The Brock-Mirman Stochastic Growth Model

Brock and Mirman (1972) provided the first optimizing growth model with unpredictable (stochastic) shocks.

The social planner’s goal is to solve the problem:

$$\max \sum_{n=0}^{\infty} \beta^n \log C_{t+n}$$  \hspace{1cm} (1)

s.t.

$$Y_t = A_t K_t^\alpha$$  \hspace{1cm} (2)

$$K_{t+1} = Y_t - C_t$$  \hspace{1cm} (3)

where $A_t$ is the level of productivity in period $t$, which is now allowed to be stochastic (alternative assumptions about the nature of productivity shocks are explored below). Note the key assumption that the depreciation rate on capital is 100 percent.

The first step is to rewrite the problem in Bellman equation form

$$V_t(K_t) = \max_{C_t} \log C_t + \beta V_{t+1}(A_t K_t^\alpha - C_t)$$  \hspace{1cm} (4)

and take the first order condition:

$$u'(C_t) = \beta E_t [A_{t+1}^\alpha K_{t+1}^{\alpha-1} u'(C_{t+1})]$$

$$\frac{1}{C_t} = \beta E_t \left[ \frac{A_{t+1}^\alpha K_{t+1}^{\alpha-1}}{C_{t+1}} \right]$$

$$1 = \beta E_t \left[ \frac{\alpha A_{t+1}^\alpha K_{t+1}^{\alpha-1}}{\gamma Y_{t+1}} \frac{C_t}{C_{t+1}} \right] \equiv R_{t+1}$$

where our definition of $R_{t+1}$ helps clarify the relationship of this equation to the usual consumption Euler equation (and you should think about why this is the right definition of the interest factor in this model).

Now we show that this FOC is satisfied by a rule that says $C_t = \gamma Y_t$, where $\gamma = 1 - \alpha \beta$. To see this, note first that the proposed consumption rule implies that $K_{t+1} = (1 - \gamma) Y_t$.

The first order condition says

$$1 = \beta E_t \left[ \frac{\alpha A_{t+1}^\alpha K_{t+1}^{\alpha-1}}{K_{t+1}} \frac{\gamma Y_t}{\gamma Y_{t+1}} \right]$$

$$= \beta E_t \left[ \frac{\alpha Y_{t+1} \gamma Y_t}{K_{t+1} \gamma Y_{t+1}} \right]$$
\[
\begin{align*}
K_{t+1} &= (1 - \gamma)Y_t \\
&= \alpha \beta A_t K_t^\alpha \\
\log k_{t+1} &= \log \alpha \beta + a_t + \alpha k_t
\end{align*}
\]
which tells us that the dynamics of the (log) capital stock have two components: One component \((a_t)\) mirrors whatever happens to the aggregate production technology; the other is serially correlated with coefficient \(\alpha\) equal to capital’s share in output.

Similarly, since log output is simply \(y = a + \alpha k\), the dynamics of output can be obtained from

\[
\begin{align*}
y_{t+1} &= a_{t+1} + \alpha k_{t+1} \\
&= \alpha (\log K_{t+1}) + a_{t+1} \\
&= \alpha (\log \alpha \beta Y_t) + a_{t+1} \\
&= \alpha (y_t + \log \alpha \beta) + a_{t+1}
\end{align*}
\]
so the dynamics of aggregate output, like aggregate capital, reflect a component that mirrors \(a\) and a serially correlated component with serial correlation coefficient \(\alpha\).

The simplest assumption to make about the level of technology is that its log follows a random walk:

\[
a_{t+1} = a_t + \epsilon_{t+1}.
\]

Under this assumption, consider the dynamic effects on the level of output from a unit positive shock to the log of technology in period \(t\) (that is, \(\epsilon_{t+1} = 1\) where \(\epsilon_s = 0 \forall s \neq t + 1\)). Suppose that the economy had been at its original steady-state level of output \(\bar{y}\) in the prior period. Then the expected dynamics of output would
be given by

\[ y_t = \bar{y} + a_t \]  \hspace{1cm} (13)

\[ \mathbb{E}_t[y_{t+1}] = \bar{y} + a_t + \alpha a_t \]  \hspace{1cm} (14)

\[ \mathbb{E}_t[y_{t+2}] = \bar{y} + a_t + \alpha a_t + \alpha^2 a_t \]  \hspace{1cm} (15)

and so on, as depicted in figure 1.

Also interesting is the case where the level of technology follows a white noise process,

\[ a_{t+1} = \bar{a} + \epsilon_{t+1}. \]  \hspace{1cm} (16)

The dynamics of income in this case are depicted in figure 2.

The key point of this analysis, again, is that the dynamics of the model are governed by two components: The dynamics of the technology shock, and the assumption about the saving/accumulation process.

For further analysis, consider a nonstochastic version of this model, with \( A_t = 1 \ \forall \ t \).

The consumption Euler equation is

\[ \frac{C_{t+1}}{C_t} = (\beta R_{t+1})^{1/\rho} \]

But this is an economy with no technological progress, so the steady-state interest rate must take on the value such that \( C_{t+1}/C_t = 1 \). Thus we must have \( \beta R = 1 \) or \( R = 1/\beta \).
In the usual model the net interest rate $r$ is equal to the marginal product of capital minus depreciation, $r = F'(K) - \delta$, so the gross interest rate is $R = 1 + F'(K) - \delta$. But in this case we have $\delta = 1$ so $R = F'(K)$.

The unconditional expectation of the interest rate, $\mathbb{E}(\log[F'(K)])$, is given by

$$
\mathbb{E}(\log[F'(K)]) = (\alpha - 1) \log[\beta] + \alpha \mathbb{E}(\log[F'(K)]) + \mathbb{E}[\epsilon]
$$

(17)

$$
\mathbb{E}(\log[F'(K)])(1 - \alpha) = (\alpha - 1) \log[\beta]
$$

(18)

$$
\mathbb{E}(\log[F'(K)]) = \log[1/\beta]
$$

(19)

$$
\mathbb{E}(\log[F'(K)\beta]) = 0
$$

(19)

Previously we derived the proposition that

$$
R\beta = 1
$$

(20)

$$
F'(K)\beta = 1
$$

(21)

$$
\log[F'(K)\beta] = 0
$$

(22)

but since this is a model in which $R = F'(K)$, this is effectively identical to the steady-state in the nonstochastic version of the model. Thus, in this special case, the modified golden rule that applies ‘in expectation’ is identical to the one that characterizes the perfect foresight model. The only difference that moving to the stochastic version of the model makes is to add an expectations operator $\mathbb{E}$ to the LHS of the nonstochastic model’s equation.
References (click to download .bib file)