1 Introduction

This handout illustrates the logic of precautionary saving by assuming that individuals face only a single, simple kind of uncertainty: A small risk of becoming permanently unemployed. More realistic assumptions yield similar conclusions (after much more work).[1]

2 The Microeconomic Consumer's Problem

The aggregate wage W_t grows by a constant factor G every period, reflecting exogenous labor productivity improvements:

$$\mathsf{W}_{t+1} = \mathsf{G}\mathsf{W}_t. \tag{1}$$

The consumer lives in a small open economy – there is a constant interest factor R. Defining $\mathbf{m}$ as **m**arket resources (net worth plus current income), $\mathbf{a}$ as end-of-period **a**ssets **a**fter **a**ll **a**ctions have been **a**ccomplished (specifically, after the consumption decision), and $\mathbf{b}$ as **b**ank **b**alances **b**efore receipt of labor income, the dynamic budget constraint (DBC) can be decomposed into the following elements:

$$\begin{aligned}
\mathbf{a}_t &= \mathbf{m}_t - \mathbf{c}_t \\
\mathbf{b}_{t+1} &= \mathsf{R}\mathbf{a}_t \\
\mathbf{m}_{t+1} &= \mathbf{b}_{t+1} + \ell_{t+1}\mathsf{W}_{t+1}\xi_{t+1}
\end{aligned} \tag{2}$$

where ℓ measures the consumer's labor productivity ('endowment') and ξ is a dummy variable indicating the consumer's employment state: Everyone is either employed (state 'e'), in which case $\xi = 1$, or unemployed (state 'u'), in which case $\xi = 0$, so that for unemployed individuals labor income is zero.[2]

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[1] The model simplifies many of the key results in Carroll (1992) and Carroll (1997) using a discrete-time version of the elegant continuous-time model of Toche (2005). For a brief summary of the precautionary saving literature, see Carroll and Kimball (2007); for a more rigorous treatment of the theoretical issues, see Carroll (2022).

[2] We could allow for unemployment insurance by modifying the values of ξ associated with the two states. The key conclusions would not change.

2.1 The Unemployed Consumer's Problem

Once a person becomes unemployed, that person can never become employed again (i.e. if $\xi_t = 0$ then $\xi_{t+1} = 0$). Consumers have a CRRA felicity function[3] $u(\bullet) = \bullet^{1-\rho}/(1-\rho)$, and they discount future felicity geometrically by β per period.

The solution to the unemployed consumer's optimization problem is[4]

$$\mathbf{c}_t^u = \underbrace{\left(1 - \overbrace{\mathsf{R}^{-1}(\mathsf{R}\beta)^{1/\rho}}^{\equiv \mathbf{P}_\mathsf{R}} \right)}_{\equiv \kappa} \mathbf{b}_t, \tag{3}$$

where the u superscript signifies the consumer's (un)employment status; κ is the marginal propensity to consume for the perfect foresight consumer, which is strictly below the MPC for the problem with uncertainty (Carroll and Kimball (1996)); and $\mathbf{P}_\mathsf{R}$ is what Carroll (2022) calls the 'return patience factor.'[5]

We now impose the 'return impatience condition' (RIC),

$$\underbrace{\left(\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}} \right)}_{=\mathbf{P}_\mathsf{R}} < 1 \tag{4}$$

which deserves its name because it is the condition that guarantees that $\kappa > 0$ – the consumer must not be *so* patient that, given the interest rate, a boost to resources fails to boost spending.[6] An alternative (equally correct) interpretation is that the condition guarantees that the PDV of consumption for the unemployed consumer is not infinity.[7]

For many purposes (not least, the calibration of the model), it turns out to be useful to alternatively express impatience conditions like (4) in terms of the upper bound of the range of time preference factors $\bar{\beta}$ that satisfy the condition; solving (4) for β, we designate this object

$$\bar{\beta}_{\mathrm{RIC}} = \mathsf{R}^{\rho-1} \tag{5}$$

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[3]'Felicity' refers explicitly to a one-period, or in a continuous-time model, an instantaneous utility function.

[4]See `PerfForesightCRRA` for a derivation, noting that human wealth is zero for the unemployed consumer.

[5]The Old English letter $\mathbf{P} = (\mathsf{R}\beta)^{1/\rho}$, which conveniently looks something like a combination of R, β, and ρ, is used to designate the 'absolute patience factor' which determines whether consumption will rise ($\mathbf{P} > 1$), stay the same ($\mathbf{P} = 1$), or fall ($\mathbf{P} < 1$) in the perfect foresight problem (`PerfForesightCRRA`). Terms like this are all defined in Carroll (2020) and citations to that paper will henceforth be omitted when new terms are introduced, on the understanding that the reader knows to see Carroll (2020) for further definition and discussion.

[6]'Pathologically patient' consumers who do not satisfy this condition can be thought of as people who would hoard any incremental resources in order to enable even more extra spending in the distant future.

[7]For a perfect foresight consumer, `PerfForesightCRRA` shows that consumption grows by the factor $\mathbf{P} = (\mathsf{R}\beta)^{1/\rho}$, so if we do not impose the RIC, consumption would 'want' to grow by a factor greater than the factor R by which it is being discounted.

and write the alternative version of the constraint as

$$\beta < \bar{\beta}_{\mathrm{RIC}}. \tag{6}$$

$\mathbf{Þ}_{\mathsf{R}}$ is the 'return patience factor' because it defines the patience factor $\mathbf{Þ}$ relative to the return factor R; correspondingly, we define the 'return patience rate' as lower-case

$$\begin{aligned} þ_{\mathsf{r}} &\equiv \log \mathbf{Þ}_{\mathsf{R}} \\ &\approx \mathbf{Þ}_{\mathsf{R}} - 1 \\ &= -\kappa \end{aligned} \tag{7}$$

and we say that a consumer is 'return impatient' if the RIC (4) holds (equivalent conditions are $þ_{\mathsf{r}} < 0$ and $\kappa > 0$).[8]

2.2 The Employed Consumer's Problem

2.2.1 Unemployment Risk as a Mean Preserving Spread in Human Wealth

If a person who is employed in period t ($\xi_t = 1$) is still employed next period ($\xi_{t+1} = 1$), market resources will be

$$\mathbf{m}^e_{t+1} = (\mathbf{m}^e_t - \mathbf{c}^e_t)\mathsf{R} + \mathsf{W}_{t+1}\ell_{t+1}. \tag{8}$$

But employed consumers face a constant risk $\mho$ of becoming unemployed. It will be convenient to define $\cancel{\mho} \equiv 1 - \mho$ as the probability that a consumer does *not* become unemployed. Whether the consumer is employed or not, the consumer's labor productivity ℓ is well-defined:[9] For convenience, ℓ is assumed to grow by a factor $\cancel{\mho}^{-1}$ every period,

$$\ell_{t+1} = \ell_t/\cancel{\mho}, \tag{9}$$

which means that for a consumer who remains employed, labor income will grow by factor

$$\mathbf{\Phi} = \mathsf{G}/\cancel{\mho} \tag{10}$$

so that the *expected* labor income growth factor for employed consumers is the same G as in the perfect foresight case:

$$\mathbb{E}_t[\mathsf{W}_{t+1}\ell_{t+1}\xi_{t+1}] = \left(\frac{\ell_t \mathsf{G}\mathsf{W}_t}{\cancel{\mho}}\right)(\mho \times 0 + \cancel{\mho} \times 1)$$

$$\left(\frac{\mathbb{E}_t[\mathsf{W}_{t+1}\ell_{t+1}\xi_{t+1}]}{\mathsf{W}_t\ell_t}\right) = \mathsf{G},$$

which is the reason for (9)'s assumption about the growth of individual labor productivity: It implies that an increase in $\mho$ is a pure increase in uncertainty with no effect on the

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[8]Throughout, we will casually treat logs of factors like $\mathbf{Þ}_{\mathsf{R}}$ as equivalent to the level minus 1; that is, we will treat expressions like those in (7) as interchangeable, which is an appropriate approximation so long as the factor is 'close' to 1.

[9]'Labor productivity' is purely hypothetical for a consumer who is unemployed; but defining it even for unemployed consumers simplifies notation and some later analysis.

PDV of expected labor income ('human wealth'); an increase in $\mho$ therefore constitutes a 'mean-preserving spread' in human wealth.

2.2.2 First Order Optimality Condition

The same solution methods used in `PerfForesightCRRA` can be applied here too (take the first order condition with respect to $\mathbf{c}$, use the `Envelope` theorem); the only difference is the need to keep the expectations operator in place. Using $\bullet$ as a placeholder for 'e' or 'u,' the usual steps lead to the standard consumption Euler equation:

$$\mathrm{u}'(\mathbf{c}_t^e) = \mathsf{R}\beta\,\mathbb{E}_t\left[\mathrm{u}'(\mathbf{c}_{t+1}^\bullet)\right]$$

$$1 = \mathsf{R}\beta\,\mathbb{E}_t\left[\left(\frac{\mathbf{c}_{t+1}^\bullet}{\mathbf{c}_t^e}\right)^{-\rho}\right]. \tag{11}$$

Defining nonbold variables as the bold equivalent divided by the level of permanent labor income for an employed consumer, e.g. $c_t^e = \mathbf{c}_t^e/(\mathsf{W}_t\ell_t)$, we can rewrite the consumption Euler equation as

$$
\begin{aligned}
1 &= \mathsf{R}\beta\,\mathbb{E}_t\left[\left(\frac{c_{t+1}^\bullet \mathsf{W}_{t+1}\ell_{t+1}}{c_t^e \mathsf{W}_t\ell_t}\right)^{-\rho}\right] \\
&= \mathsf{R}\beta\,\mathbb{E}_t\left[\left(\frac{c_{t+1}^\bullet}{c_t^e}\boldsymbol{\Phi}\right)^{-\rho}\right] \\
&= \boldsymbol{\Phi}^{-\rho}\mathsf{R}\beta\,\mathbb{E}_t\left[\left(\frac{c_{t+1}^\bullet}{c_t^e}\right)^{-\rho}\right] \\
&= \boldsymbol{\Phi}^{-\rho}\mathsf{R}\beta\left\{(1-\mho)\left(\frac{c_{t+1}^e}{c_t^e}\right)^{-\rho} + \mho\left(\frac{c_{t+1}^u}{c_t^e}\right)^{-\rho}\right\}
\end{aligned}
\tag{12}
$$

2.2.3 Analysis and Intuition of Consumption Growth

It will be useful now to define a 'growth patience factor' (terminology justified below):

$$\text{Þ}_{\boldsymbol{\Phi}} = \left(\frac{(\mathsf{R}\beta)^{1/\rho}}{\boldsymbol{\Phi}}\right), \tag{13}$$

which is the factor by which c^e would grow in the perfect foresight version of the model with permanent income growth factor $\boldsymbol{\Phi}$ (again see `PerfForesightCRRA`). Using this,

([12](#)) can be written as

$$1 = Þ_{\Phi}^{\rho} \left(\frac{c_{t+1}^e}{c_t^e} \right)^{-\rho} \left\{ (1 - \mho) + \mho \left[\left(\frac{c_{t+1}^u}{c_t^e} \right) \left(\frac{c_t^e}{c_{t+1}^e} \right) \right]^{-\rho} \right\}$$

$$= Þ_{\Phi}^{\rho} \left(\frac{c_{t+1}^e}{c_t^e} \right)^{-\rho} \left\{ 1 + \mho \left[\left(\frac{c_{t+1}^u}{c_{t+1}^e} \right)^{-\rho} - 1 \right] \right\}$$

$$\left(\frac{c_{t+1}^e}{c_t^e} \right)^{\rho} = Þ_{\Phi}^{\rho} \left\{ 1 + \mho \left[\left(\frac{c_{t+1}^e}{c_{t+1}^u} \right)^{\rho} - 1 \right] \right\}$$

$$\left(\frac{c_{t+1}^e}{c_t^e} \right) = Þ_{\Phi} \left\{ 1 + \mho \left[\left(\frac{c_{t+1}^e}{c_{t+1}^u} \right)^{\rho} - 1 \right] \right\}^{1/\rho}. \tag{14}$$

An interesting thought: Imagine the creation of an insurance system in which ⟨beginCDC⟩ the consumer pledges their next period m_{t+1}^e to the insurance company in exchange for receiving a lognormally distributed $\hat{m}_{t+1}^e$ with an amount of variation that is (according to some metric that deserves further thought) equivalent, $\log \hat{c}_{t+1}^e \sim \mathcal{N}(\mathbb{E}_t[c_{t+1}^e] - \sigma_{\hat{c}_{t+1}}^2/2, \sigma_{\hat{c}_{t+1}}^2/2)$. In that case we would be able to compute analytically the value of $\mathbb{E}_t[(c_{t+1}^e)^{-\rho}]$ in ([12](#)) and might be able to proceed with solving a version of the model with transitory as well as permanent shocks.

Second thoughts on this: This approach misses the concavity of the consumption function, which may or may not matter much depending on the magnitude of the shocks. If it does matter, it should be possible for the insurance company to provide a contract in which the amount of consumption provided in the contract is, say, $\mathbb{E}_t[c_{t+1}^e](\eta)^\bullet$ where η is lognormal and $\bullet$ gives the best approximation to the slope of the consumption function around $\mathbb{E}_t[c_{t+1}^e]$. How many people would buy the contract would depend on the profit margin required by the insurance company, as well as consumption concavity. ⟨endCDC⟩

(This is where the perfect foresight assumption is important; without it ([14](#)) would be

$$1 = Þ_{\Phi}^{\rho} \mathbb{E}_t \left[\left(\frac{c_{t+1}^e}{c_t^e} \right)^{-\rho} \left\{ 1 + \mho \left[\left(\frac{c_{t+1}^u}{c_{t+1}^e} \right)^{-\rho} - 1 \right] \right\} \right] \tag{15}$$

and we would be unable to proceed.)

Now define $\nabla_{t+1} \equiv \left(\frac{c_{t+1}^e - c_{t+1}^u}{c_{t+1}^u} \right)$ (which is the proportion by which consumption would be greater next period for an employed than for an unemployed person), and define an 'excess prudence' factor

$$\omega = \left(\frac{\rho - 1}{2} \right). \tag{16}$$

Appendix [A](#) shows that, with some approximations, we can rewrite ([11](#)) as

$$\left(\frac{c_{t+1}^e}{c_t^e} \right) \approx (1 + \mho(1 + \omega \nabla_{t+1}) \nabla_{t+1}) Þ_{\Phi} \tag{17}$$

which can be simplified in the logarithmic utility case (where $\omega = 0$) to

$$\left(\frac{c_{t+1}^e}{c_t^e}\right) \approx (1 + \mho\nabla_{t+1})\,\mathbf{P}_{\mathbf{\Phi}}. \tag{18}$$

Now since consumption if employed c_{t+1}^e is surely greater than consumption if unemployed c_{t+1}^u, ∇_{t+1} is certainly a positive number. But since $\mathbf{P}_{\mathbf{\Phi}}$ is the value that c_{t+1}^e/c_t^e would exhibit in a perfect foresight model, this equation tells us that uncertainty boosts consumption growth[10] – in the logarithmic case, consumption growth is augmented by an amount proportional to the probability of becoming unemployed $\mho$ multiplied by the size of the 'consumption risk' (the amount by which consumption would fall if unemployment occurs).

We noted above that for any given m_t^e, an increase in uncertainty constitutes a mean-preserving spread in human wealth; thus the 'human wealth effect' of an increase in $\mho$ would be zero for a consumer without a precautionary motive. In this small-open-economy model a change in $\mho$ also has no effect on the interest rate r, and so none of the conventional determinants of consumption in the perfect foresight model (the income, substitution, and human wealth effects) is affected by a change in uncertainty. The increase in consumption growth from an increase in $\mho$ in (17) or (18) therefore must be entirely the result of the precautionary motive. Furthermore, because a profile with faster consumption *growth* can only exhibit the *same PDV* if that faster growth *starts* from a lower initial consumption *level*, we know that for any given initial value of m^e, the introduction of a risk of becoming unemployed $\mho$ induces a (precautionary) decline in consumption (and corresponding increase in saving).

Furthermore, under the (compelling) assumption that $\rho > 1$, (17) implies that a consumer with a higher degree of prudence (larger ρ and therefore larger ω) will anticipate a greater increment to consumption growth as a consequence of the introduction of uncertainty. This reflects the greater precautionary saving motive induced by a higher degree of prudence.

2.2.4 Finding the Target

The target level of m^e (if one exists) will be the point of intersection between the $\Delta c^e = 0$ and $\Delta m^e = 0$ loci.

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[10]At least, for continuing-employed consumers.

The $\Delta c^e = 0$ locus can be characterized by substituting $c_{t+1}^e = c_t^e \equiv c^e$:

$$1 = \left\{ 1 + \mho \left[\left(\frac{c_{t+1}^e}{c_{t+1}^u} \right)^\rho - 1 \right] \right\} \mathdifferent{Þ}_{\boldsymbol{\Phi}}^\rho$$

$$\mathbf{Þ}_{\boldsymbol{\Phi}}^{-\rho} = (1 - \mho) + \mho \left(\frac{c^e}{c_{t+1}^u} \right)^\rho$$

$$\underbrace{\left(\frac{\mathbf{Þ}_{\boldsymbol{\Phi}}^{-\rho} - 1 + \mho}{\mho} \right)^{1/\rho}}_{\equiv \Pi} = \left(\frac{c^e}{c_{t+1}^u} \right)$$

$$\underbrace{\left(\overbrace{\mho^{-1}(\mathbf{Þ}_{\boldsymbol{\Phi}}^{-\rho} - 1)}^{\equiv \varpi} + 1 \right)^{1/\rho}}_{\Pi = (1 + \varpi)^{1/\rho}} = \left(\frac{c^e}{c_{t+1}^u} \right)$$

which boils down to

$$c^e = c_{t+1}^u \Pi. \tag{19}$$

The importance of the linearity of the consumption function of the unemployed consumer now becomes evident: It means that the RHS of (19) is linear in c_t^e:

$$c_t^e = \overbrace{(m_t^e - c_t^e) RNrm\kappa}^{c_{t+1}^u} \Pi. \tag{20}$$

Without the linearity assumption, we would have

$$\mathrm{c}^e(m_t^e) = \mathrm{c}^u((m_t^e - \mathrm{c}^e(m_t^e))RNrm)\Pi, \tag{21}$$

and even if we had a solution for c^u, this equation defines the consumption function for the employed consumer only implicitly; this explains why the linearity of the consumption function for the unemployed consumer is essential for the tractability of the solution.

We know that $m_t^e - c_t^e > 0$ because a consumer in these circumstances (facing possible perpetual unemployment) will never borrow (see below for a full discussion of this point). Since the RIC imposes $\kappa > 0$, (20) tells us that steady-state consumption (if it exists)[11] is a positive finite number so long as $\Pi > 0$.[12]

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[11]Under some parameter values, the model has solutions under which the consumer always spends less than the amount that would leave m^e constant, and so accumulates forever.

