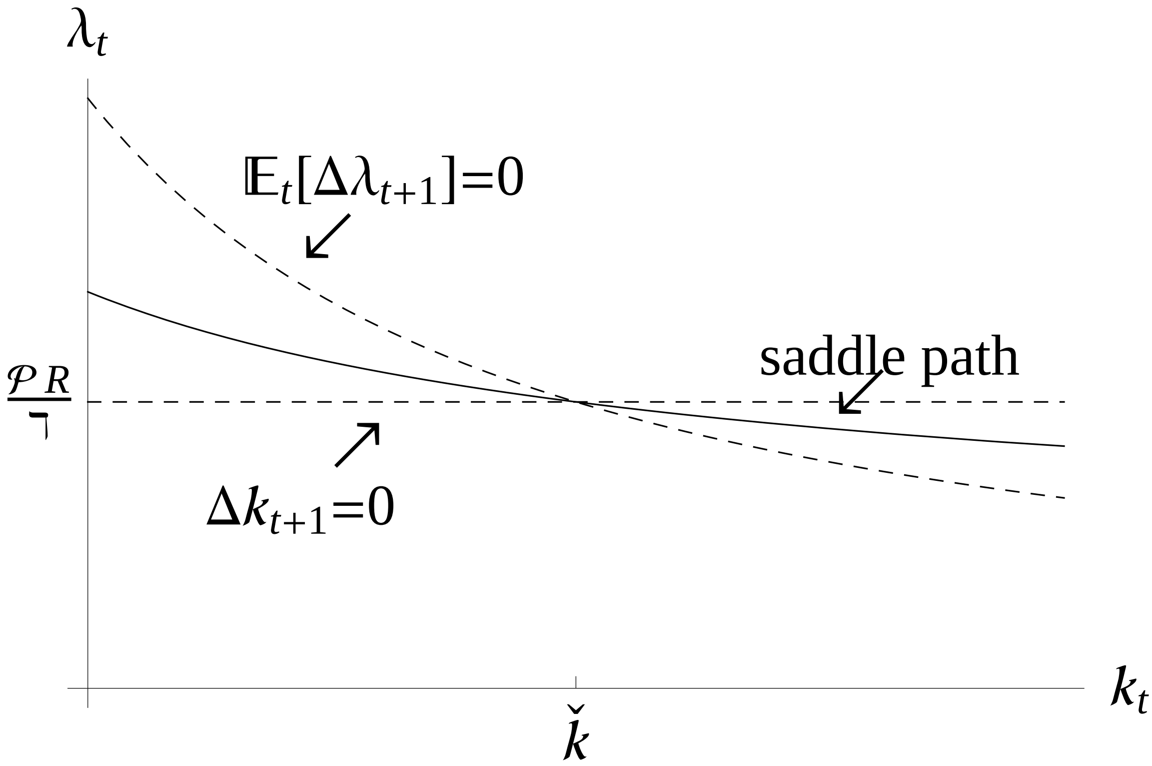
Now consider a firm of this kind that happens to have arrived in period t with positive monetary assets $m_t > 0$ and with capital equal to the steady-state target value $k_t = \check{k}$. Suppose that a thief steals all the firm's monetary assets.

> The consequences for the firm are depicted below in figure 2.
>
> Dividends follow a random walk. Thus, there is a one-time downward adjustment to the level of dividends to reflect the stolen money. Thereafter dividends are constant, as are monetary assets (which are constant at zero forever).
>
> The theft of the money has no effect on investment or the capital stock, because the firm's investment decisions are made on the basis of whether they are profitable and the theft of the money has no effect on the profitability of investments.

Now consider another kind of shock: The firm's main building gets hit by a meteor, destroying some of the firm's capital stock.

> The results are depicted below in 3.
>
> Again, because dividends follow a random walk, what the firm's managers do is to assess the effect of the meteor shock on the firm's total value and they adjust the level of dividends downward immediately to the sustainable new level of dividends. Thereafter there is no change in the level of dividends.
>
> Investment is more complicated. The firm's capital stock is obviously reduced below its steady-state value by the meteor, so there must be a period of high investment expenditures to bring capital back toward its steady state. However, the firm started out with monetary assets of zero. Therefore the high initial investment expenditures will be paid for by borrowing, driving the firm's monetary assets to a permanent negative value (the firm goes into debt to pay for its rebuilding). Gradually over time the capital stock is rebuilt back to its target level, and investment expenditures return to zero (or the level consistent with replacing depreciated capital).