[12]This will hold true iff the numerator on the LHS of (19) is a positive finite number; for this, we we need the condition:

$$\boldsymbol{\Phi}^\rho(\mathsf{R}\beta)^{-1} - (1 - \mho) > 0$$
$$\boldsymbol{\Phi}^\rho > (\mathsf{R}\beta(1 - \mho))$$
$$1 > \left(\frac{(\mathsf{R}\beta(1 - \mho))^{1/\rho}}{\boldsymbol{\Phi}} \right) = \mathbf{Þ}_{\boldsymbol{\Phi}}(1 - \mho)^{1/\rho}. \tag{22}$$

2.2.5 Upper Bounds for β, Given Other Parameters

As with the RIC, it may be useful to rewrite this as defining an upper bound to the permissible time preference rates:

$$\beta < \bar{\beta}_{\text{TBS}} = \left(\frac{\mathbf{\Phi}^\rho}{\mathsf{R}(1 - \mho)} \right). \tag{23}$$

In the limit as $\mho$ approaches zero, (22) reduces to a requirement that the growth patience factor is less than one,

$$\text{Þ}_{\mathbf{\Phi}} < 1, \tag{24}$$

which, as in `PerfForesightCRRA`, we call a 'growth impatience condition' (GIC) by analogy to the 'return impatience condition' (4) imposed earlier. `PerfForesightCRRA` shows that the limit of (22) as $\mho \downarrow 0$, $\mathsf{G} > \text{Þ}$, ensures that a consumer facing no uncertainty is sufficiently impatient that his wealth-to-permanent-income ratio will fall over time. We label the weaker condition (22) the 'GIC-TBS' (the version of the GIC required for a solution to exist in the Tractable Buffer Stock model). It will always hold if the plain-vanilla GIC_{G} holds because $\mho \geq 0$. Thus, a consumer who, in the absence of uncertainty, would satisfy both the RIC and the GIC_{G}, will have a positive finite target level of wealth when uncertainty is introduced.[13]

When it is useful to distinguish the version of the GIC that applies in the model with income growth of G from the corresponding condition when growth is $\mathbf{\Phi}$ we will label the two conditions $\text{GIC}_{\mathbf{\Phi}}$ and GIC_{G}, and the corresponding bounds on β are

$$\begin{aligned} \beta &< \bar{\beta}_{\text{GIC}_{\mathbf{\Phi}}} = \mathsf{R}^{\rho-1} \mathbf{\Phi}^\rho \\ \beta &< \bar{\beta}_{\text{GIC}_{\mathsf{G}}} = \mathsf{R}^{\rho-1} \mathsf{G}^\rho \end{aligned} \tag{25}$$

Using $\varphi \equiv \log \mathbf{\Phi}$, we similarly define the corresponding 'growth impatience rate':

$$\text{b}_\varphi \equiv \log \text{Þ}_{\mathbf{\Phi}} \approx \rho^{-1}(\mathsf{r} - \tau) - \varphi \tag{26}$$

so that the growth impatience condition (22) (the GIC-TBS) can also be written (approximately) as

$$\text{b}_\varphi - \rho^{-1}\mho < 0 \tag{27}$$

or, since $\varphi \approx \mathsf{g} + \mho$,

$$\rho^{-1}(\mathsf{r} - \tau - \mho) - (\mathsf{g} + \mho) < 0. \tag{28}$$

2.2.6 Why Increased Unemployment Risk Increases Effective Growth Impatience

Equation (28) becomes easier to satisfy (in the sense of requiring a lower τ) as $\mho$ increases, since in both places where $\mho$ appears on the LHS it is with a negative coefficient.

The reason the two appearances of $\mho$ have not been combined in (28) is that the separate terms reflect two logically distinct effects. The first appearance, where $\mho$ is

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[13]See appendix B for a proof that the $\text{GIC}_{\mathbf{\Phi}}$ condition is both necessary and sufficient for the existence of a target level of wealth.

premultiplied by $-\rho^{-1}$, can be interpreted as capturing the sense in which an increase in ℧ is like an increase in the discounting of the future (the coefficient on ℧ is the same as that on τ). This downweighting of the future occurs precisely because that future might not occur (if the consumer becomes unemployed).[14] The effect is much like the increase in discounting that occurs when a positive probability of death is introduced in consumption problems, cf. Blanchard (1985).

The second, separate, reason ℧ weakens growth impatience (that is, the GIC-TBS holds in more circumstances than the $\mathrm{GIC_G}$) is that we adjust labor productivity growth so that $\varphi = \mathrm{g} + ℧$ in order to maintain constant human wealth for different values of ℧ (eq. (9)). For higher ℧, permanent income growth is greater *conditional on remaining employed*; the continuously-employed consumer is effectively more 'impatient' in the relevant sense of desiring consumption growth slower than income growth.

This is essentially a mechanical result, which reflects our model's design for the purpose of examining thought experiments that manipulate the degree of uncertainty while leaving the perfect-foresight level of human wealth constant.

Note that although ℧ is our measure of uncertainty, neither of these effects of ℧ is in any meaningful sense directly a 'precautionary' effect; instead, they both reflect effects of ℧ on the relevant degree of growth impatience in the GIC-TBS condition.

2.2.7 The Target Level of m^e

Appendix B demonstrates that the RIC and the GIC-TBS are necessary conditions for the existence of a target value of market resources $\hat{m}^e$, and that the $\mathrm{GIC_{\Phi}}$ is sufficient. Appendix C solves for an explicit formula for that target.

Briefly, this is accomplished as follows. We can obtain the $\Delta c^e = 0$ locus by substituting $c_{t+1}^e = c_t^e = c^e$ into equation (20):

$$
\begin{aligned}
c^e &= m^e RNrm\kappa\Pi - c^e RNrm\kappa\Pi \\
c^e &= \left(\frac{RNrm\kappa\Pi}{1 + RNrm\kappa\Pi}\right) m^e.
\end{aligned}
\tag{29}
$$

Now we need to use a normalized version of the DBC (equation (8)),

$$
m_{t+1}^e = (m_t^e - c_t^e)RNrm + 1
\tag{30}
$$

to derive the $m_{t+1}^e = m_t^e = m^e$ locus (also referred to as the $\Delta m^e = 0$ locus):

$$
\begin{aligned}
RNrm^{-1}(m^e - 1) &= m^e - c^e \\
c^e &= m^e - RNrm^{-1}(m^e - 1) \\
&= (1 - RNrm^{-1})m^e + RNrm^{-1}.
\end{aligned}
\tag{31}
$$

The steady-state levels of m^e and c^e are the values of these two variables at which both (31) and (29) hold. This is just a set of two linear equations and two unknowns, and with a bit of algebra can be solved explicitly.

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[14]A different future – with the consumer unemployed – does occur; but here we are effectively analyzing the behavior of a consumer *contingent on that consumer remaining employed.*

In the special case of logarithmic utility ($\rho = 1$), the appendix shows that (under some strong assumptions) an approximation to target market resources will be given by

$$\hat{m}^e \approx 1 + \left(\frac{1}{(\varphi - \mathsf{r}) + \tau(1 + (\varphi + \tau - \mathsf{r})/\mho)} \right) \tag{32}$$

and that the GIC and the RIC guarantee that the denominator of the fraction is a positive number.

This expression encapsulates several of the key intuitions of the model. The 'human wealth effect' of growth (cf. Summers (1981)) is captured by the first φ term in the denominator; clearly, for any calibration for which the denominator is a positive number, increasing φ will increase the size of the denominator and therefore reduce the target level of wealth. The human wealth effect of interest rates is correspondingly captured by the $-\mathsf{r}$ term. An increase in the future discounting rate, τ, will also increase the size of the denominator and therefore reduce target wealth. Finally, a reduction in unemployment risk will boost $(\varphi + \tau - \mathsf{r})/\mho$ and therefore reduce target wealth.[15]

The assumption of log utility is restrictive, and probably does not capture sufficient aversion to consumption fluctuations. Fortunately, another special case helps to illuminate the effect of higher levels of prudence. The appendix shows that, in the special case where $\tau = \mathsf{r}$, the target level of wealth will be approximable by

$$\hat{m} \approx 1 + \left(\frac{1}{(\varphi - \mathsf{r}) + \tau(1 + (\varphi/\mho)(1 - (\varphi/\mho)\omega))} \right) \tag{33}$$

which is like (32) (with $\tau - \mathsf{r} = 0$) but with the addition of the final term involving ω which measures the amount by which prudence exceeds the logarithmic benchmark. An increase in ω reduces the denominator of (33) and thereby boosts the target level of wealth: Exactly what would be expected from an increase in the intensity of the precautionary motive.

Note that the different effects *interact* with each other, in the sense that the strength of, say, the human wealth effect will vary depending on the values of the other parameters. The ways in which these interactions make intuitive sense will repay deep reflection. (Hint: How much I care about the future governs the power that future events have in determining my targets; use the formula to think about why).

2.2.8 Conditions Required for a Perfect Foresight Solution; Existence of Target m^e

Interestingly, the limit of the buffer stock model as $\mho \downarrow 0$ is *not* the perfect foresight solution obtained when $\mho$ is exactly equal to zero.

The handout `PerfForesightCRRA` shows that in the perfect foresight context, it is necessary to impose the Finite Human Wealth Condition $\mathsf{R} > \mathsf{G}$ (henceforth, FHWC_G) to obtain a sensible solution.[16] But if the FHWC_G holds, the GIC_G is strictly stronger than the RIC, because the combination $\mathrm{Þ}/\mathsf{G} < 1$ and $\mathsf{R} > \mathsf{G}$ obviously implies $\mathrm{Þ}/\mathsf{R} < 1$.

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[15]$(\varphi + \tau - \mathsf{r}) > 0$ is guaranteed by (28) under log utility ($\rho = 1$).

[16]The appendix to `PerfForesightCRRA` shows that if the FHWC_G fails, the limit of the perfect foresight model is $\mathrm{c}(m) = \infty \; \forall \; m$, which is not a useful or plausible solution.

If we substitute $\boldsymbol{\Phi}$ for G, we can define the corresponding version of the condition in the case where growth is $\boldsymbol{\Phi}$: the FHWC$_{\boldsymbol{\Phi}}$.

It turns out that in the buffer stock model, we can relax the requirement that human wealth is finite.

We pointed out above that (22), which is necessary for the existence of a steady-state level of consumption, implies that the GIC$_{\boldsymbol{\Phi}}$ holds in the case being considered here, the limit as $\mho \downarrow 0$. The interesting question is therefore what happens when the FHWC$_\mathsf{G}$ does not hold (that is $\mathsf{G} > \mathsf{R}$).

Given that the GIC$_\mathsf{G}$ holds, if the FHWC$_\mathsf{G}$ does not hold the RIC may or may not hold: $\mathsf{G} > \mathsf{R}$ implies that $\text{Þ}/\mathsf{R} > \text{Þ}/\mathsf{G}$ but $1 > \text{Þ}/\mathsf{G}$ could be consistent with $\text{Þ}/\mathsf{R}$ being greater or less than one. But recall that our assumption is that the unemployed consumer is assumed to behave according to the perfect foresight model with human wealth equal to zero. We must therefore impose the RIC in order to obtain a nondegenerate solution. We therefore impose the RIC.

For any finite horizon, human wealth is finite, and there is a positive probability that income will be zero over the remainder of the horizon. This puts a strict bound on the extent to which consumers are willing to rely for current consumption upon future income that is unbounded in expectation (as the horizon extends) but potentially bounded in practice. In effect, the precautionary motive introduces a self-imposed borrowing constraint that prevents the (arbitrarily large) amount of future income from being something the consumer is willing to borrow against.

The consequence is that the limiting model (as $\mho \downarrow 0$) exhibits a solution with a unique finite target m^e so long as (22) holds, *even if human wealth is infinite*; in this case the $\Delta m^e = 0$ locus is downward sloping (because $1 - RNrm^{-1} < 1$; see (31)) while the $\Delta c^e = 0$ locus is upward sloping (as guaranteed by (22)). Thus, a target m^e will exist.
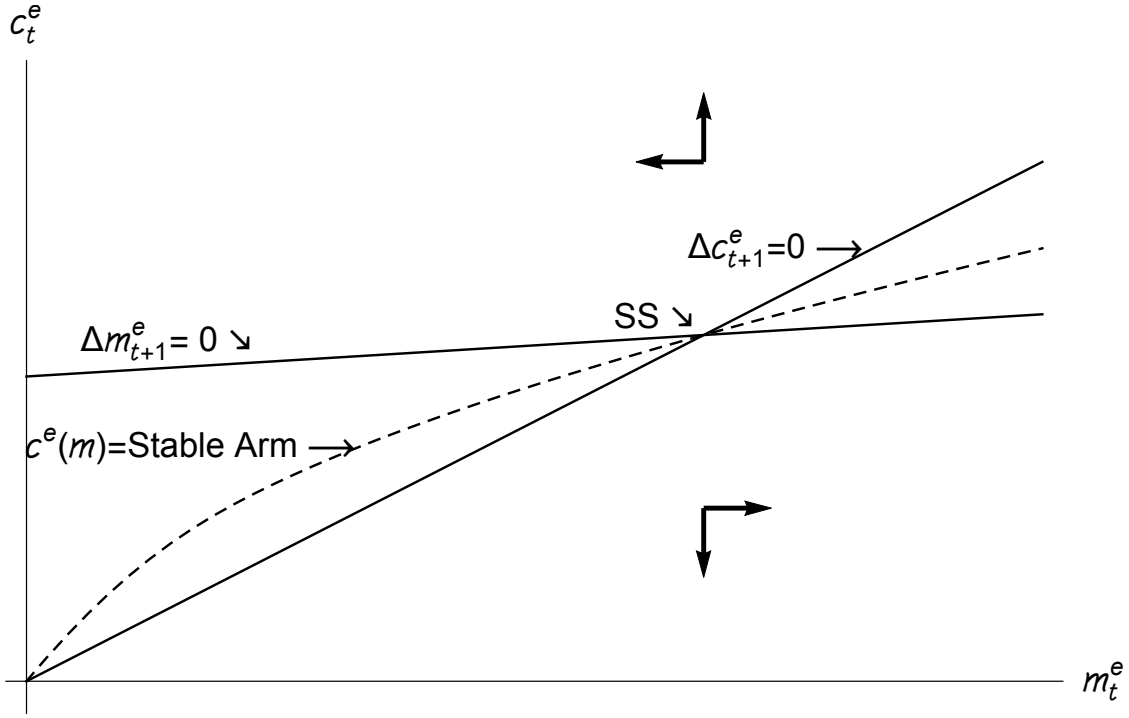
2.2.9 The Phase Diagram

Figure 1 presents the phase diagram.

The $\Delta m^e = 0$ locus, given in (31), indicates, for a given level of m^e, how much consumption c^e would be exactly the right amount to leave m^e unchanged. Call this the 'permanently sustainable consumption locus,' or for short 'sustainable consumption.'[17] For any given m^e, consuming an amount less than the 'sustainable' level will cause wealth to rise (and conversely for points above $\Delta m^e = 0$). This provides the logic for the horizontal arrows of motion in the diagram: Above the sustainable consumption locus they point left, and below they point right.

The intuition for the $\Delta c^e = 0$ locus (which comes from (29)) is a bit subtler. Take a point on the $\Delta c^e = 0$ locus, and consider how things would change if m^e were a bit higher at the same c^e. Recall that the growth rate of consumption consistent with the Euler equation (11) depends on the amount by which consumption will fall if the bad

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[17]Some authors refer to $\Delta m^e = 0$ as giving the level of 'permanent income,' but this definition differs from Friedman (1957)'s and is problematic because it could be confused with 'permanent labor income' $\mathsf{W}_t \ell_t$.

Figure 1 Phase Diagram

state is realized, $\nabla_{t+1} = c^e_{t+1}/c^u_{t+1}$. But $c^u_{t+1} = \kappa R(m^e_t - c^e_t)$ so at the same c^e_t but a greater m^e_t, c^u_{t+1} will be larger. If c^e_{t+1} were to remain unchanged, then with the larger c^u_{t+1} the ratio $\nabla_{t+1} = c^e_{t+1}/c^u_{t+1} - 1$ would be smaller.

The consequences of this are easiest to see in the logarithmic case whose consumption growth equation is derived in (18), which tells us that $c^e_{t+1} \approx c^e_t (1 + \mho\nabla_{t+1}) Þ_\Phi$, which directly implies that the lower ∇_{t+1} will yield a lower c^e_{t+1}. That is, for any point to the right of the $\Delta c^e_{t+1} = 0$ locus, the growth rate of consumption will be lower than at the corresonding point on the locus. Since on the locus, growth was zero, this means that to the right of the locus, c^e is declining (hence the down arrow in the phase diagram). Reciprocally, for any point to the left of $\Delta c^e_{t+1} = 0$, the Euler equation implies that consumption will rise.

2.2.10 The Consumption Function

The next figure shows the optimal consumption function $c(m)$ for an employed consumer (dropping the e superscript to reduce clutter). This is actually just the stable arm in the phase diagram. (Think about why). Also plotted are the 45 degree line along which $c = m^e_t$ as well as the function

$$\bar{c}(m) = (m - 1 + h)\kappa \tag{34}$$

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Figure 2 The Consumption Function
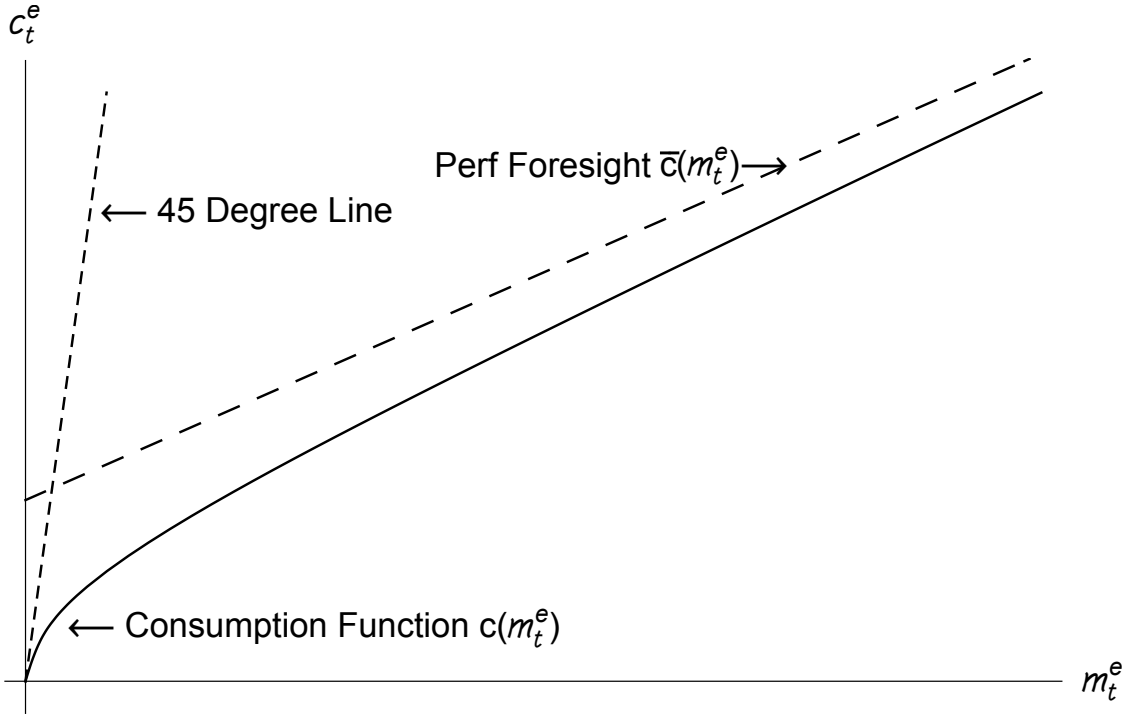
where

$$h = \left(\frac{1}{1 - \mathsf{G}/\mathsf{R}} \right) \tag{35}$$

is the level of (normalized) human wealth. $\bar{\mathrm{c}}(m)$ is the solution to a perfect foresight problem in which income grows by the factor G; it is depicted in order to introduce a final fact: As wealth approaches infinity, the solution to the problem with uncertain labor income approaches arbitrarily close to the perfect foresight solution.[18]

Note that $\mathrm{c}(m)$ is *concave*.[19] That is, the marginal propensity to consume $\boldsymbol{\kappa}(m) \equiv d\mathrm{c}(m)/dm$ is higher at low levels of m. This is because of the increase in the intensity of the precautionary motive as resources m decline; the consequences of becoming unemployed with little wealth are very painful. The MPC is high at low levels of m because at low levels of m the *relaxation* in the intensity of the precautionary motive with each extra bit of m is quite large (Kimball (1990)). This diminution in the precautionary

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[18]This limiting result requires that we impose the FHWC($\boldsymbol{\Phi} < \mathsf{R}$), because the perfect foresight consumption function is not defined if $\boldsymbol{\Phi} \geq \mathsf{R}$. Informally, the proof is as follows. Define $\underline{\mathrm{c}} \equiv (m-1)\kappa = \mathrm{c}^u(m)$ as the consumption function for the unemployed consumer who will receive no future labor income. Then $\underline{\mathrm{c}}(m) < \mathrm{c}(m) \leq \bar{\mathrm{c}}(m)$, and so $1 < \mathrm{c}(m)/\underline{\mathrm{c}}(m) < \bar{\mathrm{c}}(m)/\underline{\mathrm{c}}(m)$. In the limit as $m \uparrow \infty$, however, human wealth accounts for an arbitrarily small proportion $\kappa h/\kappa(h + m - 1)$ of consumption, so $\lim_{m \uparrow \infty} \mathrm{c}^e(m)/\mathrm{c}^u(m) = 1$ so the precautionary motive captured by ∇ vanishes.

[19]Carroll and Kimball (1996) prove that the consumption function must be concave for almost all commonly-used assumptions about risk and utility functions.

motive translates into an increase in consumption; for m-poor consumers even a modest increase in m can give a substantial boost to c.

This point is clearest as m approaches zero. Note that the consumption function always remains below the 45 degree line. This is because if the consumer were to spend all his resources in period t, $c_t = m_t$, then if he became unemployed next period he would have $m_{t+1}^u = (m_t - c_t)RNrm = 0$ which would induce $c_{t+1}^u = \kappa m_{t+1}^u = 0$, yielding negative infinite utility. Thus the consumer will never spend all of his resources - he will always leave at least a little bit for next period in case of disaster (unemployment).[20]

2.2.11 Expected Consumption Growth Is Downward Sloping in m^e

The next figure ('the growth diagram') illustrates some of the same points in a different way. It depicts the growth rate of consumption as a function of m_t^e. Since $\mho \geq 0$, the GIC$_{\mathbf{\Phi}}$ for this model implies:

$$\varphi > \rho^{-1}(\mathsf{r} - \tau), \tag{36}$$

a condition that can be visually verified for our benchmark calibration in figure 3. Now multiply both sides of (11) by $\mathbf{\Phi}$, obtaining

$$\left(\frac{\mathbf{c}_{t+1}^e}{\mathbf{c}_t^e}\right) = (\mathsf{R}\beta)^{1/\rho}\left\{1 + \mho\left[\left(\frac{c_{t+1}^e}{c_{t+1}^u}\right)^\rho - 1\right]\right\}^{1/\rho} \tag{37}$$

$$\Delta \log \mathbf{c}_{t+1}^e \approx \rho^{-1}(\mathsf{r} - \tau) + \mho\nabla_{t+1},$$

where the last line uses the same (dubious) approximations used to obtain (17).[21]

Thus consumption growth is equal to what it would be in the absence of uncertainty, plus a precautionary term. Furthermore, the precautionary contribution will become arbitrarily large as $m_t \downarrow 0$ because $c_{t+1}^u = m_{t+1}^u \kappa = (m_t - \mathsf{c}(m_t))RNrm\kappa$ approaches zero as $m_t \downarrow 0$. Sure enough, figure 3 shows that as m_t^e gets low, expected consumption growth gets very large.

Next, note that the point where the consumption growth locus meets the income growth line is labelled $\hat{m}$. This is because the place where consumption growth is equal to income growth is at the target value of m^e.

2.2.12 Summing Up the Intuition

We are finally in position to get an intuitive understanding of how the model works, and why there is a target wealth ratio. On the one hand, consumers are growth-impatient. This prevents their wealth-to-income ratio from heading off to infinity. On the other hand, consumers have a precautionary motive that intensifies more and more as the level of wealth gets lower and lower. At some point the precautionary motive gets strong enough to counterbalance impatience. The point where impatience matches prudence defines the target wealth-to-income ratio.

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[20] This is an implication not just of the CRRA utility function used here but of the general class of continuously differentiable utility functions that satisfy the usual *Inada condition* $\mathrm{u}'(0) = \infty$.

[21] We have also assumed that $(\nabla_{t+1})^2$ is 'small'.

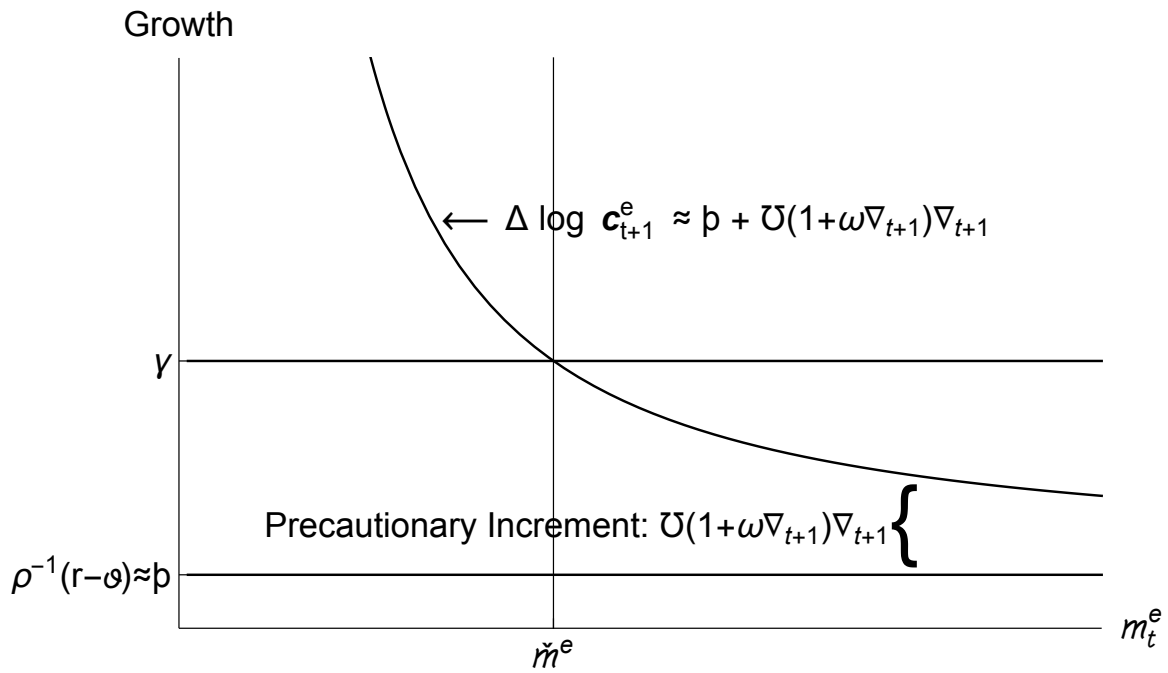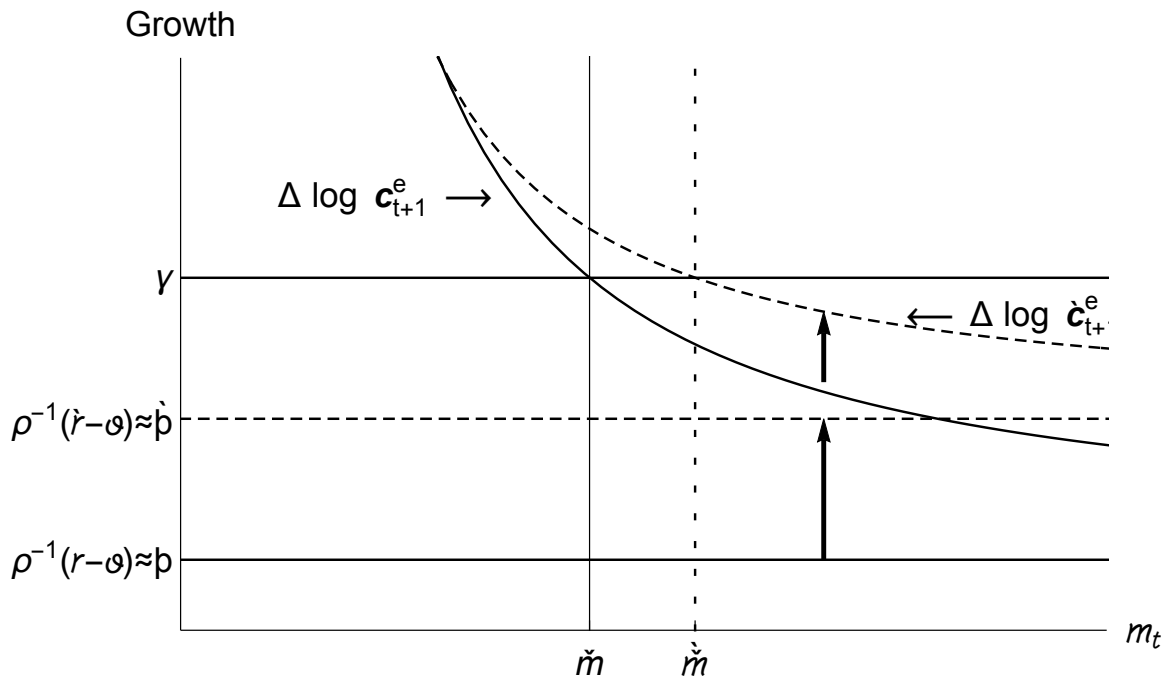**Figure 3** Income and Consumption Growth

Figure 4 Effect of An Increase In r

Now consider the results of increasing the interest rate to $\grave{r} > r$, depicted in figure 4. Obviously the perfect foresight consumption growth locus will shift up, to $\rho^{-1}(\grave{r} - \tau)$, inducing a corresponding increase in the expected consumption growth locus. But we have not changed the expected growth rate of income. It is clear from the figure, therefore, that the new target level of cash-on-hand $\grave{\check{m}}^e$ will be greater than the original target. That is, an increase in the interest rate increases the target level of wealth, as would be expected on intuitive grounds.

The next exercise is an increase in the risk of unemployment $\mho$. The principal effect we are interested in is the upward shift in the expected consumption growth locus to $\Delta\grave{\mathbf{c}}_{t+1}$. If the household starts at the original target level of resources $\grave{m}$, the size of the upward shift at that point is captured by the arrow orginating at $\{\check{m}, \varphi\}$.

In the absence of other consequences of the rise in $\mho$, the effect on the target level of m would be unambiguously positive. However, recall our adjustment to the growth rate conditional upon employment, (9); this induces the shift in the income growth locus to $\grave{\varphi}$ which has an offsetting effect on the target m ratio. Under our benchmark parameter values, the target value of m is higher than before the increase in risk even after accounting for the effect of higher φ, but in principle it is possible for the φ effect to dominate the direct effect. Note, however, that even if the target value of m is lower, it is possible that the *saving rate* will be higher; this is possible because the faster rate of φ makes a given saving rate translate into a lower ratio of wealth to income. In any case, the most useful calibrations of the model are those for which an increase in uncertainty results in either an increase in the saving rate or an increase in the target ratio of resources to permanent income. This is partly because our intent is to use the model to illustate the general features of precautionary behavior, including the qualitative effects of an increase in the magnitude of transitory shocks, which unambiguously increase both target m and saving rates.

2.2.13 Death to the Log-Linearized Consumption Euler Equation!

Figures 3 and 4 show that, so long as consumers are impatient, the steady state growth rate of consumption will be equal to the steady-state growth rate of income,

$$\Delta \log \mathbf{c}^e_{t+1} = \varphi. \tag{38}$$

Yet the approximate Euler equation for consumption growth, (37), does not contain any term explicitly involving income growth; in the logarithmic utility case, for example, the expression is
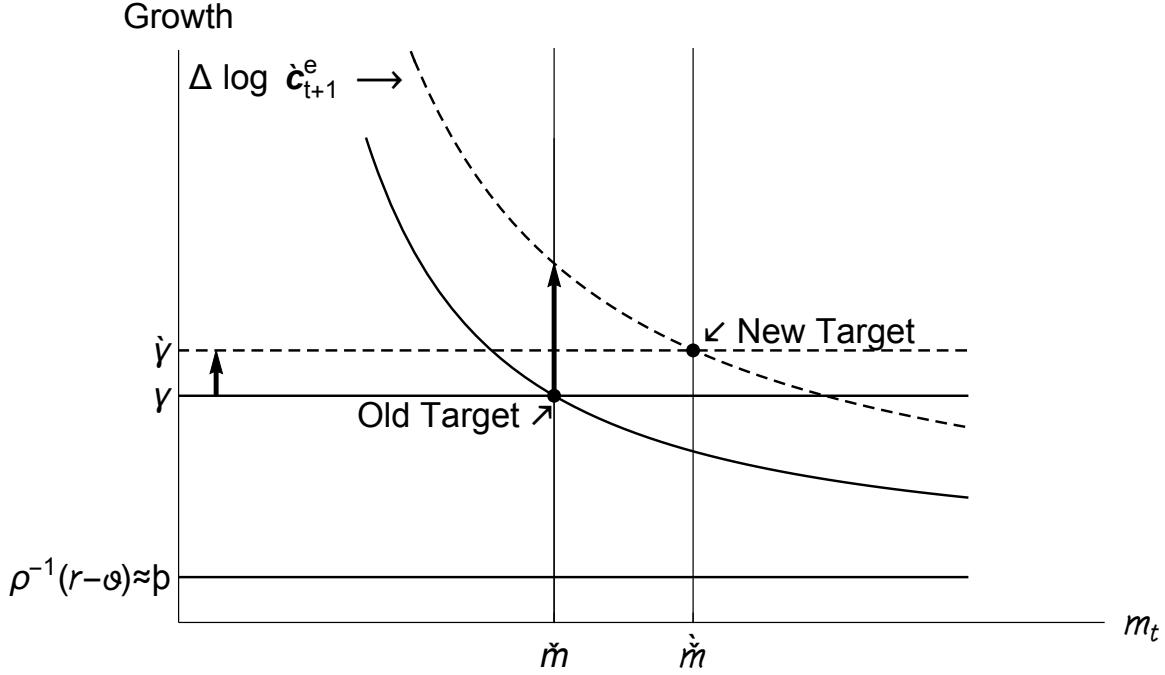
$$\Delta \log \mathbf{c}^e_{t+1} \approx \rho^{-1}(r - \tau) + \mho \nabla_{t+1}. \tag{39}$$

How can we reconcile these two expressions for consumption growth? Only by realizing that the size of the precautionary term $\mho \nabla_{t+1}$ is *endogenous*: It depends on φ. Indeed, we can solve (38) and (39) to determine that in steady-state we must have

$$\mho \check{\nabla} \approx \varphi - \rho^{-1}(r - \tau). \tag{40}$$

We can use this equation to understand the relationship between parameters and

Figure 5 Effect of an Increase in Ʊ

steady-state levels of wealth, by noting that $\nabla_{t+1}(m_t^e)$ is a downward-sloping function of m_t^e (see figure 3 again). This is because at low levels of current wealth, much of the spending of employed consumers is financed by their current income. If they lose that income, they will have no choice but to cut consumption drastically; this is reflected in a large value of ∇_{t+1}.

For example, an increase in the growth rate of income implies that the RHS of equation (40) increases. The new target level of $\hat{m}$ must be lower, because lower wealth induces greater consumption risk and a corresponding increase in the LHS of (40). This is how the human wealth effect works in this framework: Consumers who anticipate faster income growth will hold less market wealth.

The fact that consumption growth equals income growth in the steady-state poses major problems for empirical attempts to estimate the Euler equation. To see why, suppose we had a collection of countries indexed by i, identical in all respects except that they have different interest rates r_i. Then in the spirit of Hall (1988), one might be tempted to estimate an equation:

$$\Delta \log c_i = \eta_0 + \eta_1 r_i + \epsilon_i, \tag{41}$$

and to interpret the coefficient estimate on r_i as an indication of the value of ρ^{-1}.

But suppose that all of these countries contained impatient consumers and were in their steady-states where $\Delta \log c_i = \varphi_i$. Suppose further that all countries had *the same*

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steady-state income growth rate and unemployment rate.[22] Then the regression equation would return the estimates

$$\eta_0 = \varphi$$
$$\eta_1 = 0. \tag{42}$$

The econometric problem here is that there is an *omitted variable* from the regression specification, the $\mho\nabla$ term, which is (perfectly) correlated with the included variable r_i. Thus, Euler equation estimation cannot be expected to return an unbiased estimate of ρ^{-1}. For much more on this problem, see Carroll (2001). For empirical evidence that the problem is important in macroeconomic practice, see Parker and Preston (2005).

2.2.14 A Final Experiment

We now consider a final experiment: A decrease in the time preference rate. To reduce clutter, we drop the $\Delta c_{t+1}^e = 0$ locus from the phase diagram from Figure 1, and everywhere drop e the superscripts. (In exam questions, a figure like this might be referred to as the 'simplified consumption phase diagram' or just 'the consumption diagram').

Figure 6 depicts the effect on the employed consumer's spending by showing each successive point in time as a dot. Starting at time 0 from the steady-state level of consumption, the decrease in the future discounting rate (an increase in patience) causes an instantaneous drop in the level of consumption. Starting from this diminished base, consumption growth is subsequently faster than before the drop in τ.[23]

Eventually consumption approaches its new, higher equilibrium ratio to permanent income at a new, higher level of equilibrium m^e. This higher level of consumption is financed in the long run by the higher interest income earned on the higher level of wealth.

Note again, however, that equilibrium steady-state consumption growth is still equal to the growth rate of income (this follows from the fact that there is a steady-state *level* for the *ratio* of consumption to income, c). This means that the higher level of wealth in equilibrium ends up being precisely enough to reduce the precautionary term by an amount that exactly offsets the fact that the $-\rho^{-1}\tau$ term in the Euler equation is now smaller.

The final figures depict the time paths of consumption, market wealth, and the marginal propensity to consume $\boldsymbol{\kappa}(m)$ following the decline in τ. These are implicit in the phase diagram analysis, but the dots in these two new diagrams are spread out evenly over time to give a sense of the time scale over which the model adjusts toward the steady state.

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[22]The key point holds even if countries have different growth rates, but is easiest to understand if growth rates are identical.