Figure 1 Phase Diagram



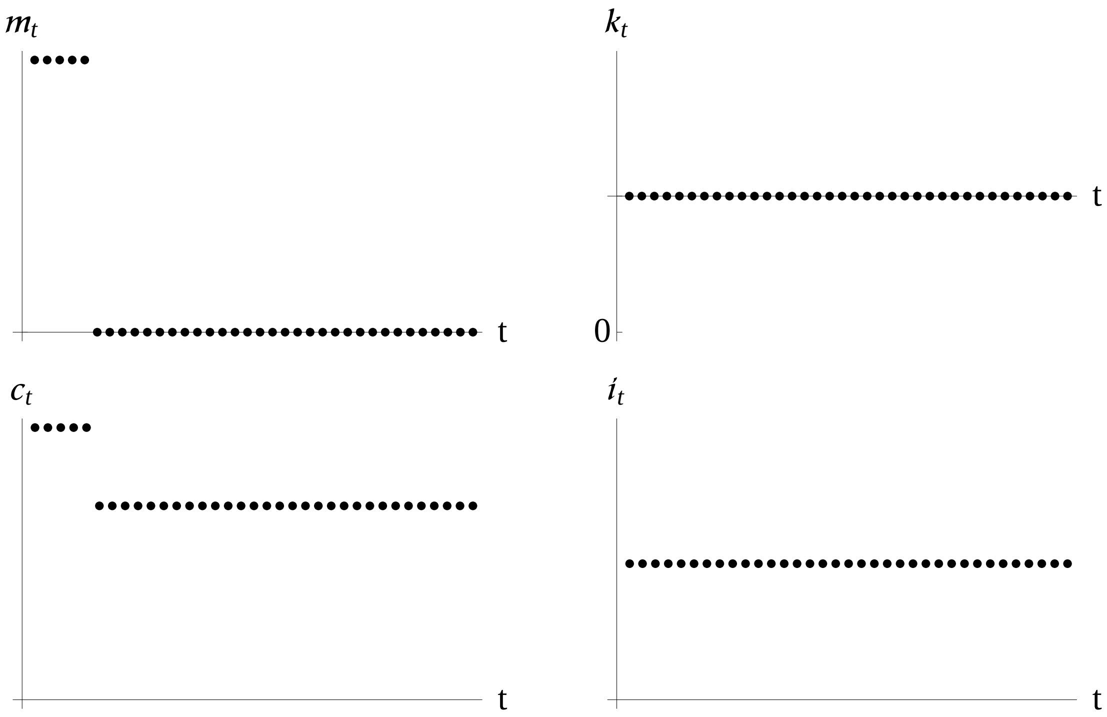


Figure 2 Negative shock to m_t

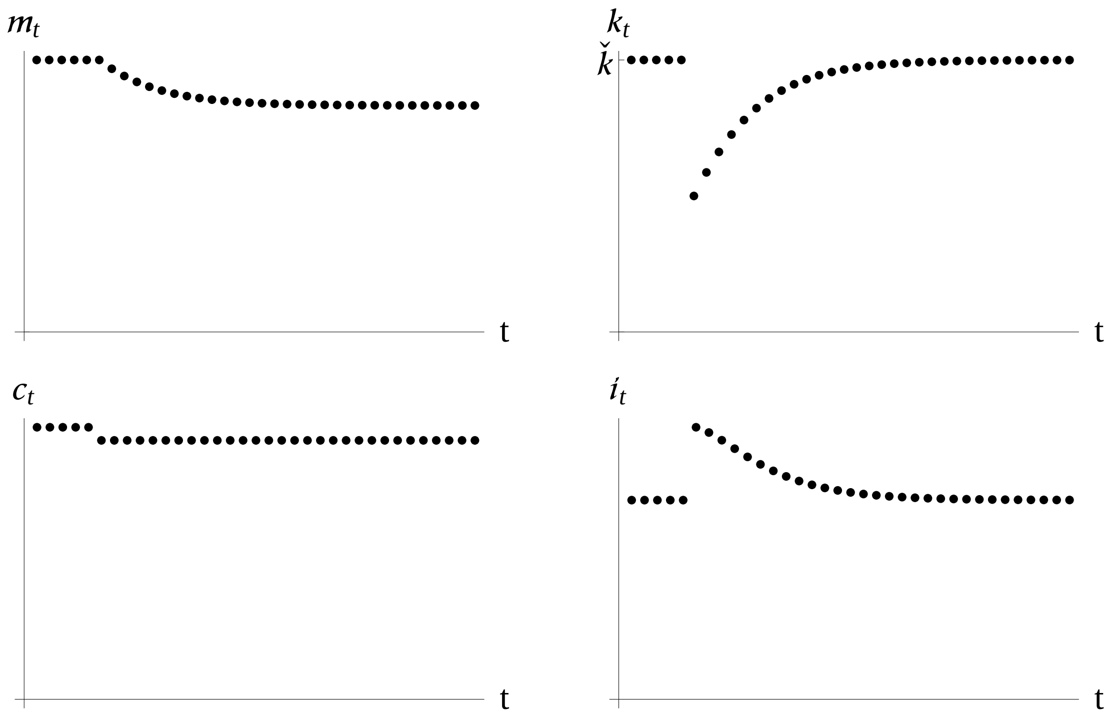


Figure 3 Negative shock to k_t