[23]We could also analyze the effects on growth, but the results would be essentially the same as in preceding figure analyzing the effect of an increase the interest rate.
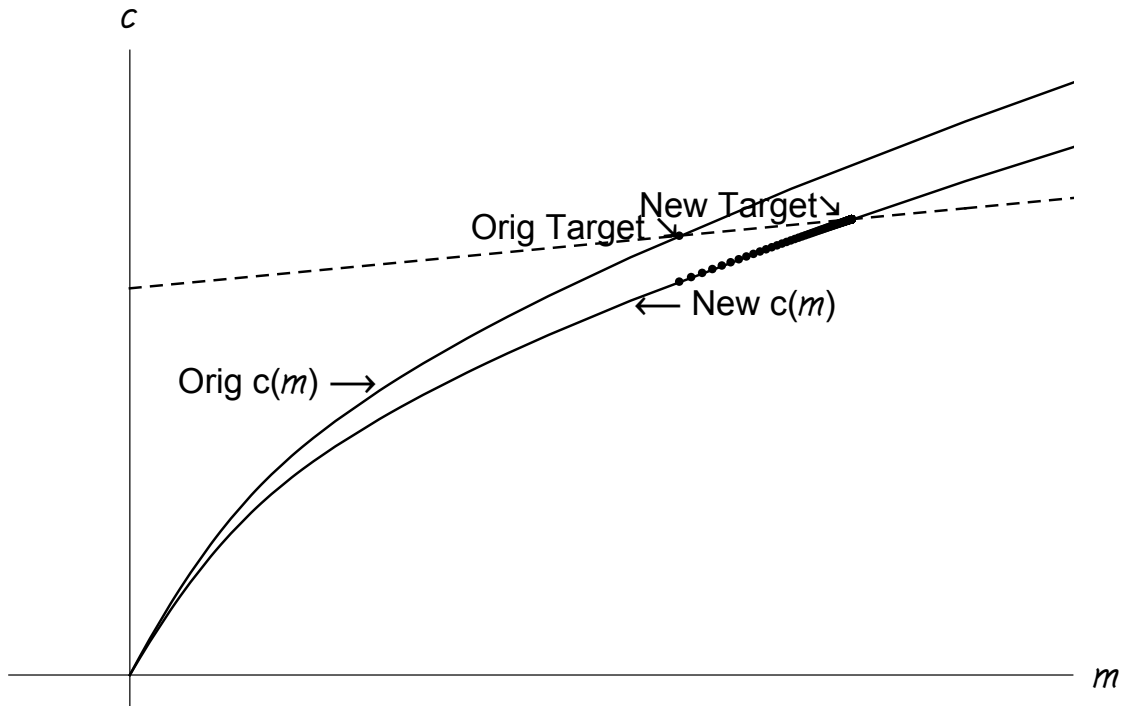
Figure 6 Effect of Lower τ On Consumption Function

c

Orig Target $\searrow$ New Target $\searrow$

$\leftarrow$ New c(m)

Orig c(m) $\rightarrow$

m

Figure 7 Path of c^e Before and After τ Decline

c^e_t

Time

0
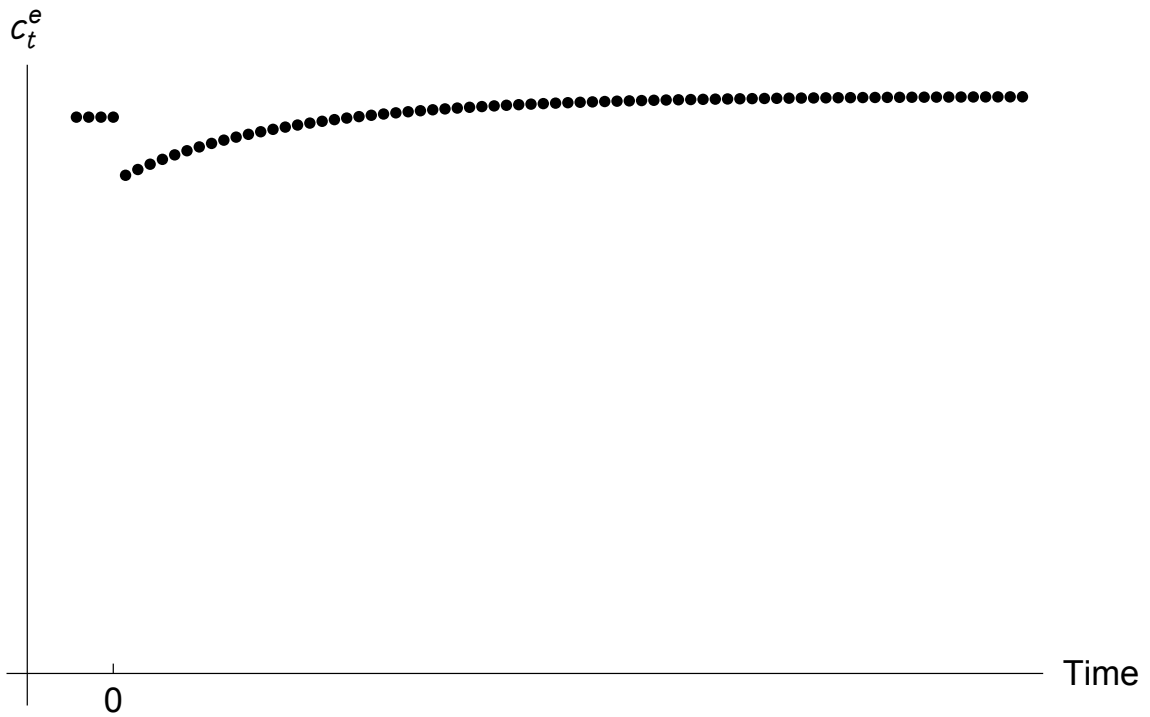
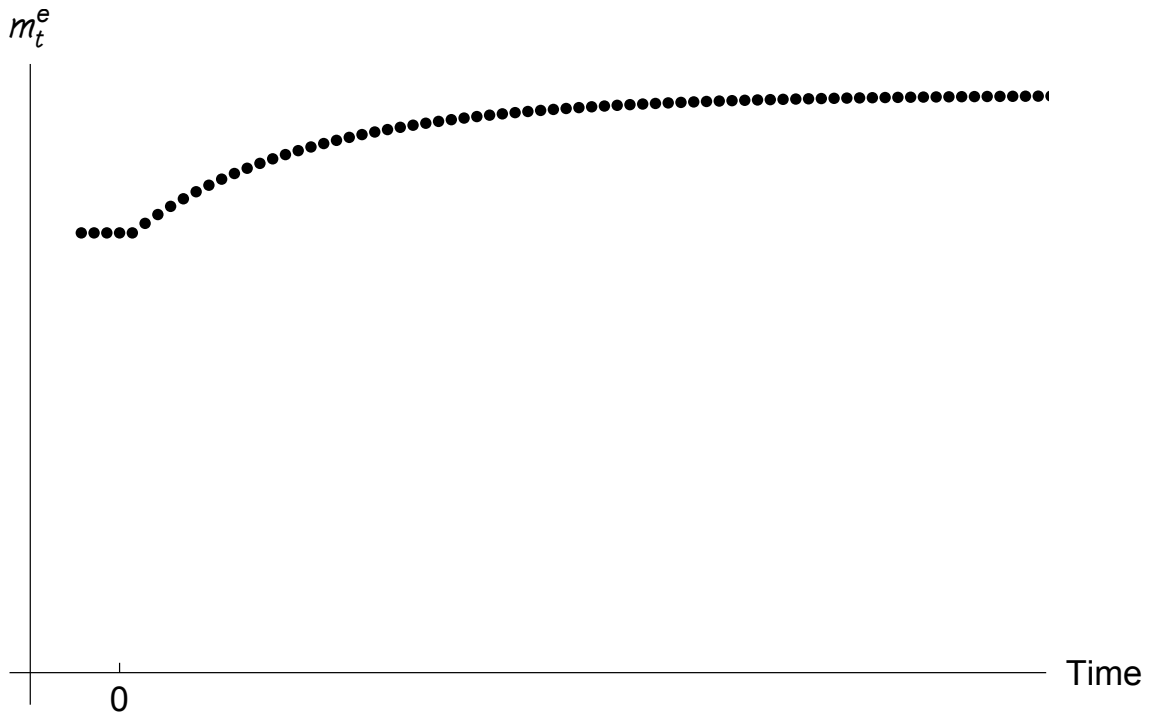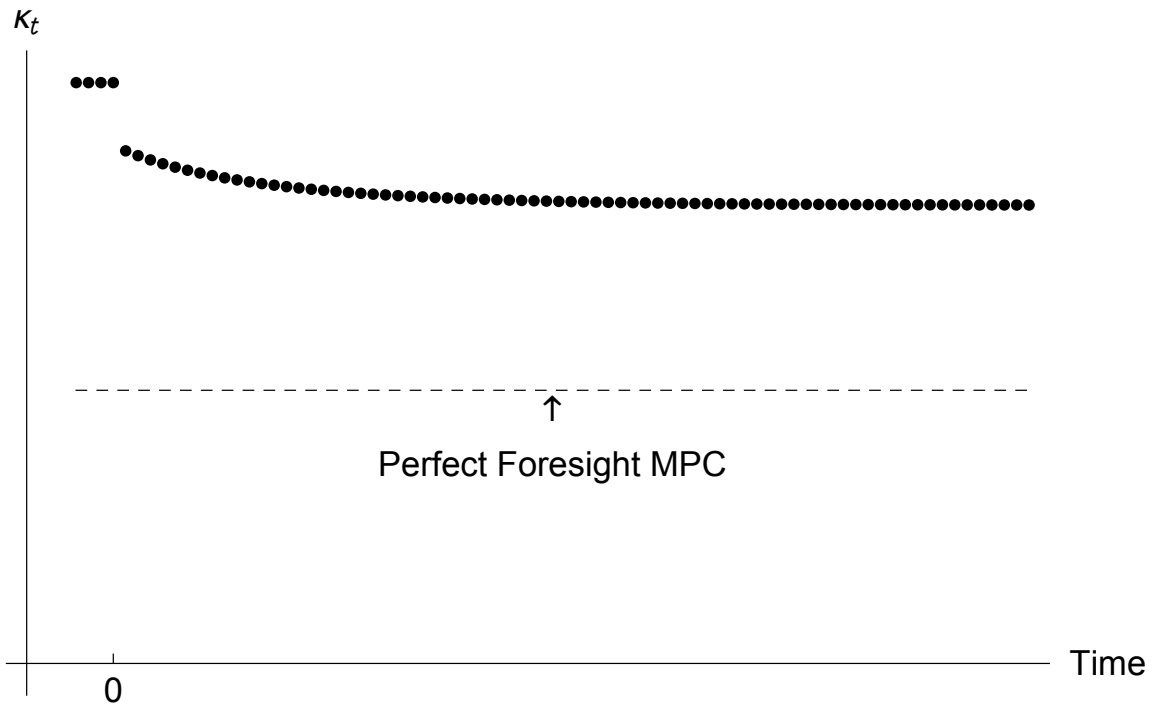**Figure 8** Path of m^e Before and After τ Decline

Figure 9 Marginal Propensity to Consume κ_t Before and After τ Decline

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3 A Macroeconomic Interpretation

Loosely following Carroll and Jeanne (2009) (with some simplifications), this section extends the model to analyze macroeconomic dynamics in a small open economy with a large number of individuals, where the population statistics reflect the fulfillment of individual consumers' *ex ante* expectations; for example, exactly proportion $\mho$ of households who are employed in period t become 'unemployed' before $t + 1$, so that the aggregate labor supply of the 'active' (still employed) members of a generation evolves according to

$$\mathbf{N}_{t+1,t} = \cancel{\mho}\mathbf{N}_{t,t}, \tag{43}$$

where the first subscript denotes the date being examined and the second denotes the period of birth of the generation being examined.

We make strong assumptions that permit straightforward aggregation. The first such assumption is that newly unemployed households immediately migrate out of the country (think of British retirees moving to southern Spain).[24] This means that macroeconomic variables will reflect only the circumstances of employed consumers, rather than a blend of the employed and the unemployed.

Each person is part of a single 'generation' of households born at the same time, and every new generation is larger by the factor Ξ than the newborn generation in the previous period:

$$\mathbf{N}_{t+1,t+1} = \Xi\mathbf{N}_{t,t}. \tag{44}$$

We assume that total production by the (surviving) members of a generation grows by the factor G every period. If total production is to grow despite a shrinking number of surviving members of the generation, production *per active capita* must grow by $\text{G}/\cancel{\mho}$ as per (9).

Consider the economy in some period 0 in which the size of the newborn population and the wage rate have been normalized to $\mathbf{N}_{0,0} = \mathsf{W}_0 = 1$. If the economy has existed for $-\tau$ periods (where τ is a negative number, indicating that the economy was created before period 0), the ratio of the total population to the population of newborns will be

$$1 + (\cancel{\mho}/\Xi) + (\cancel{\mho}/\Xi)^2 + ... + (\cancel{\mho}/\Xi)^{-\tau} = \left(\frac{1 - (\cancel{\mho}/\Xi)^{-\tau+1}}{1 - (\cancel{\mho}/\Xi)}\right) \tag{45}$$

whose limit is a finite number so long as $\cancel{\mho}/\Xi < 1$, which we require.

Relative to the labor income of period 0's newborn cohort ($\mathbf{N}_{0,0}\mathsf{W}_0 = 1$), the total labor income in period 0 of the generation born in period -1 is Ξ^{-1}; the sum of the incomes of all of the two-period-old individuals is Ξ^{-2}, and so on; total labor income for

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[24]The qualitative story is not changed if the unemployed stay at home and live off their savings; since they have a simple linear decision rule (they spend a constant proportion of their resources), accounting for their behavior is straightforward but complicates the exposition without adding much substance. See Carroll and Jeanne (2009) for a model that incorporates a stay-at-home unemployed population.

all generations in the economy in period 0 is

$$1 + \Xi^{-1} + \Xi^{-2} + ... + \Xi^{\tau} = \left(\frac{1 - (\Xi^{-1})^{-\tau+1}}{1 - \Xi^{-1}} \right), \tag{46}$$

which is finite so long as either population growth is positive $\Xi > 1$ (which we will assume) or the economy has existed for a finite period of time ($\tau > -\infty$). In either case, the *proportion* of aggregate income accounted for by a generation born at any specific moment declines toward zero as time passes (old generations never die, they just fade away).

In the balanced growth equilibrium, the growth factor for aggregate population will therefore be Ξ and output per capita will increase by G per period. Total labor income therefore grows by ΞG.

3.1 Stakes

We now examine this model under two assumptions about the initial 'stake' of newborns in the economy. (We use 'stake' to designate a transfer received by newborns). This is explicitly *not* an inheritance, as we have assumed that individuals have no bequest motive and newborns are unrelated to anyone in the existing population. Our motivation is to make the model more tractable, rather than to represent an important feature of the real world; we later perform simulations designed to show that the characteristics of the model with no 'stake' are qualitatively and quantitatively similar to those of the more tractable model with the 'stake' that makes the model tractable.

3.1.1 A 'Stake' That Yields a Representative Agent

We first consider a version of the model in which an exogenous redistribution program guarantees that the behavior of employed households can be understood by analyzing the actions of a "representative employed agent."

If a benevolent source outside the economy were to provide every newborn with an initial transfer upon birth of size $\hat{b}$, then the newborn's total monetary resources would be

$$m_{t,t}^e = \hat{b} + 1$$
$$= \hat{m}.$$

Thus, per-capita market resources for members of the newborn generation would be exactly equal to the target level of market resources for a person anticipating the future path of labor income that the members of the newborn generation actually anticipate (which is the same as the future path anticipated by all other generations as well).

If such a transfer policy had been in place forever, the economy at every point in time would consist of employed households whose consumption had been equal to its steady-state value c^e for their whole lives. That is, every individual agent in this economy would be identical in their *ratio* of consumption, market resources, etc. to permanent

labor income. The behavior of any individual would therefore be fully captured by the behavior of a representative employed agent.[25]

The foregoing scenario assumed that the 'stake' is provided by a mysterious 'benevolent source outside the economy.' Fortunately, there is an easy way to eliminate this problematic assumption: Assume that the stakes are financed by a wage tax.

The size of the required tax rate is calculated as follows. The total size of the resources transferred to the newborn generation must be

$$\hat{b}_{t,t}^{e} = \hat{b}\mathbf{N}_{t,t}\hat{\mathsf{W}}_t \tag{47}$$

where

$$\hat{\mathsf{W}}_t = \underbrace{(1-)}_{\equiv \slash}\mathsf{W}_t \tag{48}$$

is the after-tax wage rate for the economy as a whole (and $\hat{b}$ is the steady state target ratio of bank balances to after-tax wages).

From (46), the ratio of total aggregate labor income to the labor income of the newborn generation is

$$\left(\frac{1}{1 - \Xi^{-1}}\right) \tag{49}$$

so the aggregate wage tax rate required to finance a 'stake' of size $\hat{b}$ for newborns is given by

$$\hat{b} = \left(\frac{}{1 - \Xi^{-1}}\right) \tag{50}$$
$$= (1 - \Xi^{-1})\hat{b}.$$

Note, however, that in an economy where this tax has existed forever, the consequence of the tax is effectively just a permanent reduction in after-tax labor income by proportion $\slash$, compared to its value in the absence of the tax. Given the homotheticity of the model, a permanent rescaling by a constant factor leaves the scaled version of the individual's problem (and its solution) unchanged. Thus we can conclude not only that a representative agent exists in this economy, but that the steady-state characteristics of the representative agent's problem are identical (in ratio form) to the characteristics of the unrescaled individual's problem; that is, $\hat{c}(m) = c^e(m)$, $\hat{b} = \hat{b}$, and so on.

Matters are not much more complicated outside the balanced growth steady state, so long as we assume that the government always transfers the amount $\hat{b}$ to newborn households, financed by the tax derived above. Consider, for example, an economy that was in steady-state equilibrium leading up to period t, and at the beginning of t there is a sudden realization that future growth rates will be higher than those anticipated and experienced in the past: $G' > G$ after t. Since expected growth rates affect $\hat{b}$, the tax rate must be immediately and permanently changed so that the generations born after

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[25]The *level* of permanent labor income will differ for different households, depending on their age and accumulated wage seniority; the circumstances of individuals are identical only after their problem has been normalized by their varying levels of permanent labor income.

$t-1$ receive a 'stake' of the proper new size. This change in has two consequences for the generations that survive from periods prior to t. Under the old tax rate, they would have experienced $b_t^e = \mathbf{b}_t/\mathsf{W}_t = \hat{b}$; the change in expectations has no effect on $\mathbf{b}_t$ or W_t but changes the tax rate to `. Thus these households will have an actual resource ratio that differs from its new target value, $b_t^e \neq \grave{\hat{b}}$, both because the after-tax income scaling factor has changed and because the target ratio has changed from $\hat{b}$ to $\grave{\hat{b}}$.

However, if we started out in steady-state, the *ratio* problem of every member of the continuing-employed population is identical to that of every other such household (though, again, their masses differ depending on age, etc); as a result, the dynamics of the economy are fully captured by keeping track of the relative weights in the economy of the (gradually diminishing) 'representative shocked agent' and the (gradually increasing) 'representative new agent' whose behavior is locked at its steady-state value.[26]

Figure 10 illustrates the dynamics in this economy using an experiment identical to one explored above for the individual's problem: In period 0 there is a one-off decline in the future discounting rate (assuming the economy was in steady state before period 0). In the previous model, each individual consumer's consumption function shifted down, and consumption experienced a discrete jump downward, because the agent became more impatient. Here, there is a modest further effect: With more-patient consumers, the tax rate that the government sets to finance a transfer of $\hat{b}$ to the newborns must be larger (so that the ratio of initial assets to after-tax income is smaller). Qualitatively, the dynamics are indistinguishable from the individual consumer's dynamics obtainable without working through the extra complication involved in accounting for the 'stakes.'

3.1.2 No Stake

The polar alternative to assuming that newborns get a 'stake' is to assume that newborns enter the economy with zero assets. Analysis of this version of the model must be performed using simulation methods, because households of different ages will have different levels of assets. (With a concave and nonanalytical consumption function, analytical aggregation cannot be performed.)

Our simulation procedure assumes that at date 0 the economy has existed forever (so that the age distribution of relative populations and productivities are at their steady-state values), but saving has been impossible prior to period 0.[27] With everyone's $b_t^e = 0$, the ratio of market resources to permanent labor income is the same for all individuals:

$$m_{0,\tau}^e = 1. \tag{51}$$

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[26]If the economy has experienced multiple shocks, it will be necessary to retain a complete history of the shocks in order to compute the properly population-weighted dynamics. This is not too hard to do, if we start with the assumption that the economy started at its balanced growth equilibrium before the shocks began to arrive.

[27]In periods before 0, unemployment presumably would have meant immediate death by starvation; we think of this more as a starting point for the simulations than as a realistic description of a plausible economy.

Figure 10 Aggregate c in PE/SOE Economy Before and After τ Decline
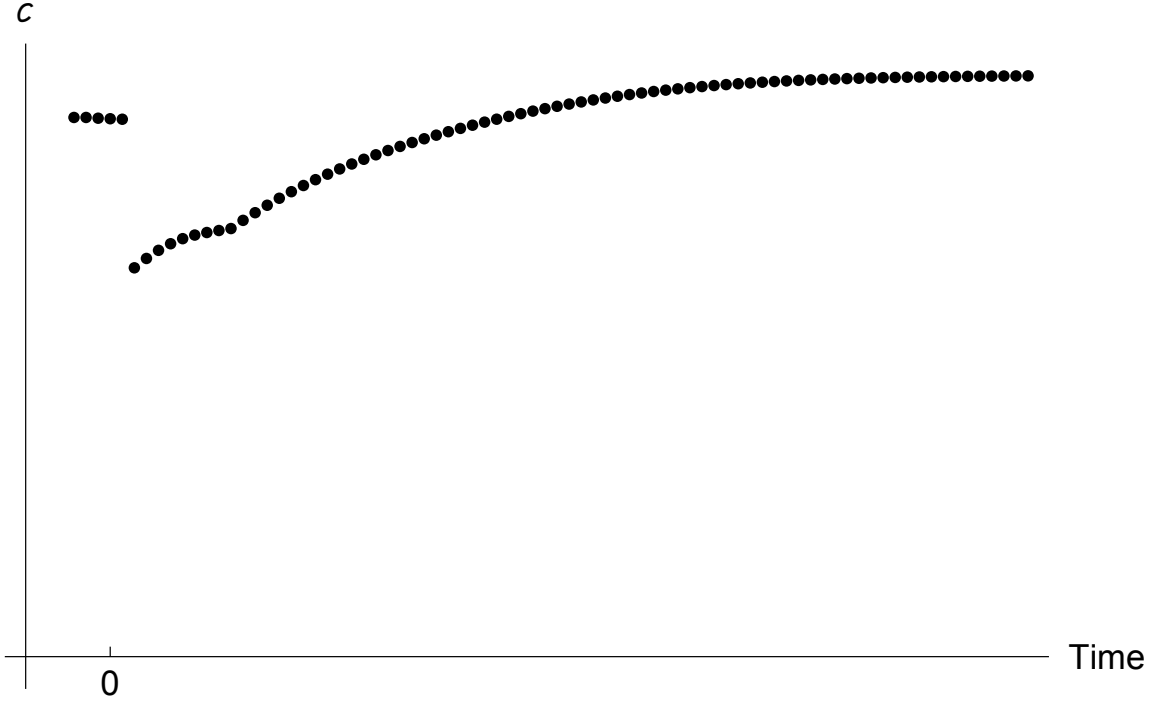
The consumption ratio in period 0 is therefore c(1) for every household (regardless of age), while the level of total labor income for a generation that is $-\tau$ periods old is $\cancel{\varnothing}^{\tau}$.[28] The population of such workers is $(\cancel{\varnothing}/\Xi)^{-\tau}$, so aggregate consumption will be given by the per-capita consumption ratio, multiplied by the per-capita level of permanent income, multiplied by the population of workers still alive:

$$
\begin{aligned}
\mathbf{c}_0 &= \sum_{\tau=0}^{-\infty} \mathrm{c}(1)\cancel{\varnothing}^{\tau}(\cancel{\varnothing}/\Xi)^{-\tau} \\
&= \mathrm{c}(1)\sum_{\tau=0}^{-\infty} \Xi^{\tau} \\
&= \mathrm{c}(1)\left(\frac{1}{1-\Xi^{-1}}\right).
\end{aligned}
\tag{52}
$$

The longer a generation lives, the more time it will have had to save toward its target level of wealth; but newborns always begin life with no assets. After period 0, therefore, age-heterogeneity in assets and consumption ratios creeps into the population.

The foregoing discussion contains (in some cases implicitly) all the assumptions necessary to conduct a simulation of this economy. Figure 11 shows the path of the ratio $\mathbf{c}_t/\mathrm{W}_t\mathbf{N}_t$ starting from period 0 for an economy under our benchmark parameterization

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[28]The absolute level of wages will have grown by $\mathrm{G}/\cancel{\varnothing}$ per period for these households since their birth, but we have normalized by the level of wages for period-0 newborns, which cancels the G from the expression.

Figure 11 Path of Aggregate c in Stakeless PE/SOE Economy From Date 0
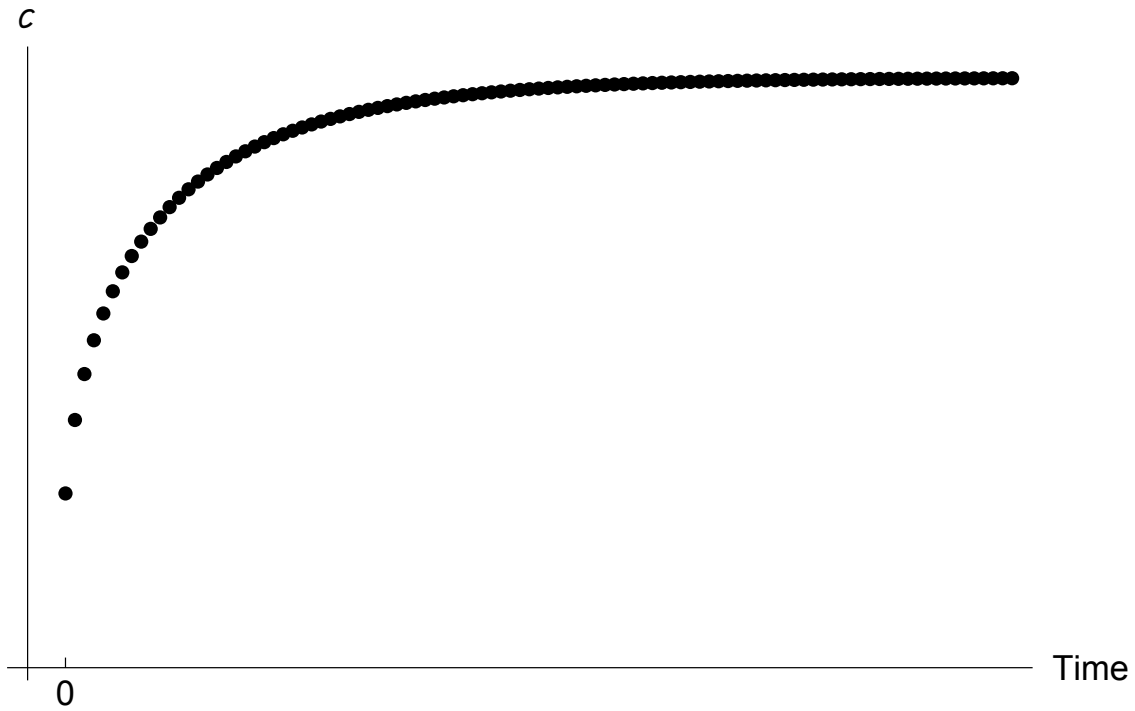
that generated our earlier figures. The only extra parameter required beyond those used before is Ξ; we choose $\Xi = 1.01$ corresponding roughly to the postwar population growth rate in the United States.

3.2 Unemployment Insurance

We consider now the consequences if the government creates a balanced-budget partial 'unemployment insurance' system. This system operates by imposing a labor income tax on the employed in order to finance transfers to the unemployed.[29]

Our definition of partial insurance starts by assuming that the 'true' labor income process is the one specified above, but the government interferes with this process by selecting a constant proportion ζ of the newly unemployed in each period who will be guaranteed a 'wage-indexed unemployment benefit' that yields the same income they would have received if they had not become unemployed. The lucky recipients of these payments, however, are subject to a risk of termination from the unemployment insurance program that matches the risk of becoming unemployed for the still-employed consumers.

---

[29]The case of perfect (full) insurance is not very interesting, because impatient agents like the ones in our model would choose to run down their assets indefinitely, leaving no role for target saving and no interesting dynamics. However, this is a knife-edge result that depends upon the complete elimination of idiosyncratic uncertainty.

Under these circumstances, the household does not care whether it remains employed, or becomes unemployed but is selected for the unemployment insurance program: The dynamics of idiosyncratic future income are identical in the two cases.

Each generation finances its own unemployment insurance (this assumption is made to keep the model as transparent as possible; it would not change things much to have the UI system financed by the general government, but this would require some extra notation and derivations, and discussion of essentially inconsequential intergenerational aspects of unemployment insurance). The government sets the overall uninsurance rate ζ ('uninsurance' rather than 'insurance' rate because $\zeta = 0$ will be 'no uninsurance' or perfect insurance, while $\zeta = 1$ will be no insurance) and enforces exogenous taxation among the 'truly employed' in the generation in such a way as to yield a modified 'post-tax-and-transfer' growth profile consistent with substituting $\grave{\mho}$ for $\mho$ in the derivations above, where

$$\grave{\mho} = \zeta \mho \tag{53}$$

and $\zeta < 1$ guarantees that the insurance program results in a reduction of idiosyncratic income risk.

Substituting $\grave{\mho}$ for $\mho$ in the prior derivations, this scheme is consistent with the generational budget constraint because, as noted above, the individual income growth process was constructed so that the present discounted value of income remains invariant to the size of $\mho$, and the aggregate income of a generation grows by G regardless of the underlying idiosyncratic unemployment risk.

This scheme is attractive because in practice it simply requires solving the model specified above for a different value of $\mho$. This will make solution and simulation of the model particularly simple.

In addition to the consumption function, the solution procedure produces an estimate of the representative employed agent's value function.[30] Value depends on current resources, but also (numerically) on all of the parameters in the calibration of the model. We are particularly interested in how value relates to expected labor productivity growth G, to the 'true' unemployment risk $\mho$, and to the generosity of the unemployment insurance program ζ. Writing

$$v(m_t^e; G, \mho, \zeta), \tag{54}$$

(where we separate the state variable m_t^e from the parameters using a semicolon) we can investigate a variety of interesting questions. Some examples:

- For an economy starting at the steady state without unemployment insurance, how valuable is a given improvement in insurance? That is, what is the change in monetary resources that would make the representative consumer indifferent to the change in insurance? To answer this, we can find the Δm^e such that

$$v(m^e + \Delta m^e; G, \mho, 1) = v(m^e; G, \mho, \zeta) \tag{55}$$

---

[30]See the appendix for a sketch of the method.

- Suppose a policymaker considering is considering increasing the economy's openness or degree of 'globalization.' The benefit is faster productivity growth G. The cost is increased unemployment risk $\mho$. For a given generosity of unemployment insurance, how much extra growth is necessary to offset the utility cost of greater risk? That is, supposing the relationship between G and $\mho$ is indexed by a parameter φ, e.g. $\grave{\mho} = \mho + (\dot{G} - G)\varphi$, what is the φ for which

$$v(m^e; G, \mho, 1) = v(m^e; \grave{G}, \grave{\mho}, 1) \tag{56}$$

- If the policymaker can increase the generosity of unemployment insurance in the face of increased unemployment risk stemming from globalization (perhaps 'trade adjustment assistance' is politically feasible even if increased unemployment insurance *per se* is not), by how much would the unemployment insurance program need to change in order to offset the utility consequences of a given increase in $\mho$?

Many more such questions can be imagined.   endCDC

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Appendix

A Approximate Formula for Consumption Growth

Using from `MathFacts` the second-order and then the first-order Taylor approximations [`TaylorTwo`]$(1 + \epsilon)^\zeta \approx 1 + \zeta\epsilon + (1/2)\zeta(\zeta - 1)\epsilon^2$ and then [`TaylorOne`]$(1 + \epsilon)^\zeta \approx 1 + \zeta\epsilon$, the expression in braces in (11) can be rewritten

$$
\left\{1 + \mho\left[\left(\frac{c^e_{t+1}}{c^u_{t+1}}\right)^\rho - 1\right]\right\}^{1/\rho} = \left\{1 + \mho\left[\left(\frac{c^u_{t+1} + c^e_{t+1} - c^u_{t+1}}{c^u_{t+1}}\right)^\rho - 1\right]\right\}^{1/\rho}
$$

$$
= \left\{1 + \mho\left[(1 + \nabla_{t+1})^\rho - 1\right]\right\}^{1/\rho}
$$

$$
\approx \left\{1 + \mho\left[1 + \rho\nabla_{t+1} + \rho(\nabla_{t+1})^2\omega - 1\right]\right\}^{1/\rho}
$$

$$
= \left\{1 + \rho\mho(\nabla_{t+1} + (\nabla_{t+1})^2\omega)\right\}^{1/\rho}
$$

$$
\approx 1 + \mho\left(1 + \nabla_{t+1}\omega\right)\nabla_{t+1},
$$

which leads directly to (17) in the main text.

B Conditions for a Target to Exist

B.1 Using the Phase Diagram Loci

At a steady-state value of m^e, both $\Delta c^e = 0$ and $\Delta m^e = 0$ hold (equations (29) and (31)); for convenience defining $\mu = RNrm\kappa\Pi + 1$,

$$
0 = \overbrace{\left(\frac{\mu - 1}{\mu}\right)m^e}^{\Delta c^e} = \overbrace{\left(\frac{RNrm - 1}{RNrm}\right)m^e + RNrm^{-1}}^{\Delta m^e} = 0. \tag{57}
$$

But since $RNrm^{-1}$ is a positive number, at $m^e = 0$ the $\Delta m^e = 0$ locus's value is $RNrm^{-1}$ while the value of the $\Delta c^e = 0$ locus is zero, the two loci can intersect for a positive m^e only if the slope of the $\Delta c^e = 0$ locus is greater:[31]

$$
\left(\frac{\mu - 1}{\mu}\right) > \left(\frac{RNrm - 1}{RNrm}\right) \tag{58}
$$

which is equivalent to

$$
\overbrace{RNrm\kappa\Pi}^{=\mu-1} > RNrm - 1 \tag{59}
$$

where the LHS is (proportional to) the slope of $\Delta c^e = 0$ and the RHS is (proportional to) the slope of $\Delta m^e = 0$.

beginCDC

---

[31]We also need μ to be nonnegative.

It is not totally obvious why these are equivalent, so here's the derivation.

$$\left(\frac{\mu - 1}{\mu}\right) > \left(\frac{RNrm - 1}{RNrm}\right)$$
$$RNrm(\mu - 1) > \mu(RNrm - 1)$$
$$RNrm\mu - RNrm > \mu RNrm - \mu$$
$$-RNrm > -\mu \tag{60}$$
$$RNrm < \mu$$
$$RNrm < RNrm(\kappa\Pi + RNrm^{-1})$$
$$1 < \kappa\Pi + RNrm^{-1}$$
$$RNrm - 1 < \kappa RNrm\Pi.$$

For any fixed $\mho$ and G and R we can find some α for which $\mathsf{G} = \mathsf{R}(1 - \alpha\mho)$, and using this α it turns out to be useful to rewrite

$$
\begin{aligned}
RNrm^{-1} &= \boldsymbol{\Phi}/\mathsf{R} \\
&= \mathsf{G}/\mathsf{R}(1 - \mho) \\
&= \mathsf{R}(1 - \alpha\mho)/\mathsf{R}(1 - \mho) \\
&= (1 - \alpha\mho + \mho - \mho)/(1 - \mho) \\
&= (1 - \mho + (1 - \alpha)\mho)/(1 - \mho) \\
&= 1 + (1 - \alpha)\mho/(1 - \mho).
\end{aligned}
\tag{61}
$$

Note for future use that (61) implies that whenever $\alpha \leq 1$, the FHWC$_{\boldsymbol{\Phi}}$ fails ('human wealth is infinite') because $RNrm^{-1} > 1 \Rightarrow \mathsf{R}/\boldsymbol{\Phi} = RNrm < 1 \Rightarrow \mathsf{R} < \boldsymbol{\Phi}$.

Multiplying both sides of (59) by $RNrm^{-1}$ then substituting the expression for $RNrm^{-1}$ from (61) gives

$$
\begin{aligned}
1 - RNrm^{-1} &< \kappa\Pi \\
-(1 - \alpha)\mho/(1 - \mho) &< \kappa\Pi
\end{aligned}
\tag{62}
$$

B.2 A Target Always Exists When Human Wealth Is Infinite

Since $0 < \mho < 1$ and $\kappa > 0$ (as guaranteed by the RIC), (62) is satisfied whenever the FHWC$_{\boldsymbol{\Phi}}$ fails ($\alpha \leq 1$) and $\Pi > 0$. We now show that under these conditions, $(1 + \varpi)^{1/\rho} = \Pi > 0$.

Π from (19) is:

$$\Pi = \left(1 + \overbrace{\mho^{-1}(\Þ_{\boldsymbol{\Phi}}^{-\rho} - 1)}^{\equiv \varpi}\right)^{1/\rho} \tag{63}$$

but note that

$$\Þ_{\boldsymbol{\Phi}} = \Þ_{\mathsf{R}} \overbrace{(\mathsf{R}/\boldsymbol{\Phi})}^{=RNrm} \tag{64}$$

and in the case where $\alpha = 1$, $RNrm$ must also be 1, implying that $\text{Þ}_{\Phi} = \text{Þ}_{\mathsf{R}} < 1$ (the RIC) so that $\text{Þ}_{\Phi}^{-\rho} > 1$ and so $\varpi > 0$ and hence $\Pi > 1 > 0$. The other interesting case is when $\alpha = 0$ so that $\mathsf{G} = \mathsf{R}$ and $RNrm = \mathsf{R}/\Phi = \mathsf{R}(1 - \mho)/\mathsf{G} = (1 - \mho) < 1$. In this case $\text{Þ}_{\Phi} < \text{Þ}_{\mathsf{R}}$ and so $\text{Þ}_{\Phi}^{-\rho} > \text{Þ}_{\mathsf{R}}^{-\rho} > 1$ and so ϖ is even more positive so that Π is even more strongly > 0. Similar logic holds for any $\alpha \le 1$.

Thus, we can conclude that, when human wealth is infinite (that is, if $\alpha \le 1$), a target $\hat{m}^e$ will exist.

B.3 Conditions Under Which a Target Exists When Human Wealth is Finite

In the case where human wealth is finite ($\alpha > 1$), we need the RHS of (62) not merely to be positive, but to exceed a specific positive number, $(\alpha - 1)\mho/(1 - \mho)$:

$$\kappa(1 + \varpi)^{1/\rho} > (\alpha - 1)\mho/(1 - \mho)$$

$$(1 + \varpi)^{1/\rho} > \left(\frac{(\alpha - 1)\mho}{\kappa(1 - \mho)} \right)$$

$$(1 + \varpi) > \left(\frac{(\alpha - 1)\mho}{\kappa(1 - \mho)} \right)^{\rho}$$

$$(\text{Þ}_{\Phi}^{-\rho} - 1)\mho^{-1} = \varpi > \left(\frac{(\alpha - 1)\mho}{\kappa(1 - \mho)} \right)^{\rho} - 1$$

$$(\text{Þ}_{\Phi}^{-\rho} - 1) > \mho \left[\left(\frac{(\alpha - 1)\mho}{\kappa(1 - \mho)} \right)^{\rho} - 1 \right] \tag{65}$$

$$\text{Þ}_{\Phi}^{-\rho} > 1 + \mho \left[\left(\frac{(\alpha - 1)\mho}{\kappa(1 - \mho)} \right)^{\rho} - 1 \right]$$

$$\text{Þ}_{\Phi} < \left\{ 1 + \mho \underbrace{\left[\left(\frac{(\alpha - 1)\mho}{\kappa(1 - \mho)} \right)^{\rho} - 1 \right]}_{\equiv \chi} \right\}^{-1/\rho}$$

and the boundary will be the point at which this expression holds with equality.

An increase in impatience caused by an increase in the pure time preference rate τ (equivalently, a reduction in β) has the effect of reducing growth-patience (the LHS of (65)) and of increasing the RHS. This means that there will be some time preference rate sufficiently large (some β sufficiently small) to guarantee that the condition holds with equality. Then (65) will always be satisfied by any β satisfying

$$\beta < \bar{\beta}_{\text{FHW}}. \tag{66}$$

Here's a more careful exposition of the above: Rewrite the boundary condition as

$$\text{Þ}_{\mathsf{R}} \overbrace{\left(\frac{1 - \mho}{1 - \alpha\mho} \right)}^{=\mathsf{R}/\Phi} = \text{Þ}_{\Phi} = \left\{ 1 + \mho \left[\left(\frac{(\alpha - 1)\mho}{\kappa(1 - \mho)} \right)^{\rho} - 1 \right] \right\}^{-1/\rho}$$

31

and note that if we reduce α incrementally the denominator on the LHS increases so the LHS decreases, while the RHS increases. Under the new value of α, therefore, $Þ_{\Phi}$, if the condition held with equality before it now holds with inequality.

Since we have assumed the RIC (so that $\kappa > 0$), as $℧ \downarrow 0$ or $\alpha \downarrow 1$, (65) asymptotes to the GIC_{Φ} for any given value of β.

The apparently harder case is when $\alpha > 1$ and $℧ > 0$. But note that we will have found $\bar{\beta}_{\text{FHW}}$ if we can find the corresponding κ at which the first term in χ reaches 1:

$$\left(\frac{(\alpha - 1)℧}{(1 - Þ_{\text{R}})(1 - ℧)}\right)^{\rho} = 1$$

$$\left(\frac{(\alpha - 1)℧}{(1 - Þ_{\text{R}})(1 - ℧)}\right) = 1$$

$$\left(\frac{(\alpha - 1)℧}{(1 - ℧)}\right) = 1 - Þ_{\text{R}}$$

$$1 - \left(\frac{(\alpha - 1)℧}{(1 - ℧)}\right) = Þ_{\text{R}} \tag{67}$$

$$\left[1 - \left(\frac{(\alpha - 1)℧}{(1 - ℧)}\right)\right] = (\text{R}\beta)^{1/\rho}/\text{R}$$

$$\text{R}^{\rho}\left[1 - \left(\frac{(\alpha - 1)℧}{(1 - ℧)}\right)\right]^{\rho} = (\text{R}\beta)$$

$$\text{R}^{\rho - 1}\left[1 - \left(\frac{(\alpha - 1)℧}{(1 - ℧)}\right)\right]^{\rho} = \bar{\beta}_{\text{FHW}}.$$

Somewhat miraculously, at this value of β, because $\chi = 0$, (65) holds with equality, which means that $\bar{\beta}_{\text{FHW}} = \bar{\beta}_{\text{GIC}_{\Phi}}$. This means that the GIC_{Φ} defines the definitive boundary condition: A finite target m^e exists so long as $\beta < \bar{\beta}_{\text{GIC}_{\Phi}} = \Phi^{\rho}/\text{R}$.

B.4 Solutions Exist Even When Growth Impatience Fails

We have just demonstrated that satisfying the GIC_{Φ} condition is necessary and sufficient to guarantee existence of a target $\hat{m}^e$. But we suggested earlier that a weaker condition, the GIC-TBS, guarantees the existence of a well-defined consumption function.

This can be understood as follows. Rewrite the requirement for existence of a target, (59), as

$$\kappa(1 + \varpi)^{1/\rho} + 1 > RNrm, \tag{68}$$

or taking logs we have approximately

$$\kappa(1 + \varpi)^{1/\rho} > \text{r} - \varphi. \tag{69}$$

The LHS captures the slope of the $\Delta c^e = 0$ locus, which is κ modified by ϖ whose difference from $\varpi = 0$ captures the degree of growth (im)patience.[32] The RHS captures the slope of the $\Delta m^e = 0$ locus. Recall that the inequality captures the fact that a target

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[32]'Captures' does not mean 'is equal to.' Equation (59) provides the actual formula for the slope.

$\hat{m}^e$ exists if these two loci intercept, which happens if the slope of $\Delta c^e = 0$ exceeds that of $\Delta m^e = 0$.

If the consumer is 'growth patience poised' (that is, $Þ_\Phi = 1$), then $\varpi = 0$ and the slope of the $\Delta c^e = 0$ locus is identical to the κ that characterizes the perfect foresight consumption function. In this case (69) becomes

$$\mathsf{r} - \rho^{-1}(\mathsf{r} - \tau) > \mathsf{r} - \varphi$$
$$\varphi > \rho^{-1}(\mathsf{r} - \tau), \tag{70}$$

which is the (log version of) the GIC$_\Phi$. The condition cannot hold both as an equality $Þ_\Phi = 1$ (our starting assumption) and an inequality $Þ_\Phi < 1$ (the conclusion of (70)). This contradiction constitutes a proof that exactly at $Þ_\Phi = 1$ a target does not exist.

As noted above, if the consumer is growth-impatient ($Þ_\Phi < 1$) then $\varpi > 0$ and the slope of $\Delta c^e = 0$ is monotonically increased as the degree of growth-impatience increases (so that target $\hat{m}^e$ is diminished).

But if the consumer is growth-patient ($Þ_\Phi > 1$) then $\varpi < 0$ and the slope of $\Delta c^e = 0$ is diminished (which reflects the fact that the greater the degree of patience, the lower will consumption be for any given m^e).[33] The lower bound is defined by the point at which the degree of growth patience becomes so strong that the slope of $\Delta c^e = 0$ reaches zero (when $\Pi = 0$; equivalently, ϖ reaches -1). This restricts the permissible degree of growth patience, because $\Pi > 0$ requires (rewrite (22)):

$$\left(\frac{(\mathsf{R}\beta(1 - \mho))^{1/\rho}}{\mathsf{G}/(1 - \mho)} \right) = \left(\frac{(\mathsf{R}\beta(1 - \mho))^{1/\rho}}{\Phi} \right) < 1. \tag{71}$$

Expanding on a discussion in the main text, the numerator in the leftmost expression reflects the sense in which the unemployment risk acts in a manner similar to the effect of an extra degree of discounting (reflecting the fact that the relevant condition applies only so long as the consumer remains in employment – a condition whose probability is $(1 - \mho)$), while the denominator reflects the mechanical effect in which the relevant measure of growth is boosted by the adjustment that preserves human wealth. Writing the perfect foresight version of the growth patience factor as $Þ_\mathsf{G}$ (which is just the limit as $\mho \downarrow 0$), we can see that the expression on the LHS is just $Þ_\mathsf{G}(1 - \mho)^{1+1/\rho}$ which is smaller than $Þ_\mathsf{G}$ because $\mho > 0$ and $1 + \rho^{-1} > 0$. So, the GIC-TBS holds whenever the plain-vanilla GIC$_\Phi$ holds, but not vice-versa; there are parametric configurations in which a perfect-foresight consumer with income growth rate G would not satisfy the relevant GIC$_\mathsf{G}$ (so, their wealth-to-income ratio would go to infinity), but the same consumer faced the same human wealth but with an unemployment risk $\mho$ would have a finite target wealth-to-income ratio.

The easiest way to understand all of this is graphically. A notebook Carroll (Ongoing) (see references for details) in the code archive associated with these lecture notes shows how this works for alternative values of β.

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[33]Note that if the GIC$_\Phi$ fails so that $\Pi < 1$, the slope of the $\Delta c^e = 0$ locus is shallower than the slope of the perfect foresight consumption function. The fact that these two loci never intersect reflects the fact that the consumer will behave in such a way as to accumulate m^e forever.

C The Exact Formula for Target m

To simplify the expressions in the derivations below, we define $\zeta \equiv RNrm\kappa\Pi$ so that $R\kappa\Pi = \zeta\mathbf{\Phi}$ and we drop the e superscripts, allowing (29) to be rewritten as

$$c = \left(\frac{\zeta}{1+\zeta}\right) m. \tag{72}$$

If a target value $\hat{m}$ exists it will be at the point of intersection between the $\Delta c^e = 0$ and the $\Delta m^e = 0$ loci:

$$\left(\frac{\zeta}{1+\zeta}\right)\hat{m} = (1 - RNrm^{-1})\hat{m} + RNrm^{-1}$$

$$\left(RNrm\frac{\zeta}{1+\zeta}\right)\hat{m} = (RNrm - 1)\hat{m} + 1$$

$$\left(RNrm\left\{\frac{\zeta}{1+\zeta} - 1\right\} + 1\right)\hat{m} = 1$$

$$\left(RNrm\left\{\frac{\zeta - (1+\zeta)}{1+\zeta}\right\} + \frac{1+\zeta}{1+\zeta}\right)\hat{m} = 1$$

$$\left(\frac{1+\zeta - RNrm}{1+\zeta}\right)\hat{m} = 1 \tag{73}$$

$$\hat{m} = \left(\frac{1+\zeta}{1+\zeta - RNrm}\right)$$

$$\hat{m} = \left(\frac{1+\zeta + RNrm - RNrm}{1+\zeta - RNrm}\right)$$

$$= 1 + \left(\frac{RNrm}{1+\zeta - RNrm}\right)$$

$$= 1 + \left(\frac{R}{\mathbf{\Phi} + \zeta\mathbf{\Phi} - R}\right).$$

A first point about this formula is suggested by the fact that

$$\zeta\mathbf{\Phi} = R\kappa\left(1 + \left(\frac{\mathbf{P}_{\mathbf{\Phi}}^{-\rho} - 1}{\mho}\right)\right)^{1/\rho} \tag{74}$$

which is likely to increase as $\mho$ approaches zero.[34] Note that the limit as $\mho \to 0$ is infinity, which implies that $\lim_{\mho \to 0} \hat{m} = 1$. This is precisely what would be expected from this model in which consumers are impatient but self-constrained to have $m^e > 1$: As the risk gets infinitesimally small, the amount by which target m^e exceeds its minimum possible value shrinks to zero.

We now show that the RIC and GIC$_{\mathbf{\Phi}}$ ensure that the denominator of the fraction in

---

[34]'Likely' but not certain because of the fact that $\mho$ affects $\mathbf{P}_{\mathbf{\Phi}}$ as well as appearing in the denominator of (73); however, for plausible calibrations the effect of the denominator predominates.

([73](#)) is positive:

$$\mathbf{\Phi} + \zeta\mathbf{\Phi} - \mathsf{R} = \mathbf{\Phi} + \mathsf{R}\kappa\Pi - \mathsf{R}$$

$$= \mathbf{\Phi} + \mathsf{R}\left(1 - \frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)\left(\frac{(\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathbf{\Phi}})^{-\rho} - 1}{\mho} + 1\right)^{1/\rho} - \mathsf{R}$$

$$> \mathbf{\Phi} + \mathsf{R}\left(1 - \frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)\left(\frac{(\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathbf{\Phi}})^{-\rho} - 1}{1} + 1\right)^{1/\rho} - \mathsf{R}$$

$$= \mathbf{\Phi} + \mathsf{R}\left(1 - \frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)\frac{\mathbf{\Phi}}{(\mathsf{R}\beta)^{1/\rho}} - \mathsf{R}$$

$$= \mathbf{\Phi} + \mathsf{R}\frac{\mathbf{\Phi}}{(\mathsf{R}\beta)^{1/\rho}} - \mathbf{\Phi} - \mathsf{R}$$

$$= \mathsf{R}\left(\frac{\mathbf{\Phi}}{(\mathsf{R}\beta)^{1/\rho}} - 1\right)$$

$$> 0.$$

However, note that $\mho$ also affects $\mathbf{\Phi}$; thus, the first inequality above does not necessarily imply that the denominator is decreasing as $\mho$ moves from 0 to 1.

D Approximating Target m

Now defining

$$\aleph = \left(\frac{\mathbf{Þ}_{\mathbf{\Phi}}^{-\rho} - 1}{\mho}\right), \tag{75}$$

under certain conditions we can obtain further insight into ([73](#)) using a judicious mix of first- and second-order Taylor expansions (along with $\kappa = -\text{þ}_\mathsf{r}$):[35]

$$\zeta\mathbf{\Phi} = \mathsf{R}\kappa\left(1 + \aleph\right)^{1/\rho}$$
$$\approx -\mathsf{R}\text{þ}_\mathsf{r}\left(1 + \rho^{-1}\aleph + (\rho^{-1})(\rho^{-1} - 1)(\aleph^2/2)\right) \tag{76}$$
$$= -\mathsf{R}\text{þ}_\mathsf{r}\left(1 + \rho^{-1}\aleph\left\{1 + \left(\frac{1 - \rho}{\rho}\right)(\aleph/2)\right\}\right).$$

But

$$\aleph = \left(\frac{(1 + \text{þ}_\varphi)^{-\rho} - 1}{\mho}\right)$$
$$\approx \left(\frac{1 - \rho\text{þ}_\varphi - 1}{\mho}\right) \tag{77}$$
$$\approx -\left(\frac{\rho\text{þ}_\varphi}{\mho}\right)$$

---

[35]See below for caveats.

which is guaranteed to be positive by the GIC$_\mathbf{\Phi}$, but which can take any value in the interval $(0, \infty)$. Note, however, that the approximations above are valid only if $\aleph$ is 'small' which requires that the degree of growth impatience be small relative to the size of the unemployment risk. Thus, the formulae derived above (and used below) are reliable only in rather special circumstances, in particular when the consumer is only very slightly growth-impatient.[36] Under these circumstances, this approximation can be substituted into (76) to obtain

$$\zeta\mathbf{\Phi} \approx -\mathsf{R}\text{þ}_{\mathsf{r}}\left(1 - (\text{þ}_\varphi/\mho)(1 + (1-\rho)(-\text{þ}_\varphi/\mho)/2)\right)$$

$$\approx \underbrace{-\mathsf{R}\text{þ}_{\mathsf{r}}}_{>0}\left\{1 \underbrace{-(\text{þ}_\varphi/\mho)}_{>0}\left(1 + \underbrace{(1-\rho)}_{<0}\underbrace{(-\text{þ}_\varphi/\mho)}_{>0}/2\right)\right\}. \tag{78}$$

and inspired by Kimball (1990) defining a term related to the excess of prudence over the logarithmic case,

$$\omega = \left(\frac{\rho - 1}{2}\right), \tag{79}$$

(73) can be approximated by

$$\hat{m} \approx 1 + \left(\frac{1}{\mathbf{\Phi}/\mathsf{R} - \text{þ}_{\mathsf{r}}\left(1 - (\text{þ}_\varphi/\mho)(1 - (-\text{þ}_\varphi/\mho)\omega)\right) - 1}\right)$$

$$\approx 1 + \left(\frac{1}{(\varphi - \mathsf{r}) + (-\text{þ}_{\mathsf{r}})\left(1 + (-\text{þ}_\varphi/\mho)(1 - (-\text{þ}_\varphi/\mho)\omega)\right)}\right) \tag{80}$$

where negative signs have been preserved in front of the þ_{r} and þ_φ terms as a reminder that the GIC$_\mathbf{\Phi}$ and the RIC imply these terms are themselves negative (so that $-\text{þ}_{\mathsf{r}}$ and $-\text{þ}_\varphi$ are positive). *Ceteris paribus*, an increase in relative risk aversion ρ will increase ω and thereby decrease the denominator of (80). This suggests that greater risk aversion will result in a larger target level of wealth.[37]

The formula also provides insight about how the human wealth effect works in equilibrium. All else equal, the human wealth effect is captured by the $(\varphi - \mathsf{r})$ term in the denominator of (80), and it is obvious that a larger value of φ will result in a smaller target value for m. But it is also clear that the size of the human wealth effect will depend on the magnitude of the patience and prudence contributions to the denominator, and that those terms can easily dominate the human wealth effect. This reduction in the human wealth effect is interesting because practitioners have known at least since Summers (1981) that the human wealth effect is implausibly large in the perfect foresight model.

For (80) to make sense, we need the denominator of the fraction to be a positive

---

[36]Other approximations are better for consumers who are highly impatient, relative to their unemployment risk; in this case a better approximation to (76) is obtained by rewriting it as $\zeta\mathbf{\Phi} = \mathsf{R}\kappa\aleph^{1/\rho}\left(\aleph^{-1}1 + 1\right)^{1/\rho}$ and approximating using $(1 + \aleph^{-1})^{1/\rho} \approx 1 + \rho^{-1}\aleph^{-1} + \rho^{-1}(\rho^{-1} - 1)\aleph^{-2}/2$.

[37]"Suggests" because this derivation used some dubious approximations; the suggestion is verified, however, for plausible numerical calibrations.

number; defining

$$\hat{\text{þ}}_\varphi = \text{þ}_\varphi(1 - (-\text{þ}_\varphi/\mho)\omega), \tag{81}$$

this means that we need:

$$
\begin{aligned}
(\varphi - \text{r}) &> \text{þ}_\text{r} - \text{þ}_\text{r}\hat{\text{þ}}_\varphi/\mho \\
&= \left(\rho^{-1}(\text{r} - \tau) - \text{r}\right) - \text{þ}_\text{r}\hat{\text{þ}}_\varphi/\mho \\
\varphi &> \rho^{-1}(\text{r} - \tau) - \text{þ}_\text{r}\hat{\text{þ}}_\varphi/\mho \\
0 &> \underbrace{\rho^{-1}(\text{r} - \tau) - \varphi}_{\text{þ}_\varphi} - \text{þ}_\text{r}(\hat{\text{þ}}_\varphi/\mho) \\
0 &> \text{þ}_\varphi - \text{þ}_\text{r}(\hat{\text{þ}}_\varphi/\mho).
\end{aligned}
\tag{82}
$$

But since the RIC guarantees $\text{þ}_\text{r} < 0$ and the GIC$_\Phi$ guarantees $\text{þ}_\varphi < 0$ (which, in turn, guarantees $\hat{\text{þ}}_\varphi < 0$), this condition must hold.[38]

The same set of derivations imply that we can replace the denominator in (80) with the negative of the RHS of (82), yielding a more compact expression for the target level of resources,

$$
\begin{aligned}
\hat{m} &\approx 1 + \left(\frac{1}{\text{þ}_\text{r}(\hat{\text{þ}}_\varphi/\mho) - \text{þ}_\varphi}\right) \\
&= 1 + \left(\frac{1/(-\text{þ}_\varphi)}{1 + (-\text{þ}_\text{r}/\mho)(1 + (-\text{þ}_\varphi/\mho)\omega)}\right).
\end{aligned}
\tag{83}
$$

This formula makes plain the fact that an increase in either form of impatience, by increasing the denominator of the fraction in (83), will reduce the target level of assets.

We are now in position to discuss (80), understanding that the impatience conditions guarantee that its denominator is a positive number.

Two specializations of the formula are particularly useful. The first is the case where $\rho = 1$ (logarithmic utility). In this case

$$
\begin{aligned}
\text{þ}_\text{r} &= -\tau \\
\text{þ}_\varphi &= \text{r} - \tau - \varphi \\
\omega &= 0
\end{aligned}
\tag{84}
$$

and the approximation becomes

$$\hat{m} \approx 1 + \left(\frac{1}{(\varphi - \text{r}) + \tau(1 + (\varphi + \tau - \text{r})/\mho)}\right) \tag{85}$$

which neatly captures the effect of an increase in human wealth (via either increased

---

[38] In more detail: For the second-order Taylor approximation in (76), we implicitly assume that the absolute value of the second-order term is much smaller than that of the first-order one, i.e. $|\rho^{-1}\aleph| \geq |(\rho^{-1})(\rho^{-1} - 1)(\aleph^2/2)|$. Substituting (77), the above could be simplified to $1 \geq (-\text{þ}_\varphi/\mho)\omega$, therefore we have $\hat{\text{þ}}_\varphi < 0$. This simple justification is based on the confidence that we have proved above that RIC and GIC$_\Phi$ guarantee the denominator of the fraction in (73) is positive.

φ or reduced r), the effect of increased impatience τ, or the effect of a reduction in unemployment risk $\mho$ in reducing target wealth.

The other useful case to consider is where r $= \tau$ but $\rho > 1$. In this case, we have

$$
\begin{aligned}
\flat_{\mathsf{r}} &= -\tau \\
\flat_{\varphi} &= -\varphi \\
\hat{\flat}_{\varphi} &= -\varphi(1 - (\varphi/\mho)\omega)
\end{aligned} \tag{86}
$$

so that

$$
\hat{m} \approx 1 + \left(\frac{1}{(\varphi - \mathsf{r}) + \tau(1 + (\varphi/\mho)(1 - (\varphi/\mho)\omega))} \right) \tag{87}
$$

where the additional term involving ω in this equation captures the fact that an increase in the prudence term ω shrinks the denominator and thereby boosts the target level of wealth.[39]

E Numerical Solution

E.1 The Consumption Function

To solve the model by the method of *reverse shooting*,[40] we need c_t^e as a function of c_{t+1}^e. Starting with (11):

$$
\left(\frac{c_{t+1}^e}{c_t^e} \right) = \mathbf{\Phi}^{-1}(\mathsf{R}\beta)^{1/\rho} \left\{ 1 + \mho \left[\left(\frac{c_{t+1}^e}{c_{t+1}^u} \right)^\rho - 1 \right] \right\}^{1/\rho}
$$

$$
c_t^e = \left(\frac{c_{t+1}^e}{\mathbf{\Phi}^{-1}(\mathsf{R}\beta)^{1/\rho} \left\{ 1 + \mho \left[\left(\frac{c_{t+1}^e}{\kappa(m_{t+1}^e - 1)} \right)^\rho - 1 \right] \right\}^{1/\rho}} \right) \tag{88}
$$

$$
= \mathbf{\Phi}(\mathsf{R}\beta)^{-1/\rho} c_{t+1}^e \left\{ 1 + \mho \left[\left(\frac{c_{t+1}^e}{\kappa(m_{t+1}^e - 1)} \right)^\rho - 1 \right] \right\}^{-1/\rho}.
$$

Inverting (30), the reverse shooting equation for m_t^e is

$$
m_t^e = RNrm^{-1}(m_{t+1}^e - 1) + c_t^e. \tag{89}
$$

The reverse shooting approximation will be more accurate if we use it to obtain estimates of the marginal propensity to consume as well. These are obtained by differentiating the consumption Euler equation with respect to m_t:

---

[39]It would be inappropriate to use the equation to consider the effect of an increase in r because the equation was derived under the assumption $\tau = \mathsf{r}$ so r is not free to vary.

[40]See Judd (1998) for a presentation of shooting methods of solution for numerical difference and differential equations.

$$u'(c^e(m_t)) = \overbrace{RNrm\beta\mathbf{\Phi}^{1-\rho}}^{\beth}\,\mathbb{E}_t[u'(c^\bullet(m_{t+1}))] \tag{90}$$
$$u''(c^e(m_t))\boldsymbol{\kappa}^e(m_t) = \beth RNrm(1-\boldsymbol{\kappa}^e(m_t))\,\mathbb{E}_t[u''(c^\bullet(m_{t+1}))\boldsymbol{\kappa}^\bullet(m_{t+1})]$$

so that defining, e.g., $\kappa^e_t = \boldsymbol{\kappa}^e(m_t)$ we have

$$\kappa^e_t = (1-\kappa^e_t)\underbrace{\beth RNrm(1/u''(c^e_t))\,\mathbb{E}_t\left[u''(c^\bullet_{t+1})\kappa^\bullet_{t+1}\right]}_{\equiv\,\natural_{t+1}}$$

$$(1+\natural_{t+1})\kappa^e_t = \natural_{t+1}$$

$$\kappa^e_t = \left(\frac{\natural_{t+1}}{1+\natural_{t+1}}\right).$$

At the target level of m^e we have

$$\overbrace{(1/u''(\hat{c}^e))\,\mathbb{E}_t\left[u''(c^\bullet)\kappa^\bullet\right]}^{\natural/RNrm\beth} = \mho\overbrace{(u''(\hat{c}^e)/u''(\hat{c}^e))}^{=1}\kappa^e + \mho(u''(\hat{c}^u)/u''(\hat{c}^e))\kappa$$

so that

$$\natural = \beth RNrm(\mho\kappa^e + \mho(\hat{c}^u/\hat{c}^e)^{-\rho-1}\kappa) \tag{91}$$

yielding from (91) a quadratic equation in κ^e:

$$\left(1+\beth RNrm(\mho\kappa^e + \mho(\hat{c}^u/\hat{c}^e)^{-\rho-1}\kappa)\right)\kappa^e = \beth RNrm(\mho\kappa^e + \mho(\hat{c}^u/\hat{c}^e)^{-\rho-1}\kappa) \tag{92}$$

which has one solution for κ^e in the interval $[0,1]$, which is the MPC at target wealth.[41]

The limiting MPC as consumption approaches zero, $\bar{\kappa}^e$, will also be useful; this is obtained by noting that utility in the employed state next year becomes asymptotically irrelevant as c^e_t approaches zero, so that

$$\lim_{c^e_t\to 0}\overbrace{\beth RNrm\kappa^e_{t+1}\left(\mho(c^e_{t+1}/c^e_t)^{-\rho-1} + \mho(c^u_{t+1}/c^e_t)^{-\rho-1}\kappa\right)}^{\natural_{t+1}} = \beth RNrm\mho(c^u_{t+1}/c^e_t)^{-\rho-1}\kappa$$

$$= \beth RNrm\mho(\kappa RNrm a^e_t/(a^e_t(\bar{\kappa}^e/(1-\bar{\kappa}^e))))^{-\rho-1}\kappa$$

$$= \beth RNrm\mho(\kappa RNrm((1-\bar{\kappa}^e)/\bar{\kappa}^e))^{-\rho-1}\kappa$$

so that from (91) we have

$$\bar{\kappa}^e \equiv \lim_{m_t\to 0}\boldsymbol{\kappa}^e(m_t) = \left(\frac{\beth RNrm\mho(\kappa RNrm((1-\bar{\kappa}^e)/\bar{\kappa}^e))^{-\rho-1}\kappa}{1+\beth RNrm\mho(\kappa RNrm((1-\bar{\kappa}^e)/\bar{\kappa}^e))^{-\rho-1}\kappa}\right) \tag{93}$$

which implicitly defines $\bar{\kappa}^e$. An explicit solution is not available, but after parameter values have been defined a numerical rootfinder can calculate a solution almost instantly.

Finally, it will be useful to have an estimate of the curvature (second derivative) of the consumption function. This can be obtained by a procedure analogous to the one used to obtain the MPC: differentiate the differentiated Euler equation (90) again. Noting

---

[41] The *Mathematica* code constructs this derivative and solves the quadratic equation analytically; the Matlab code simply copies the analytical formula generated by *Mathematica* .

that $\kappa^{u\prime} = 0$ we can obtain:

$$(\boldsymbol{\kappa}_t^e)^2 \mathrm{u}'''(\mathrm{c}_t^e) + \boldsymbol{\kappa}_t^{e\prime} \mathrm{u}''(\mathrm{c}_t^e) =$$
$$\beth RNrm \left\{ (-\boldsymbol{\kappa}_t^{e\prime}) \, \mathbb{E}_t[\mathrm{u}''(\mathrm{c}_{t+1}^\bullet)\boldsymbol{\kappa}_{t+1}^\bullet] + RNrm(1-\boldsymbol{\kappa}_t^e)^2 \left(\mathbb{E}_t[(\boldsymbol{\kappa}_{t+1}^\bullet)^2\mathrm{u}'''(\mathrm{c}_{t+1}^\bullet)] + \wp\mathrm{u}''(\mathrm{c}_{t+1}^e)\boldsymbol{\kappa}_{t+1}^{e\prime} \right) \right\}$$

so that

$$\boldsymbol{\kappa}_t^{e\prime} = \left(\frac{\beth RNrm^2(1-\boldsymbol{\kappa}_t^e)^2 \left(\mathbb{E}_t[(\boldsymbol{\kappa}_{t+1}^\bullet)^2\mathrm{u}'''(\mathrm{c}_{t+1}^\bullet)] + \wp\mathrm{u}''(\mathrm{c}_{t+1}^e)\boldsymbol{\kappa}_{t+1}^{e\prime} \right) - (\boldsymbol{\kappa}_t^e)^2\mathrm{u}'''(\mathrm{c}_t^e)}{\mathrm{u}''(\mathrm{c}_t^e) + \beth RNrm \, \mathbb{E}_t[\mathrm{u}''(\mathrm{c}_{t+1}^\bullet)\boldsymbol{\kappa}_{t+1}^\bullet]} \right)$$

which can be further simplified at the target because $\boldsymbol{\kappa}_t^{e\prime}(\hat{m}) = \boldsymbol{\kappa}_{t+1}^{e\prime}(\hat{m}) = \kappa^{e\prime}$ so that

$$\kappa^{e\prime} = \left(\frac{\beth RNrm^2(1-\kappa^e)^2 \, \mathbb{E}_t[(\kappa^\bullet)^2\mathrm{u}'''(\mathrm{c}^\bullet)] - (\kappa^e)^2\mathrm{u}'''(\hat{c}^e)}{\mathrm{u}''(\hat{c}^e) + \beth RNrm \, \mathbb{E}_t[\mathrm{u}''(\mathrm{c}^\bullet)\kappa^\bullet] - \beth RNrm^2(1-\kappa^e)^2\wp\mathrm{u}''(\hat{c}^e)} \right). \tag{94}$$

Another differentiation of (94) similarly allows the construction of a formula for the value of $\kappa^{e\prime\prime}$ at the target $\hat{m}$; in principle, any number of derivatives can be constructed in this manner.[42]

It is plausible to hypothesize that in the limit as m_t^e approaches infinity, the ratio $\kappa_{t+1}^{e\prime}/\kappa_t^{e\prime}$ approaches a constant μ. If we make such an assumption, then we can use other results to show that

$$\lim_{m_t^e \to \infty} \boldsymbol{\kappa}_t^{e\prime} = \left(\frac{\beth RNrm^2(1-\kappa)^2 \left(\kappa^2\mathrm{u}'''(c_{t+1}) + \wp\mathrm{u}''(c_{t+1})\mu\boldsymbol{\kappa}_t^{e\prime} \right) - \kappa^2\mathrm{u}'''(c_t)}{\mathrm{u}''(c_t) + \beth RNrmu''(c_{t+1})\kappa} \right)$$
$$= \left(\frac{\beth RNrm^2(1-\kappa)^2 \left(\kappa^2\mathrm{u}'''(c_{t+1}) \right) - \kappa^2\mathrm{u}'''(c_t)}{\mathrm{u}''(c_t) + \beth RNrmu''(c_{t+1})\kappa - \beth RNrm^2(1-\kappa)^2\wp\mathrm{u}''(c_{t+1})\mu} \right)$$
$$= \left(\frac{\beth RNrm^2(1-\kappa)^2 \left(\kappa^2\mathrm{u}'''(c_t\Phi) \right) - \kappa^2\mathrm{u}'''(c_t)}{\mathrm{u}''(c_t) + \beth RNrmu''(c_t\Phi)\kappa - \beth RNrm^2(1-\kappa)^2\wp\mathrm{u}''(c_t\Phi)\mu} \right) \tag{95}$$
$$= \left(\frac{\mathrm{u}'''(c_t)}{\mathrm{u}''(c_t)} \right) \left(\frac{\beth RNrm^2(1-\kappa)^2 \left(\kappa^2\Phi^{-\rho-2} \right) - \kappa^2}{1 + \beth RNrm\Phi^{-\rho-1}\kappa - \beth RNrm^2(1-\kappa)^2\wp\Phi^{-\rho-1}\mu} \right)$$
$$= -\left(\frac{(1+\rho)}{c_t} \right) \left(\frac{\beth RNrm^2(1-\kappa)^2 \left(\kappa^2\Phi^{-\rho-2} \right) - \kappa^2}{1 + \beth RNrm\Phi^{-\rho-1}\kappa - \beth RNrm^2(1-\kappa)^2\wp\Phi^{-\rho-1}\mu} \right)$$

Separately, the limit of (94) as $c_t^e \to 0$ will be dominated by the outcomes for the unemployed consumer so that

$$\lim_{c_t^e \to 0} \left\{ (\bar{\kappa}^e)^2\mathrm{u}'''(c_t^e) + \kappa_t^{e\prime}\mathrm{u}''(c_t^e) \right\} = \lim_{c_t^e \to 0} \left\{ \beth RNrm(-\kappa_t^{e\prime})\mho\mathrm{u}''(c_{t+1}^u)\kappa + \beth RNrm^2(1-\bar{\kappa}^e)^2\mho(\kappa)^2\mathrm{u}'''(c_{t+1}^u) \right\}$$

---

[42] *Mathematica* permits the convenient computation of the analytical derivatives, and then the substitution of constant target values to obtain analytical expressions like (94). These solutions are simply imported by hand into the Matlab code.

becomes

$$\kappa_t^{e\prime}(\mathrm{u}''(c_t^e) + \beth RNrm\mho\mathrm{u}''(c_{t+1}^u)) = \beth RNrm^2(1-\bar{\kappa}^e)^2\mho(\kappa)^2\mathrm{u}'''(c_{t+1}^u) - (\bar{\kappa}^e)^2\mathrm{u}'''(c_t^e)$$

$$\kappa_t^{e\prime} = \left(\frac{\beth RNrm^2(1-\bar{\kappa}^e)^2\mho(\kappa)^2\mathrm{u}'''(c_{t+1}^u) - (\bar{\kappa}^e)^2\mathrm{u}'''(c_t^e)}{\mathrm{u}''(c_t^e) + \beth RNrm\mho\mathrm{u}''(c_{t+1}^u)} \right) \tag{96}$$

which approaches a function γ/c_t^e as c_t^e approaches zero. This is inconvenient, because an infinite second derivative cannot be used for a data point in the set of interpolating points.

endCDC

Reverse shooting requires us to solve separately for an approximation to the consumption function above the steady state and another approximation below the steady state. Using the approximate steady-state κ^e and $\kappa^{e\prime}$ obtained above, we begin by picking a very small number for ▲ and then creating a Taylor approximation to the consumption function near the steady state:

$$m_t^e = \hat{m} + \blacktriangle$$
$$\tilde{\mathbf{c}}(\blacktriangle) = \hat{c}^e + \blacktriangle\kappa^e + (\blacktriangle^2/2)\kappa^{e\prime} + (\blacktriangle^3/6)\kappa^{e\prime\prime}$$

and then iterate the reverse-shooting equations until we reach some period n in which m_{t-n}^e escapes some pre-specified interval $[\underline{m}^e, \bar{m}^e]$ (where the natural value for $\underline{m}^e$ is 1 because this is the m that would be owned by a consumer who had saved nothing in the prior period and therefore is below any feasible value of m that could be realized by an optimizing consumer). This generates a sequence of points all of which are on the consumption function. A parallel procedure (substituting $-$ for $+$ in (97) and where appropriate in the corresponding equation for c generates the sequence of points for the approximation below the steady state. Taken together with the already-derived characterization of the function at the target level of wealth, these points constitute the basis for a piecewise second-order interpolating approximation to the consumption function on the interval $[\underline{m}^e, \bar{m}^e]$.

E.2 The Value Function

As a preliminary, note that since $\mathrm{u}(xy) = \mathrm{u}(x)y^{1-\rho}$, value for an unemployed consumer is

$$\begin{aligned}
\mathrm{V}_t^u &= \mathrm{u}(C_t^u) + \beta\mathrm{u}(C_{t+1}^u) + \beta^2\mathrm{u}(C_{t+2}^u) + \dots \\
&= \mathrm{u}(C_t^u)\left(1 + \beta\{(\mathsf{R}\beta)^{1/\rho}\}^{1-\rho} + \beta^2\left\{(\mathsf{R}\beta)^{2/\rho}\right\}^{1-\rho} + \dots\right) \\
&= \mathrm{u}(C_t^u)\underbrace{\left(\frac{1}{1 - \beta(\mathsf{R}\beta)^{(1/\rho)-1}}\right)}_{\equiv\mathfrak{v}}
\end{aligned} \tag{97}$$

where the RIC guarantees that the denominator in the fraction is a positive number.

From this we can see that value for the normalized problem is similarly:

$$\mathrm{v}^u(m_t) = \mathrm{u}(\kappa m_t)\mathfrak{v}. \tag{98}$$

Turning to the problem of the employed consumer, we have

$$\mathrm{v}^e(m_t) = \mathrm{u}(c_t^e) + \beta \mathbf{\Phi}^{1-\rho} \, \mathbb{E}_t[\mathrm{v}^\bullet(m_{t+1})] \tag{99}$$

and at the target level of market resources this will be unchanging for a consumer who remains employed so that

$$
\begin{aligned}
\hat{v}^e &= \mathrm{u}(\hat{c}^e) + \beta \mathbf{\Phi}^{1-\rho} \left(\mho \hat{v}^e + \mho \mathrm{v}^u(a^e RNrm) \right) \\
(1 - \beta \mathbf{\Phi}^{1-\rho} \mho) \hat{v}^e &= \mathrm{u}(\hat{c}^e) + \beta \mathbf{\Phi}^{1-\rho} \mho \mathrm{v}^u(a^e RNrm) \\
\hat{v}^e &= \left(\frac{\mathrm{u}(\hat{c}^e) + \beta \mathbf{\Phi}^{1-\rho} \mho \mathrm{v}^u(a^e RNrm)}{(1 - \beta \mathbf{\Phi}^{1-\rho} \mho)} \right).
\end{aligned}
\tag{100}
$$

Given these facts, our recursion for generating a sequence of points on the consumption function can be used at the same time to generate corresponding points on the value function from

$$v_t^e = \mathrm{u}(c_t^e) + \beta \mathbf{\Phi}^{1-\rho} \left(\mho v_{t+1}^e + \mho \mathrm{v}^u(a_t^e RNrm) \right) \tag{101}$$

with the first iteration point generated by numerical integration from

$$v_{\check{t}}^e = \hat{v}^e + \int_0^{\blacktriangle} \mathrm{u}'(\tilde{\mathbf{c}}(\bullet)) d\bullet \tag{102}$$

An alternative method (which does not require numerical integration but does not work as well) is to use the Envelope theorem to compute an approximation to the value function in the vicinity of the steady state:

$$
\begin{aligned}
\mathrm{v}^{e\prime}(m_t^e) &= \mathrm{u}'(\mathrm{c}(m_t^e)) \\
\mathrm{v}^{e\prime\prime}(m_t^e) &= \mathrm{u}''(\mathrm{c}(m_t^e))\kappa_t^e \\
\mathrm{v}^{e\prime\prime\prime}(m_t^e) &= \mathrm{u}''(\mathrm{c}(m_t^e))\kappa_t^{e\prime} + \mathrm{u}'''(\mathrm{c}(m_t^e))(\kappa_t^e)^2
\end{aligned}
\tag{103}
$$

which leads to an approximation of the level of value at $\hat{m} + \blacktriangle$ using

$$\mathrm{v}^e(\hat{m} + \blacktriangle) \approx \hat{v}^e + \mathrm{u}'(\hat{c}^e)\blacktriangle + (\blacktriangle^2/2)\mathrm{u}''(\hat{c}^e)\kappa^e + (\blacktriangle^3/6) \left(\mathrm{u}''(\hat{c}^e)\kappa^{e\prime} + \mathrm{u}'''(\hat{c}^e)(\kappa^e)^2 \right).$$

F The Algorithm

With the above results in hand, the model is solved and the various functions constructed as follows. Define $\star_t = \{m_t^e, c_t^e, \kappa_t^e, v_t^e, \kappa_t^{e\prime}\}$ as a vector of points that characterizes a particular situation that an optimizing employed household might be in at any given point in time. Using the backwards-shooting functions derived above, for any point $\star_{\check{t}}$ we can construct the sequence of points that must have led up to it: $\star_{\check{t}-1}$ and $\star_{\check{t}-2}$ and so on. And using the approximations near the steady state like (97), we can construct a vector-valued function $\circ(\blacktriangle)$ that generates, e.g., $\{\hat{m} + \blacktriangle, \tilde{\mathbf{c}}(\blacktriangle), ...\}$.

Now define an operator $\cdots$ as follows: $\cdots$ applied to some starting point $\star_t$ uses the backwards dynamic equations defined above to produce a vector of points $\star_{t-1}, \star_{t-2}, ...$ consistent with the model until the m_{t-n}^e that is produced goes outside of the pre-defined bounds for solving the problem.

We can merge the points below the steady state with the steady state with the points above the steady state to produce $\ddot{\divideontimes} = \cdots (\mathbf{o}(-\varepsilon)) \cup \mathbf{o}(0) \cup \cdots (\mathbf{o}(\varepsilon))$. These points can then be used to generate appropriate interpolating approximations to the consumption function and other desired functions.

Designate, e.g., the vector of points on the consumption function generated in this manner by $\ddot{\divideontimes}[c]$, so that

$$\{\ddot{\divideontimes}[m], \{\ddot{\divideontimes}[c], \ddot{\divideontimes}[\kappa^e], \ddot{\divideontimes}[\kappa^{e\prime}]\}^{\intercal}\}^{\intercal} = \begin{pmatrix} m[1] & \{c[1], \kappa^e[1], \kappa^{e\prime}[1]\} \\ m[2] & \{c[2], \kappa^e[2], \kappa^{e\prime}[2]\} \\ ... & ... \\ m[N] & \{c[N], \kappa^e[N], \kappa^{e\prime}[N]\} \end{pmatrix} \tag{104}$$

where N is the number of points that have been generated by the merger of the backward shooting points described above.

The object (104) is not an arbitrary example; it reflects a set of values that uniquely define a fourth order piecewise polynomial spline such that at every point in the set the polynomial matches the level and first derivative included in the list. Standard numerical mathematics software can produce the interpolating function with this input; for example, the syntax in *Mathematica* is simply

$$\texttt{cE = Interpolation}[\{\ddot{\divideontimes}[m], \{\ddot{\divideontimes}[c], \ddot{\divideontimes}[\kappa^e], \ddot{\divideontimes}[\kappa^{e\prime}]\}^{\intercal}\}^{\intercal}]. \tag{105}$$

which creates a function `cE` that is a $\mathbf{C}^4$ interpolating polynomial connecting these points. beginCDC

With these points it is feasible to construct an approximating interpolation that will be quite accurate within the bounds $[0, \bar{m}^e]$.[43] This approximation, however, becomes highly problematic when evaluated outside of this range. (Polynomial interpolations tend to 'blow up' when extended outside the interval on which they are constructed.) endCDC

The reverse shooting algorithm terminates at some finite maximum point $\bar{m}$, but for completeness it is useful to have an approximation to the consumption function that is reasonably well behaved for any $\hat{m}$ no matter how large.[44]

Since we know that the consumption function in the presence of uncertainty asymptotes to the perfect foresight function, we adopt the following approach. Defining the level of precautionary saving as[45]

$$\not{c}(m) = \bar{c}(m) - c(m), \tag{106}$$

we know (see the discussion below in appendix section G) that

$$\lim_{m \to \infty} \not{c}(m) = 0. \tag{107}$$

Defining $\vec{m} = m - \bar{m}$, a convenient functional form to postulate for the propensity to

---

[43]A little bit of work is required to obtain a good representation all the way down to 0; see the software for the solution.

[44]An extrapolation of the approximating interpolation will not perform well; a polynomial approximation will inevitably "blow up" if evaluated at large enough $\hat{m}$.

[45]Mnemonic: This is the amount of consumption that is cancelled as a result of uncertainty.

precautionary-save is

$$\not{c}(m) = e^{\phi_0 - \phi_1 \vec{m}} + e^{\gamma_0 - \gamma_1 \vec{m}} \tag{108}$$

with derivatives

$$\begin{aligned}
\not{c}'(m) &= -\phi_1 e^{\phi_0 - \phi_1 \vec{m}} - \gamma_1 e^{\gamma_0 - \gamma_1 \vec{m}} \\
\not{c}''(m) &= \phi_1^2 e^{\phi_0 - \phi_1 \vec{m}} + \gamma_1^2 e^{\gamma_0 - \gamma_1 \vec{m}} \\
\not{c}'''(m) &= -\phi_1^3 e^{\phi_0 - \phi_1 \vec{m}} - \gamma_1^3 e^{\gamma_0 - \gamma_1 \vec{m}}.
\end{aligned} \tag{109}$$

Evaluated at $\bar{m}$ (for which $\not{c}$ and its derivatives will have numerical values assigned by the reverse-shooting solution method described above), this is a system of four equations in four unknowns and, though nonlinear, can be easily solved for values of the ϕ and γ coefficients that match the level and first three derivatives of the "true" $\not{c}$ function.[46]

beginCDC

An alternative approximation can be obtained by defining an 'inverted' value function which is much closer to being linear. Using $\not{p} = 1 - \rho$, we define

$$\mathrm{n}(m) = (\not{p}\mathrm{v})^{1/\not{p}} \tag{110}$$

which has many useful properties, among them that as $\mathrm{v} \to \infty$, $\mathrm{n} \to 0$. The values of $\{\mathrm{n}, \mathrm{n}', \mathrm{n}''\}$ can be computed at the same points for which values of v and its derivatives were constructed using the reverse-shooting method. Outside of the interval, we can use the consumption function to produce n by numerical integration from the boundaries of the interval.

To complete this scheme, we will need the derivatives:

$$\begin{aligned}
\mathrm{n}' &= (\not{p}\mathrm{v})^{(1/\not{p})-1}\mathrm{v}' \\
&= (\mathrm{n}^{\not{p}})^{(1/\not{p})-1}\mathrm{u}' \\
\mathrm{n}'' &= \not{p}((1/\not{p})-1)(\mathrm{n}^{\not{p}})^{(1/\not{p})-2}(\mathrm{u}')^2 + (\not{p}\mathrm{v})^{(1/\not{p})-1}\mathrm{u}''\boldsymbol{\kappa}.
\end{aligned} \tag{111}$$

along with the limits:

$$\begin{aligned}
\lim_{m \to 0} \mathrm{n}(m) &= 0 \\
\lim_{m \to 0} \mathrm{n}'(m) &= \lim_{m \to 0} (\bar{\kappa}^e m)^{-\rho}(\not{p}\mathrm{v})^{\frac{1-\not{p}}{\not{p}}} \\
&= \lim_{m \to 0} (\bar{\kappa}^e m)^{-\rho}(\not{p}(\mathrm{u}(\bar{\kappa}^e m) + \beta\Phi^{1-\rho}\,\mathbb{E}_t[\mathrm{v}^\bullet_{t+1}]))^{\frac{\rho}{1-\rho}} \\
&= \lim_{m \to 0} (\bar{\kappa}^e m)^{-\rho}(\not{p}(\mathrm{u}(\bar{\kappa}^e m) + \beta\Phi^{1-\rho}\mho\mathrm{v}^u_{t+1}(m(1 - \bar{\kappa}^e))))^{\frac{\rho}{1-\rho}} \\
&= \lim_{m \to 0} (\bar{\kappa}^e m)^{-\rho}(\not{p}(\mathrm{u}(\bar{\kappa}^e m) + \beta\Phi^{1-\rho}\mho\mathrm{u}(m(1 - \bar{\kappa}^e)\bar{\kappa}^e)\mathfrak{v}))^{\frac{\rho}{1-\rho}} \\
&= \lim_{m \to 0} (\bar{\kappa}^e m)^{-\rho}((\bar{\kappa}^e m)^{1/(1-\rho)}(1 + \beta\Phi^{1-\rho}\mho(1 - \bar{\kappa}^e)^{1-\rho}\mathfrak{v}))^{\frac{\rho}{1-\rho}} \\
&= \lim_{m \to 0} (1 + \beta\Phi^{1-\rho}\mho(1 - \bar{\kappa}^e)^{1-\rho}\mathfrak{v})^{\frac{\rho}{1-\rho}}
\end{aligned} \tag{112}$$

Carroll (2022) shows that the value function can be bounded for $m > 0$ by a function

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[46]The exact symmetry in the treatment of γ and ϕ means that there will actually be two symmetrical solutions; either can be used.

of the form

$$F(m) = \eta + m^{1-\rho}$$
$$F'(m) = (1 - \rho)m^{-\rho}$$
$$F''(m) = -\rho(1 - \rho)m^{-\rho-1}$$

(113)

for any $\eta > \underline{\eta} = 2\beth/(1 - \beth)$.

Thus, dropping arguments we define

$$\mathrm{w} = \mathrm{v}F^{-1}$$
$$\mathrm{w}' = \mathrm{v}'F^{-1} - \mathrm{v}F^{-2}F'$$
$$= (\mathrm{v}' - \mathrm{v}F^{-1}F')F^{-1}$$
$$\mathrm{w}'' = \mathrm{v}''F^{-1} - \mathrm{v}'F^{-2}F' - (\mathrm{v}'F^{-2}F' - 2\mathrm{v}F^{-3}(F')^2 + \mathrm{v}F^{-2}F'')$$
$$= \left(\mathrm{v}'' - 2\mathrm{v}'F^{-1}F' + 2\mathrm{v}F^{-2}(F')^2 - \mathrm{v}F^{-1}F''\right)F^{-1}$$

(114)

Thus, dropping arguments we define

$$\mathrm{w}F = \mathrm{v}$$
$$\mathrm{w}'F + \mathrm{w}F' = \mathrm{v}'$$
$$\mathrm{w}''F + \mathrm{w}'F' + \mathrm{w}'F' + \mathrm{w}F'' = \mathrm{v}''$$
$$\mathrm{w}''F + 2\mathrm{w}'F' + \mathrm{w}F'' = \mathrm{v}''$$
$$\mathrm{w}'''F + \mathrm{w}''F' + 2\left(\mathrm{w}''F' + \mathrm{w}'F''\right) + \mathrm{w}'F'' + \mathrm{w}F''' = \mathrm{v}'''$$
$$\mathrm{w}'''F + 3\mathrm{w}''F' + 3\mathrm{w}'F'' + \mathrm{w}F''' = \mathrm{v}'''$$

(115)

and $\mathrm{w}(0)$ must be computed by application of L'Hopital's rule from

$$\lim_{m \to 0} \mathrm{w}(m) = \lim_{m \to 0} \left(\frac{\mathrm{u}(m\bar{\kappa}) + \beta \mathrm{v}^u(m(1 - \bar{\kappa}))}{\eta + m + m^{1-\rho}} \right)$$
$$= \mathrm{u}(\bar{\kappa}) + \mathrm{v}^u(1 - \bar{\kappa})$$

(116)

and

$$\mathrm{w}'(0) = \left(\frac{\mathrm{u}'(\mathrm{c}(m)) - \mathrm{w}(0)F'}{F} \right)$$
$$= \left(\frac{\mathrm{u}'(\mathrm{c}(m)) - \mathrm{w}(0)(1 - (1 - \rho)m^{-\rho})}{\eta + m + m^{1-\rho}} \right)$$

(117)

$$\mathrm{w}''' = \left(\mathrm{v}''' - 3\mathrm{w}''F' - 3\mathrm{w}'F'' - \mathrm{w}F'''\right)F^{-1}$$

(118)

Finally, the limit as m goes to zero will be useful:

$$\underline{w} \equiv \lim_{m \to 0} \left(\frac{\mathrm{v}(m)}{\eta + m + m^{1-\rho}} \right) = \lim_{m \to 0} \left(\frac{\mathrm{v}'(m)}{1 + (1 - \rho)m^{-\rho}} \right)$$

$$= \lim_{m \to 0} \left(\frac{\mathrm{u}'(\mathrm{c}(m))}{1 + (1 - \rho)m^{-\rho}} \right)$$

$$= \lim_{m \to 0} \left(\frac{(m\bar{\kappa}^e)^{-\rho}}{1 + (1 - \rho)m^{-\rho}} \right)$$

$$= \left(\frac{(\bar{\kappa}^e)^{-\rho}}{(1 - \rho)} \right)$$

$$\lim_{m \to 0} \mathrm{w}'(m) = \lim_{m \to 0} \left(\frac{\mathrm{u}'(\mathrm{c}(m)) - (1 + (1 - \rho)m^{-\rho})\mathrm{v}/(\eta + m + m^{1-\rho})}{\eta + m + m^{1-\rho}} \right)$$

$$= \lim_{m \to 0} \left(\frac{\mathrm{u}'(\mathrm{c}(m)) - (1 + (1 - \rho)m^{-\rho})\underline{w}}{\eta + m + m^{1-\rho}} \right)$$

$$= \lim_{m \to 0} \left(\frac{-\rho(\bar{\kappa}^e m)^{-\rho-1}\bar{\kappa}^e + \rho(1 - \rho)m^{-\rho-1}\underline{w}}{(1 + (1 - \rho)m^{-\rho})} \right)$$

$$= \lim_{m \to 0} \left(\frac{-\rho(\bar{\kappa}^e m)^{-1}\bar{\kappa}^e + \rho(1 - \rho)m^{-1}\underline{w}}{(1 - \rho)} \right)$$

$$= \lim_{m \to 0} \left(\frac{-\rho m^{-1} + \rho(1 - \rho)m^{-1}\underline{w}}{(1 - \rho)} \right) \tag{119}$$

Another solution is to note that value for this problem is bounded below by the value that would be experienced by an unemployed consumer and above by the value that would be experienced by a consumer with the same PDV of income but whose income was perfectly certain,

$$\mathrm{v}(m) < \mathrm{v}^e(m) < \bar{\mathrm{v}}(m)$$
$$0 < \mathrm{v}^e(m) - \mathrm{v}(m) < \bar{\mathrm{v}}(m) - \mathrm{v}(m)$$
$$0 < \underbrace{\left(\frac{\mathrm{v}^e(m) - \mathrm{v}(m)}{\bar{\mathrm{v}}(m) - \mathrm{v}(m)} \right)}_{\equiv \varsigma(m)} < 1 \tag{120}$$

so that for any m value can be represented by

$$\mathrm{v}(m) = \varsigma(m)(\bar{\mathrm{v}}(m) - \mathrm{v}(m)) + \mathrm{v}(m) \tag{121}$$

Since $\bar{\mathrm{v}}$ and v are analytical functions, this means that our problem is effectively reduced to approximating the function $\varsigma(m)$ whose value is bounded in the $[0, 1]$ interval. _{endCDC}

G Modified Formulas For Case Where $\Phi \geq \mathsf{R}$

The text asserts that if $\Phi < \mathsf{R}$ the consumption function for a finite-horizon employed consumer approaches the $\bar{\mathrm{c}}_t(m)$ function that is optimal for a perfect-foresight consumer

with the same horizon,

$$\lim_{m\uparrow\infty} \bar{c}_t(m) - c_t(m) = 0. \tag{122}$$

This proposition can be proven by careful analysis of the consumption Euler equation, noting that as m approaches infinity the proportion of consumption will be financed out of (uncertain) labor income approaches zero, and that the magnitude of the precautionary effect is proportional to the square of the proportion of such consumption financed out of uncertain labor income.

A footnote also claims that for employed consumers, $c(m)$ approaches a different, but still well-defined, limit even if $\Phi \geq R$, so long as the impatience condition holds.

It turns out that the limit in question is the one defined by the solution to a perfect foresight problem with liquidity constraints. A semi-analytical solution does exist in this case, but it requires formidable notation and analysis to present and understand, so the details are not presented here. A continuous-time treatement can be found in Park (2006).

References

BLANCHARD, OLIVIER J. (1985): "Debt, Deficits, and Finite Horizons," *Journal of Political Economy*, 93(2), 223–247.

CARROLL, CHRISTOPHER (2020): "Theoretical Foundations of Buffer Stock Saving," *Econ-ARK REMARK*, Available at https://econ-ark.github.io/BufferStockTheory.

CARROLL, CHRISTOPHER D. (1992): "The Buffer-Stock Theory of Saving: Some Macroeconomic Evidence," *Brookings Papers on Economic Activity*, 1992(2), 61–156, https://www.econ2.jhu.edu/people/ccarroll/BufferStockBPEA.pdf.

——— (1997): "Buffer Stock Saving and the Life Cycle/Permanent Income Hypothesis," *Quarterly Journal of Economics*, CXII(1), 1–56.

——— (2001): "Death to the Log-Linearized Consumption Euler Equation! (And Very Poor Health to the Second-Order Approximation)," *Advances in Macroeconomics*, 1(1), Article 6, https://www.econ2.jhu.edu/people/ccarroll/death.pdf.

——— (2022): "Theoretical Foundations of Buffer Stock Saving," *Submitted*.

——— (Ongoing): "Mathematica Notebook Illustrating Target Wealth In Cases Where FHWC-TBS Fails," *./Code/Mathematica/Examples/ManipulateParameters/When-FHWC-Holds.nb*, Download archive and open Mathematica notebook.

CARROLL, CHRISTOPHER D., AND OLIVIER JEANNE (2009): "A Tractable Model of Precautionary Reserves, Net Foreign Assets, or Sovereign Wealth Funds," *NBER Working Paper Number 15228*, https://www.econ2.jhu.edu/people/ccarroll/papers/cjSOE.

CARROLL, CHRISTOPHER D., AND MILES S. KIMBALL (1996): "On the Concavity of the Consumption Function," *Econometrica*, 64(4), 981–992, https://www.econ2.jhu.edu/people/ccarroll/concavity.pdf.

CARROLL, CHRISTOPHER D., AND MILES S. KIMBALL (2007): "Precautionary Saving and Precautionary Wealth," *Palgrave Dictionary of Economics and Finance, 2nd Ed.*, https://www.econ2.jhu.edu/people/ccarroll/papers/PalgravePrecautionary.pdf.

FRIEDMAN, MILTON A. (1957): *A Theory of the Consumption Function*. Princeton University Press.

HALL, ROBERT E. (1988): "Intertemporal Substitution in Consumption," *Journal of Political Economy*, XCVI, 339–357, Available at http://www.stanford.edu/~rehall/Intertemporal-JPE-April-1988.pdf.

JUDD, KENNETH L. (1998): *Numerical Methods in Economics*. The MIT Press, Cambridge, Massachusetts.

KIMBALL, MILES S. (1990): "Precautionary Saving in the Small and in the Large," *Econometrica*, 58, 53–73.

PARK, MYUNG-HO (2006): "An Analytical Solution to the Inverse Consumption Function with Liquidity Constraints," *Economics Letters*, 92, 389–394.

PARKER, JONATHAN A., AND BRUCE PRESTON (2005): "Precautionary Saving and Consumption Fluctuations," *American Economic Review*, 95(4), 1119–1143.

SUMMERS, LAWRENCE H. (1981): "Capital Taxation and Accumulation in a Life Cycle Growth Model," *American Economic Review*, 71(4), 533–544, http://www.jstor.org/stable/1806179.

TOCHE, PATRICK (2005): "A Tractable Model of Precautionary Saving in Continuous Time," *Economics Letters*, 87(2), 267–272.